Primitive higher order embeddings
of abelian surfaces

Th. Bauer[*] and T. Szemberg[†]

January 24, 1996

0. Introduction

The aim of this note is to study higher order embeddings of abelian surfaces defined by primitive line bundles.

In recent years several concepts of higher order embeddings have been studied, in particular k-spannedness, k-very ampleness and k-jet ampleness (see [7] for references on this topic). Recall that a line bundle L on a smooth projective variety X is called k-jet ample, if for any choice of distinct points $x_1, \ldots, x_r$ in X and positive integers $k_1, \ldots, k_r$ with $\sum k_i = k + 1$ the evaluation map

$$H^0(X, L) \longrightarrow H^0\left(X, L \otimes \mathcal{O}_X/(\mathfrak{m}_{x_1}^{k_1} \otimes \ldots \otimes \mathfrak{m}_{x_r}^{k_r})\right)$$

is surjective. Further, L is called k-very ample (resp. k-spanned), if for any zero-dimensional subscheme $(Z, \mathcal{O}_Z)$ of X of length $k + 1$ (resp. for any curvilinear zero-dimensional subscheme $(Z, \mathcal{O}_Z)$ of X of length $k + 1$) the natural map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective. Here a subscheme is called *curvilinear*, if it is locally contained in a smooth curve.

There are interesting geometrical interpretations of k-very ampleness and k-spannedness in terms of secant varieties and embeddings of Hilbert schemes. On the other hand, k-jet ampleness has recently gained interest because of its close connection to Seshadri constants of ample line bundles.

In [4] higher order embeddings of abelian varieties defined by tensor products of ample line bundles were studied. In particular, a bound for the k-jet ampleness of tensor powers of ample line bundles was given:

Theorem ([4]). *Let X be an abelian variety, L an ample line bundle on X and $k \geq 0$. Then mL is k-jet ample for $m \geq k + 2$.*

On the other extreme, one is lead to consider primitive line bundles, i.e. line bundles which are not algebraically equivalent to some nontrivial power of an ample line bundle. The problem then is to give criteria for such a line bundle L to define a higher order embedding in terms of suitable numerical invariants of L. In the case of abelian varieties certainly the *type* of L is the obvious data. Unfortunately even for very ampleness (i.e. embeddings of order $k = 1$) it seems to be a difficult problem to find sharp numerical criteria on abelian varieties of arbitrary dimension (see [11] for the best bounds available at present). In this note we focus on the surface case, where a complete classification can be given.

We show:

Theorem. *Let X be an abelian surface with Picard number 1 and let L be an ample line bundle of type $(1, d)$, $d \geq 1$. Fix a non-negative integer k. Then*
  (a) *L is k-jet ample if and only if $d > \frac{1}{2}(k + 2)^2$,*
  (b) *L is k-very ample if and only if $d \geq 2k + 3$,*
  (c) *L is k-spanned if and only if it is k-very ample.*

For $k = 0$ and $k = 1$ these are the well-known bounds for global generation and very ampleness of primitive line bundles on abelian surfaces. It is known that in these cases the assumption on the Picard number can be weakened by the assumption that X is simple (see Sect. 1). However – as will be shown in Sect. 4 – this is not possible in general. In fact the numerical criteria of the theorem may fail as soon as the Picard number is at least 2.

Notation and Conventions. We work throughout over the field $\mathbb{C}$ of complex numbers.

For a $\mathbb{Q}$-divisor D we denote by $\lceil D \rceil$ its round-up and by $\lfloor D \rfloor$ its round-down or integer part.

We would like to thank the referee for his comments on the first version of this paper.

1. Global generation and very ampleness

To put our investigation of higher order embeddings into perspective we briefly review the situation for global generation and very ampleness of line bundles of type $(1, d)$, $d \geq 1$, on an abelian surface X.

For $d = 1$ the polarized abelian surface (X, L) is either the Jacobian of a smooth curve of genus 2 or the polarized product of elliptic curves – at any rate $H^0(X, L)$ consists of just one divisor.

If $d \geq 2$, then as a consequence of the Decomposition Theorem (see [18, Theorem 4.3.1]) the linear system $|L|$ has fixed components if and only if X

is a product of elliptic curves E_1, E_2 and L is the product polarization $L = \mathcal{O}_X(E_1 \times \{0\} + d(\{0\} \times E_2))$.

From now on assume that L is not of this form.

For $d = 2$ the system $|L|$ is a pencil with exactly four base points. Look at [2] for more on abelian surfaces of type $(1, 2)$.

For $d \geq 3$ the linear system $|L|$ has no base points, since the base locus of $|L|$ is invariant under the action of the kernel $K(L)$ of the canonical homomorphism

$$X \longrightarrow \mathrm{Pic}^0(X), \quad x \longmapsto t_x^* L - L \ ,$$

where $t_x : X \longrightarrow X$, $y \longmapsto x + y$, denotes the translation by x.

In the case $d = 3$ the mapping $X \longrightarrow \mathbb{P}_2$ defined by L is a $6 : 1$-covering branched over a curve of degree 18. See [9] for a detailed investigation of a family of abelian surfaces of type $(1, 3)$.

For $d = 4$ the bundle L cannot be very ample, since there are no abelian surfaces in $\mathbb{P}_3$ by the Lefschetz hyperplane theorem. Nevertheless for generic (X, L) of type $(1, 4)$ the map $X \longrightarrow \mathbb{P}_3$ is birational onto a singular octic in $\mathbb{P}_3$ (see [10]).

Finally, it follows from Reider's theorem that for $d \geq 5$ the line bundle L is very ample if and only if there is no elliptic curve $E \subset X$ with $LC = 2$ (see [18, Theorem 10.4.1]). Actually one can get much more precise information on the elliptic curves obstructing the very ampleness of L (see [17]).

In sum, one has uniform bounds for global generation and very ampleness on abelian surfaces without elliptic curves:

Theorem 1.1 *Let X be a simple abelian surface and let L be an ample line bundle of type $(1, d)$, $d \geq 1$. Then*

(a) *L is globally generated if and only if $d \geq 3$,*

(b) *L is very ample if and only if $d \geq 5$.*

2. Generation of jets

The aim of this section is to give a sharp criterion for k-jet ampleness of primitive line bundles. We begin by showing:

Proposition 2.1 *Let X be an abelian surface with Picard number 1, let L be an ample line bundle of type $(1, d)$, $d \geq 1$ and let k be a non-negative integer. If $d > \frac{1}{2}(k + 2)^2$, then L is k-jet ample.*

A proof of Proposition 2.1 could be given using results of Demailly obtained by analytic methods (cf. [13, 8.12]). However, we prefer to give an algebraic proof using $\mathbb{Q}$-divisors based on the

Kawamata-Viehweg Vanishing Theorem 2.2 *Let Y be a smooth projective surface and let M be a nef and big $\mathbb{Q}$-divisor on Y. Then*

$$H^i(Y, \mathcal{O}_Y(K_Y + \lceil M \rceil)) = 0 \qquad \text{for } i > 0 \ .$$

Note that there is no normal crossing hypothesis in the surface case of Kawamata-Viehweg vanishing thanks to Sakai's lemma (see [15, Lemma 1.1]).

Proof of Proposition 2.1. Let points $x_1, \ldots, x_r$ in X and positive integers $k_1, \ldots, k_r$ with $\sum k_i = k + 1$ be given. We will show that

$$H^1 \left(X, L \otimes \bigotimes_{i=1}^{r} \mathfrak{m}_{x_i}^{k_i} \right) = 0 \ . \tag{$*$}$$

To prove $(*)$ we are going to use $\mathbb{Q}$-divisors much as in the proof of the variant of Reider's theorem given in [15, Sect. 1]. The first step thus consists in producing an effective divisor with high multiplicities at the points x_i via an elementary dimension count: For $n \geq 1$ one has by Riemann-Roch

$$h^0(X, nL) = \frac{1}{2}n^2 L^2 = n^2 d > \frac{1}{2}(k+2)^2 n^2 \ .$$

On the other hand, it imposes at most

$$\sum_{i=1}^{r} \binom{n(k_i + 1) + 1}{2} = \frac{1}{2}n^2 \sum (k_i + 1)^2 + O(n) \leq \frac{1}{2}(k+2)^2 n^2 + O(n)$$

conditions on a divisor to have multiplicities bigger than $n(k_i + 1)$ at the points x_i for $1 \leq i \leq r$. This shows that for $n \gg 0$ there is a divisor $D \in |nL|$ with

$$m_i =_{\mathrm{def}} \mathrm{mult}_{x_i} D > n(k_i + 1) \qquad \text{for } 1 \leq i \leq r \ .$$

Now let $f : \widetilde{X} \longrightarrow X$ be the blow-up of X in the points $x_1, \ldots, x_r$ with exceptional divisors $E_1, \ldots, E_r$. Consider the $\mathbb{Q}$-divisor

$$M =_{\mathrm{def}} f^* L - \lambda f^* D, \text{ where } \lambda =_{\mathrm{def}} \max \left\{ \frac{k_i + 1}{m_i} \ \Big| \ 1 \leq i \leq r \right\} \ .$$

We have $M \equiv (1 - \lambda n)f^* L$, so M is nef and big by choice of λ. Next we determine the round-up of M. Writing $D = \sum_{j=1}^{s} d_j D_j$ with $d_j > 0$ and irreducible curves D_j we get

$$\lceil M \rceil = f^* L - \sum_{j=1}^{s} \lfloor \lambda d_j \rfloor D_j' - \sum_{i=1}^{r} \lfloor \lambda m_i \rfloor E_i$$

where D_j' is the proper transform of D_j. Since $\mathrm{NS}(X) \cong \mathbb{Z}$ and L is primitive, any component D_j of D is algebraically equivalent to some positive multiple of L, hence we must have $d_j \leq n$. This gives the estimate $\lambda d_j \leq \lambda n < 1$, hence $\lfloor \lambda d_j \rfloor = 0$. Further, we have

$$\lambda m_i = \max_{\ell} \frac{k_\ell + 1}{m_\ell} \cdot m_i \geq k_i + 1 \ ,$$

hence $\lfloor \lambda m_i \rfloor \geq k_i + 1$. Thus we found

$$K_{\widetilde{X}} + \lceil M \rceil = f^*L - \sum_{i=1}^{r} \left(\lfloor \lambda m_i \rfloor - 1 \right) E_i = f^*L - \sum_{i=1}^{r} b_i E_i \ ,$$

where $b_i \geq k_i$. The Kawamata-Viehweg vanishing theorem then implies

$$H^1 \left(\widetilde{X}, f^*L - \sum_{i=1}^{r} b_i E_i \right) = 0 \ . \qquad (**)$$

Finally, taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \left(f^*L - E_j - \sum_{i=1}^{r} k_i E_i \right) \longrightarrow \mathcal{O}_{\widetilde{X}} \left(f^*L - \sum_{i=1}^{r} k_i E_i \right) \longrightarrow \mathcal{O}_{E_j} \left(f^*L - \sum_{i=1}^{r} k_i E_i \right) \longrightarrow 0$$

and using induction we see that we may replace the coefficients b_i in the vanishing $(**)$ by the possibly smaller numbers k_i and the Leray spectral sequence gives the desired vanishing $(*)$ on X. $\qquad \square$

Next we show that the condition $d > \frac{1}{2}(k+2)^2$ is also necessary for k-jet ampleness. This holds without the assumption on the Picard number:

Proposition 2.3 *Let L be an ample line bundle of type $(1, d)$, $d \geq 1$, on an abelian surface X. If L is k-jet ample, then $d > \frac{1}{2}(k+2)^2$.*

Proof. Let L be k-jet ample and suppose to the contrary that $d \leq \frac{1}{2}(k+2)^2$. Let $e \in X$ be a halfperiod. Taking cohomology from the exact sequence

$$0 \longrightarrow L \otimes \mathfrak{m}_e^{k+1} \longrightarrow L \longrightarrow L \otimes \mathcal{O}_X / \mathfrak{m}_e^{k+1} \longrightarrow 0$$

and using the surjectivity of $H^0(X, L) \longrightarrow H^0 \left(X, L \otimes \mathcal{O}_X / \mathfrak{m}_e^{k+1} \right)$, we find that it imposes exactly $\binom{k+2}{2}$ conditions on a divisor in $|L|$ to have multiplicity $k+1$ in e, i.e.

$$h^0 \left(X, L \otimes \mathfrak{m}_e^{k+1} \right) = d - \binom{k+2}{2} \ . \qquad (*)$$

Passing to a translate of L we may assume that L is symmetric, i.e. $\iota^*L \cong L$, where $\iota : X \longrightarrow X$ is the (-1)-involution. We will derive a contradiction with $(*)$ considering the number of conditions imposed on even sections of L. For this recall that the space of global sections of L admits a decomposition

$$H^0(X, L) = H^0(X, L)^+ \oplus H^0(X, L)^-$$

into even and odd sections. After possibly replacing L by another symmetric translate the respective dimensions are given by

$$h^0(X, L)^{\pm} = \begin{cases} \dfrac{d \pm 1}{2} & \text{, if } d \text{ is odd} \\ \dfrac{d \pm 2}{2} & \text{, if } d \text{ is even} \end{cases}$$

(see [18, Corollary 4.6.6]).

We first consider the case that k is even. Note that the multiplicity of an even section in an odd halfperiod is always odd. So if we choose the halfperiod e to be odd, then the number of conditions on an even section imposed by vanishing in e to the order $k + 1$ is at most

$$2 + 4 + \ldots + k = 2 \cdot \binom{\frac{k}{2} + 1}{2} .$$

Thus we have the estimate

$$h^0 \left(X, L \otimes \mathfrak{m}_e^{k+1} \right)^+ \geq h^0(X, L)^+ - 2 \cdot \binom{\frac{k}{2} + 1}{2} = \left\lceil \frac{d+1}{2} \right\rceil - \frac{1}{4}(k^2 + 2k) .$$

Using $(*)$ we find the inequality

$$0 \leq h^0 \left(X, L \otimes \mathfrak{m}_e^{k+1} \right) - h^0 \left(X, L \otimes \mathfrak{m}_e^{k+1} \right)^+ \leq d - \left\lceil \frac{d+1}{2} \right\rceil - \frac{(k+2)^2}{4} ,$$

which gives a contradiction with the assumption $d \leq \frac{1}{2}(k + 2)^2$. This completes the proof in case k is even. For k odd a similar argument works considering an even halfperiod. □

3. k-very ample and k-spanned line bundles

In this section we prove an analogue of Theorem 1.1 for k-very ampleness. We begin by recalling the numerical criterion for k-very ampleness given by Beltrametti and Sommese.

Theorem 3.1 ([5]) *Let L be a nef line bundle on a smooth projective surface Y with $L^2 \geq 4k + 5$, $k \geq 0$. Then the following conditions are equivalent:*

(i) *The adjoint bundle $K_Y + L$ is not k-very ample.*

(ii) *There is an effective divisor D on Y such that $L - 2D$ is $\mathbb{Q}$-effective, D contains some 0-dimensional subscheme where the k-very ampleness fails, and*

$$LD - (k + 1) \leq D^2 < \frac{LD}{2} < k + 1 .$$

As an easy consequence we get the following criterion for k-very ampleness:

Proposition 3.2 *Let X be an abelian surface with Picard number 1, let L be an ample line bundle of type $(1, d)$, $d \geq 1$ and let k be a non-negative integer. If $d \geq 2k + 3$, then L is k-very ample.*

For the proof of the proposition note first that $L^2 = 2d \geq 4k + 6$ by assumption. Further, if D is any effective divisor on X, then we have $D \equiv_{\mathrm{alg}} pL$ for some $p \geq 1$, so

$$LD = pL^2 \geq 4k + 6$$

which shows that a divisor as in condition (ii) of Theorem 3.1 cannot exist. $\qquad\square$

The harder part is to show that the bound $2k + 3$ is sharp. Actually this is already the sharp bound for k-spannedness:

Proposition 3.3 *Let L be an ample line bundle of type $(1, d)$, $d \geq 1$, on an abelian surface X. If L is k-spanned, then $d \geq 2k + 3$.*

Proof. The analogous statement for k-very ampleness follows from [1]. However, the argument extends to the case of k-spannedness. We briefly sketch it to make this clear.

Theorem 1.1 implies that the proposition holds for $k = 0$ and $k = 1$. So in particular L may be assumed to be very ample. Suppose then that L is k-spanned and $d \leq 2k + 2$. Let

$$\varphi : A \longrightarrow \mathbb{P}(H^0(L)) = \mathbb{P}_{d-1}$$

be the embedding given by L and let $C \in |L|$ be a smooth curve. The exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L \otimes \mathcal{O}_C \longrightarrow 0$$

and Riemann-Roch imply that C is not projectively normal in $\mathbb{P}_{d-1}$. Let $D \in \mathbb{P}(H^0(L \otimes \mathcal{O}_C))$ be the image of C under the embedding defined by the complete linear system $|L \otimes \mathcal{O}_C|$. The restriction map $\alpha : H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_C)$ induces the projection

$$\alpha^* : \mathbb{P}\left(H^0(L \otimes \mathcal{O}_C)\right) \longrightarrow \mathbb{P}\left(H^0(L)\right)$$

from the line $Q = \mathbb{P}(\mathrm{coker}\,\alpha)$. The restriction of this projection to D gives an isomorphism $D \cong C$.

Now consider the k-th secant variety $\mathrm{Sec}^k(D)$ of D. By assumption $d - 1 \leq 2k + 1$, so Zak's formula (see [21]) implies

$$\dim \mathrm{Sec}^k(D) = \min\{d, 2k + 1\} \geq d - 1 \ ,$$

hence there is a point $P \in \mathrm{Sec}^k(D) \cap Q$. This means that there is a 0-dimensional subscheme Z_D of D of length $k + 1$ such that

$$P \in \mathbb{P}\left(H^0\left(L \otimes \mathcal{O}_C \otimes \mathcal{I}_{Z_D}\right)\right) \ .$$

Then $Z = \alpha^*(Z_D)$ is a curvilinear subscheme of length $k + 1$ on A. Since $L \otimes \mathcal{O}_C$ is k-spanned and P lies in the center of the projection α^* we get

$$
\begin{aligned}
d - (k + 1) - 1 &= \dim \mathbb{P}\left(H^0\left(L \otimes \mathcal{I}_Z\right)\right) = \dim \alpha^*\left(\mathbb{P}\left(H^0\left(L \otimes \mathcal{O}_C \otimes \mathcal{I}_{Z_D}\right)\right)\right) \\
&< \dim \mathbb{P}\left(H^0\left(L \otimes \mathcal{O}_C \otimes \mathcal{I}_{Z_D}\right)\right) = d - (k + 1) - 1 \ ,
\end{aligned}
$$

a contradiction. $\qquad\square$

Remark 3.4 We would like to point out here that Theorem 3.1 also leads to a bound for k-jet ampleness. First recall the following relation between k-very ampleness and k-jet ampleness (see [6, Proposition 2.1]):

If L is k'-very ample, then L is k-jet ample, where k is the biggest integer such that

$$\binom{k+2}{2} \leq k' + 1 \ . \tag{$*$}$$

Suppose now that we want to show the k-jet ampleness of a line bundle of type $(1, d)$ on an abelian surface X for suitable $d \geq 1$. According to $(*)$ it is enough to show k'-very ampleness, where $k' = \binom{k+2}{2} - 1$. Assuming $\rho(X) = 1$ and using Theorem 3.1 this will follow as soon as $L^2 \geq 4k' + 5$, which is equivalent to

$$d \geq k^2 + 3k + 3 \ .$$

So the bound one gets in this way is also quadratic in k, but it is considerably bigger than the optimal bound $d > \frac{1}{2}(k + 2)^2$ that we gave in Sect. 2.

4. Surfaces with higher Picard number

One might be tempted to hope that uniform numerical criteria for k-jet ampleness and k-very ampleness could be given for simple abelian surfaces, rather than for abelian surfaces with Picard number 1 – just as it is the case for global generation and very ampleness in Theorem 1.1. However, for a line bundle of type $(1, d)$ the implications

(i) if $d > \frac{1}{2}(k + 2)^2$, then L is k-jet ample

(ii) if $d \geq 2k + 3$, then L is k-very ample

may fail as soon as the Picard number gets bigger than 1, even for simple abelian surfaces. We will illustrate this by considering a two-dimensional family of simple abelian surfaces, where this happens.

4.1. In our discussion we are going to consider Seshadri constants. Recall that for a nef line bundle L on a smooth projective variety X the *Seshadri constant* of L at a point $x \in X$ is by definition the real number

$$\varepsilon(L, x) = \inf_{C \ni x} \frac{LC}{\mathrm{mult}_x(C)} \ ,$$

where the infimum is taken over all irreducible curves $C \subset X$ containing x. The Seshadri constant is to be thought of as measuring the local positivity of L at x. It is bounded from above by $\varepsilon(L, x) \leq \sqrt{L^d}$, where $d = \dim X$ (see [14, Remark 1.8]). There is an interesting connection between Seshadri constants and the generation

of jets: Denote by $s(L, x)$ the maximal number s such that L generates s-jets at x, i.e. such that

$$H^0(X, L) \longrightarrow H^0\left(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1}\right)$$

is onto. If L is ample, then

$$\varepsilon(L, x) = \limsup_{n \longrightarrow \infty} \frac{s(nL, x)}{n}$$

(see [12, Theorem 6.4]).

Finally, note that if X is an abelian variety, then by homogeneity $\varepsilon(L) = \varepsilon(L, x)$ is actually independent of the point x.

4.2. Now we turn to the examples. Let X be a principally polarized abelian surface with endomorphism ring $\mathbb{Z}[\sqrt{7}]$. (There is a two-dimensional family of such surfaces, cf. [8, Sect. 2].) The abelian surface X does not contain elliptic curves and its Néron-Severi group is generated by line bundles L_0 and L_1 of types $(1, 1)$ resp. $(1, 9)$ with $L_0 L_1 = 8$ (cf. [3, Sect. 3]).

We consider the line bundles

$$M_n =_{\text{def}} nL_0 - L_1 \qquad (n \geq 1)$$

on X. For $n \gg 0$ these bundles are certainly ample and primitive. M_n is of type $(1, d_n)$ with $d_n = n^2 - 8n + 9$. Put

$$k_n =_{\text{def}} \left\lfloor \sqrt{2n^2 - 16n + 18} \right\rfloor - 3 .$$

Then we have $d_n > \frac{1}{2}(k_n + 1)^2$, so if we assume the criterion (i) to hold on X, then M_n had to be k_n-jet ample. In particular it should generate k_n-jets at any point. Since we have an inclusion

$$H^0(X, M_n) \hookrightarrow H^0(X, nL_0)$$

we get $s(nL_0) \geq s(M_n) \geq k_n$, which using $\varepsilon(L_0) \leq \sqrt{L_0^2} = \sqrt{2}$ implies that the Seshadri constant of L_0 is maximal:

$$\varepsilon(L_0) = \limsup_{n \longrightarrow \infty} \frac{s(nL_0)}{n} = \sqrt{2} . \qquad (*)$$

On the other hand, one can use Steffens' argument [20] to bound $\varepsilon(L_0)$ from below: There is a curve C in $|4L_0|$ with $\text{mult}_0(C) = 6$, namely the image of the unique divisor $\Theta \in |L_0|$ under the endomorphism $2_X : X \longrightarrow X$ given as multiplication by 2, so we have

$$\varepsilon(L_0) \leq \frac{L_0 C}{\text{mult}_0(C)} = \frac{4}{3} ,$$

a contradiction with $(*)$.

Next assume that criterion (ii) holds on X. This would imply that M_n is k'_n-very ample, where

$$k'_n =_{\text{def}} \left\lfloor \frac{n^2 - 8n + 6}{2} \right\rfloor .$$

On the other hand, for a divisor $D \in |L_1|$ we have

$$M_n D = 8n - 18 ,$$

so for $n \gg 0$ we certainly have $M_n D < k'_n$, which is clearly impossible for a k'_n-very ample line bundle.

References

[1] Ballico, E., Sommese, A.J.: Projective surfaces with k-very ample line bundles of degree $\leq 4k + 4$. Nagoya Math. J. 136, 57-79 (1994)

[2] Barth, W.: Abelian Surfaces with $(1, 2)$-Polarisation. Algebraic Geometry (Sendai, 1985), Advanced Studies in Pure Mathematics 10, 1987, pp. 41-84.

[3] Bauer, Th.: Quartic surfaces with 16 skew conics. J. reine angew. Math. 464, 207-217 (1995)

[4] Bauer, Th., Szemberg, T.: Higher order embeddings of abelian varieties. Math. Z. (to appear)

[5] Beltrametti, M.C., Sommese, A.J.: Zero cycles and k-th order embeddings. Projective surfaces and their classification, Symp. Math., INDAM, vol. 32, Academic Press, New York London, 1988, pp. 33-48.

[6] Beltrametti, M.C., Sommese, A. J.: On k-jet ampleness. In: Complex Analysis and Geometry, edited by V. Ancona and A. Silva, Plenum Press, New York, 1993, pp. 355-376.

[7] Beltrametti, M.C., Sommese, A.J.: The adjunction theory of complex projective varieties. De Gruyter Expositions in Math. 16, Berlin New York, 1995.

[8] Birkenhake, Ch.: Tensor products of ample line bundles on abelian varieties. Manuscripta math. 84, 21-28 (1994)

[9] Birkenhake, Ch., Lange, H.: A family of abelian surfaces and curves of genus four. Manuscripta math. 85, 393-407 (1994)

[10] Birkenhake, Ch., Lange, H., van Straten, D.: Abelian surfaces of type $(1, 4)$. Math. Ann. 285, 625-646 (1989)

[11] Debarre, O., Hulek, K., Spandaw, J.: Very ample linear systems on abelian varieties. Math. Ann. 300, 181-202 (1994)

[12] Demailly, J.-P.: Singular Hermitian metrics on positive line bundles. Complex algebraic varieties (Bayreuth, 1990), Lect. Notes Math. 1507, Springer-Verlag, 1992, pp. 87-104.

[13] Demailly, J.-P.: L^2 vanishing theorems for positive line bundles and adjunction theory. To appear.

[14] Ein, L., Küchle, O., Lazarsfeld, R.: Local positivity of ample line bundles. J. Diff. Geom. (to appear)

[15] Ein, L., Lazarsfeld, R.: Global generation of pluricanonical and adjoint linear series on smooth projective threefolds. J. Am. Math. Soc., Vol.6, No.4, 875-903 (1993)

[16] Ein, L., Lazarsfeld, R.: Seshadri constants on smooth surfaces. Astérisque 218, 177-186 (1993)

[17] Hulek, K., Lange, H.: Examples of abelian surfaces in $\mathbb{P}_4$. J. reine angew. Math. 363, 200-216 (1985)

[18] Lange, H., Birkenhake, Ch.: Complex Abelian Varieties. Grundlehren der math. Wiss. 302, Springer-Verlag, 1992.

[19] Lazarsfeld, R.: Lectures on linear series. Park City / IAS Mathematics series vol. 3, 1-56 (1993)

[20] Steffens, A.: Remarks on Seshadri constants. Preprint

[21] Zak, F.: Linear system of hyperplane sections on varieties of low codimension. Functional Anal. Appl. 19, 165-173 (1985)

Thomas Bauer, Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstraße $1\frac{1}{2}$, D-91054 Erlangen, Germany

Tomasz Szemberg, Instytut Matematyki, Uniwersytet Jagielloński, Reymonta 4, PL-30-059 Kraków, Poland