THE GAUSSIAN MEASURE ON ALGEBRAIC VARIETIES

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Abstract. We prove that the ring $\mathbf{R}[M]$ of all polynomials defined on a real algebraic variety $M \subset \mathbf{R}^n$ is dense in the Hilbert space $L^2(M, e^{-|x|^2}\mathrm{d}\mu)$, where $\mathrm{d}\mu$ denotes the volume form of M and $\mathrm{d}\nu = e^{-|x|^2}\mathrm{d}\mu$ the Gaussian measure on M.

1. Introduction

The aim of the present note is to prove that the ring $\mathbf{R}[M]$ of all polynomials defined on a real algebraic variety $M \subset \mathbf{R}^n$ is dense in the Hilbert space $L^2(M, e^{-|x|^2}\mathrm{d}\mu)$, where $\mathrm{d}\mu$ denotes the volume form of M and $\mathrm{d}\nu = e^{-|x|^2}\mathrm{d}\mu$ is the Gaussian measure on M. In case $M = \mathbf{R}^n$, the result is well known since the Hermite polynomials constitute a complete orthonormal basis of $L^2(\mathbf{R}^n, e^{-|x|^2}\mathrm{d}\mu)$.

2. The volume growth of an algebraic variety and some consequences

We consider a smooth algebraic variety $M \subset \mathbf{R}^n$ of dimension d. Then M has polynomial volume growth: there exists a constant C depending only on the degrees of the polynomials defining M such that for any euclidian ball B_r with center $0 \in \mathbf{R}^n$ and radius $r > 0$ the inequality

$$\mathrm{vol}_d(M \cap B_r) \ \leq \ C \cdot r^d$$

holds (see [Brö]). Via Crofton formulas the mentioned inequality is a consequence of Milnor's results concerning the Betti numbers of an algebraic variety (see [Mi1], [Mi2], in which the stated inequality is already implicitly contained). This estimate yields first of all that the restrictions on M of the polynomials on $\mathbf{R}^n$ are square-integrable with respect to the Gaussian measure on M.

Proposition 1. *Let M be a smooth submanifold of the euclidian space $\mathbf{R}^n$. Suppose that M has polynomial volume growth,i.e., there exist constants C and $l \in \mathbf{N}$ such that for any ball B_r*

$$\mathrm{vol}_d(M \cap B_r) \ \leq \ C \cdot r^l$$

holds. Denote by $\mathrm{d}\mu$ the volume form of M. Then:

1. *The ring $\mathbf{R}[M]$ of all polynomials on M is contained in the Hilbert space $L^2(M, e^{-|x|^2}\mathrm{d}\mu)$;*
2. *all functions $e^{\alpha|x|^2}$ for $\alpha < 1/2$ belong to $L^2(M, e^{-|x|^2}\mathrm{d}\mu)$.*

Proof. Throughout this article, denote the distance of the point $x \in \mathbf{R}^n$ to the origin by $r^2 = |x|^2$. We shall prove that the integrals

$$I_m(M) \ := \ \int_M r^m e^{-r^2}\mathrm{d}\mu < \infty, \quad m = 1, 2, \ldots$$

are finite. However,

$$I_m(M) \ = \ \sum_{j=0}^{\infty} \int_{M \cap (B_{j+1} - B_j)} r^m e^{-r^2}\mathrm{d}\mu$$

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and consequently we can estimate $I_m(M)$ as follows:

$$I_m(M) \;\leq\; \sum_{j=0}^{\infty}(j+1)^m e^{-j^2}\left[\mathrm{vol}(M\cap B_{j+1})-\mathrm{vol}(M\cap B_j)\right] \;\leq\; \sum_{r=0}^{\infty}(r+1)^m e^{-r^2}\mathrm{vol}(M\cap B_{r+1})\,.$$

Using the assumption on the volume growth of M we immediately obtain

$$I_m(M) \;\leq\; C\cdot\sum_{r=0}^{\infty}(r+1)^{m+l}e^{-r^2}\,.$$

Denoting the summands of the latter series by a_r, we readily see that it converges, since

$$\frac{a_{r+1}}{a_r} \;=\; \frac{(r+1)^{m+l}e^{-r^2-2r-1}}{(r)^{m+l}e^{-r^2}} \;=\; \left(\frac{r+1}{r}\right)^{m+l}\frac{1}{e^{2r+1}} \longrightarrow 0.$$

A similar calculation yields the result for the functions $e^{\alpha r^2}$ with $\alpha < 1/2$. □

3. A dense subspace in $\mathcal{C}^0_\infty(S^n)$

The aim of this section is to verify that a certain linear subspace of $\mathcal{C}^0(S^n)$ is dense therein. Since the family of functions we have in mind cannot be made into an algebra, we have to replace the standard Stone-Weierstraß argument by something different. The idea for overcoming this problem is to use a combination of the well-known theorems of Hahn-Banach, Riesz and Bochner.

To begin with, we uniformly approximate the function $e^{-r^2}e^{i\langle k,x\rangle}$ for a fixed vector $k\in\mathbf{R}^n$.

Lemma. *Denote by $p_m(x)$ the polynomial*

$$p_m(x) = \sum_{\alpha=0}^{m-1} i^\alpha \left\langle k,x\right\rangle^\alpha/\alpha!\,.$$

Then the sequence $e^{-r^2}p_m(x)$ converges uniformly to $e^{-r^2}e^{i\langle k,x\rangle}$ on $\mathbf{R}^n$.

Proof. The inequality

$$|\,p_m(x)-e^{i\langle k,x\rangle}\,| \;\leq\; \frac{\|k\|^m\|x\|^m}{m!}\;e^{\|k\|\cdot\|x\|}$$

implies (set $y=\|k\|\cdot\|x\|$)

$$\sup_{x\in\mathbf{R}^n}|\,e^{-r^2}p_m(x)-e^{-r^2}e^{i\langle k,x\rangle}\,| \;\leq\; \sup_{0\leq y}\frac{y^m}{m!}\;e^{y-y^2/\|k\|^2} \;=:\; C_m\,.$$

Therefore, we have to check that for any fixed vector $k\in\mathbf{R}^n$ the sequence C_m tends to zero as $m\to\infty$. For simplicity, denote by k the length of the vector $k\in\mathbf{R}^n$. A direct calculation yields the following formula:

$$C_m \;=\; \frac{1}{m!}\left(\frac{k^2}{4}+\frac{k}{4}\sqrt{k^2+8m}\right)^m \exp\left(\frac{k^2}{4}+\frac{k}{4}\sqrt{k^2+8m}-\frac{1}{k^2}\left(\frac{k^2}{4}+\frac{k}{4}\sqrt{k^2+8m}\right)^2\right)\,.$$

We are only interested in the asymptotics of C_m. We will thus ignore all constant factors not depending on m. In this sense, we obtain

$$C_m \;\approx\; \frac{1}{m!}\left(\frac{k^2}{4}+\frac{k}{4}\sqrt{k^2+8m}\right)^m \exp\left(\frac{k}{8}\sqrt{k^2+8m}-\frac{k^2+8m}{16}\right)\,.$$

The Stirling formula $m!\approx\sqrt{m}\,m^m e^{-m}$ allows us to rewrite the asymptotics of C_m:

$$C_m \;\approx\; \frac{1}{\sqrt{m}\,m^m}\left(\frac{k^2}{4}+\frac{k}{4}\sqrt{k^2+8m}\right)^m \exp\left(\frac{k}{8}\sqrt{k^2+8m}+\frac{m}{2}\right)\,.$$

Since

$$\lim_{m\to\infty}(\sqrt{k^2+8m}-\sqrt{8m}) \;=\; 0\,,$$

we can furthermore replace $\sqrt{k^2+8m}$ by $2\sqrt{2m}$:

$$C_m \;\approx\; \frac{1}{\sqrt{m}\,m^m}\left(\frac{k^2}{4}+\frac{k}{4}\sqrt{k^2+8m}\right)^m \exp\left(\frac{k}{4}\sqrt{2m}+\frac{m}{2}\right) \;=:\; e^{C_m^*}$$

with

$$C_m^* \;=\; m\ln\left(\frac{k^2}{4}+\frac{k}{4}\sqrt{k^2+8m}\right)+\frac{k}{2\sqrt{2}}\sqrt{m}+\frac{m}{2}-m\ln(m)-\frac{1}{2}\ln(m)\,.$$

In case m is sufficiently large with respect to k, we can estimate $\ln(k^2/4+k/4\cdot\sqrt{k^2+8m})$ by $\frac{1}{2}\ln(m)+\alpha$ for some constant α:

$$\begin{aligned} C_m^* &\lessapprox \frac{m}{2}\ln(m)+\alpha m+\frac{k}{2\sqrt{2}}\sqrt{m}+\frac{m}{2}-m\ln(m)-\frac{1}{2}\ln(m)\\ &\leq -\frac{m}{2}\ln(m)+(\alpha+1/2)m+\frac{k}{2\sqrt{2}}\sqrt{m}\\ &\leq -\frac{m}{2}\ln(m)+(\alpha+1/2+\frac{k}{2\sqrt{2}})m\\ &= m\left(\alpha+1/2+\frac{k}{2\sqrt{2}}-\frac{1}{2}\ln(m)\right)\,. \end{aligned}$$

Finally, $C_m=\exp(C_m^*)$ converges to zero. □

Proposition 2. *Denote by $\mathcal{P}(\mathbf{R}^n)$ the ring of all polynomials on $\mathbf{R}^n$. Then the linear space $\Sigma_\infty := \mathcal{P}(\mathbf{R}^n)\cdot e^{-r^2}$ is dense in the space $\mathcal{C}^0_\infty(S^n)$ of all continuous functions on $S^n=\mathbf{R}^n\cup\{\infty\}$ vanishing at infinity.*

Proof. Suppose the closure $\overline{\Sigma_\infty}$ of the linear space Σ_∞ does not coincide with $\mathcal{C}^0_\infty(S^n)$. Then the Hahn-Banach Theorem implies the existence of a linear continuous functional $L:\mathcal{C}^0(S^n)\to\mathbf{R}$ such that

1. $L|_{\Sigma_\infty}=0$;
2. $L(g_0)\neq 0$ for at least one $g_0\in\mathcal{C}^0_\infty(S^n)$.

According to Riesz' Theorem (see [Rud, Ch.6, p.129 ff.]), L may be represented by two regular Borel measures μ_+, μ_- on S^n:

$$L(f) \;=\; \int_{S^n} f(x)\,\mathrm{d}\mu_+(x)-\int_{S^n} f(x)\,\mathrm{d}\mu_-(x)\,.$$

In particular, μ_+ and μ_- are finite. The first property $L|_{\Sigma_\infty}=0$ of L implies

$$\int_{S^n} e^{-r^2}p(x)\,\mathrm{d}\mu_+(x) \;=\; \int_{S^n} e^{-r^2}p(x)\,\mathrm{d}\mu_-(x)$$

for any polynomial $p(x)$. Let us introduce the measures $\nu_\pm=e^{-r^2}\mu_\pm$ on the subset $\mathbf{R}^n\subset S^n$. Then

$$\int_{\mathbf{R}^n} p(x)\,\mathrm{d}\nu_+(x) \;=\; \int_{\mathbf{R}^n} p(x)\,\mathrm{d}\nu_-(x)$$

holds and remains true for any complex-valued polynomial. We may thus choose $p(x)=p_m(x)$ as in the previous lemma

$$p_m(x) \;=\; \sum_{\alpha=0}^{m-1} i^\alpha\left\langle k,x\right\rangle^\alpha/\alpha!\,.$$

But, then

$$\int_{S^n} p_m(x)e^{-r^2}\,\mathrm{d}\mu_+(x) \;=\; \int_{\mathbf{R}^n} p_m(x)\,\mathrm{d}\nu_+(x) \;=\; \int_{\mathbf{R}^n} p_m(x)\,\mathrm{d}\nu_-(x) \;=\; \int_{S^n} p_m(x)e^{-r^2}\,\mathrm{d}\mu_-(x)$$

together with the uniform convergence of $p_m(x)e^{-r^2}$ to $e^{i\langle k,x\rangle}e^{-r^2}$ implies

$$\int_{S^n} e^{i\langle k,x\rangle}e^{-r^2}\,\mathrm{d}\mu_+(x) \;=\; \int_{S^n} e^{i\langle k,x\rangle}e^{-r^2}\,\mathrm{d}\mu_-(x)\,,$$

i.e.,

$$\int_{\mathbf{R}^n} e^{i\langle k,x\rangle}\,\mathrm{d}\nu_+(x) \;=\; \int_{\mathbf{R}^n} e^{i\langle k,x\rangle}\,\mathrm{d}\nu_-(x)\,.$$

Therefore, the Fourier transforms of the measures ν_+ and ν_- coincide. Consequently, by Bochner's Theorem (see [Mau, Ch.XIX, p.774 ff.]) we conclude that $\nu_+ = \nu_-$ on $\mathbf{R}^n$. The linear functional $L : \mathcal{C}^0(S^n) \to \mathbf{R}$ must thus be the evaluation of a function at infinity:

$$L(f) \;=\; c\cdot f(\infty)\,,$$

a contradiction to the existence of a function $g_0 \in \mathcal{C}^0_\infty(S^n)$ satisfying $L(g_0) \neq 0$. □

4. The main result

Theorem 1. *Let the closed subset $M \subset \mathbf{R}^n$ be a smooth submanifold satisfying the polynomial volume growth condition. Then the ring $\mathbf{R}[M]$ of all polynomials on M is a dense subspace of the Hilbert space $L^2(M, e^{-r^2}\mathrm{d}\mu)$.*

Proof. Consider the one-point-compactification $\hat{M} \subset S^n$ of $M \subset \mathbf{R}^n$. Then Proposition 2 of Section 3 implies that

$$\Sigma_\infty(\hat{M}) \;:=\; \mathbf{R}[M]\cdot e^{-r^2/4}$$

is dense in $\mathcal{C}^0_\infty(\hat{M})$. We introduce the measure $\mathrm{d}\nu = e^{-r^2/2}\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the volume form of M. Since

$$\int_M \mathrm{d}\nu \;=\; \int_M e^{-r^2/2}\mathrm{d}\mu \;=\; \int_M (e^{r^2/4})^2 e^{-r^2}\mathrm{d}\mu \;=:\; V \;<\; \infty\,,$$

$\mathrm{d}\nu$ defines a regular Borel measure $\mathrm{d}\hat{\nu}$ on $\hat{M}$ (by setting $\mathrm{d}\hat{\nu}(\infty) = 0$). Therefore, the algebra $\mathcal{C}^0_\infty(\hat{M})$ of all continuous functions on $\hat{M}$ vanishing at infinity is dense in $L^2(\hat{M}, \mathrm{d}\hat{\nu})$:

$$\overline{\mathcal{C}^0_\infty(\hat{M})} \;=\; L^2(\hat{M}, \mathrm{d}\hat{\nu})\,.$$

For any function f in $L^2(M, e^{-r^2}\mathrm{d}\mu)$ we have

$$\int_M |fe^{-r^2/4}|^2 e^{-r^2/2}\mathrm{d}\mu \;=\; \int_M |f|^2 e^{-r^2}\mathrm{d}\mu \;<\; \infty$$

and, therefore, $fe^{-r^2/4}$ lies in $L^2(\hat{M}, \mathrm{d}\hat{\nu})$. Thus, for a fixed $\varepsilon > 0$, there exists a function $g \in \mathcal{C}^0_\infty(\hat{M})$ such that

$$\int_M |fe^{-r^2/4} - g(x)|^2 e^{-r^2/2}\mathrm{d}\mu \;<\; \varepsilon/2\,.$$

According to Proposition 2 we can find a polynomial $p(x) \in \mathbf{R}[M]$ approximating g:

$$\sup_{x\in\hat{M}} |g(x) - p(x)e^{-r^2/4}|^2 \;<\; \varepsilon/2V\,.$$

Using the inequality $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ we conclude

$$\int_M |f(x)e^{-r^2/4} - p(x)e^{-r^2/4}|^2 e^{-r^2/2}\mathrm{d}\mu \;<\; \varepsilon\,;$$

but this is equivalent to

$$\int_M |f(x) - p(x)|^2 e^{-r^2}\mathrm{d}\mu \;<\; \varepsilon\,.$$

□

5. Examples and final remarks

We shall give a few simple examples. Notice that we recover, of course, that the polynomials are dense in $L^2(\mathbf{R}^n, e^{-r^2}\mathrm{d}\mu)$ (Hermite polynomials) or in $L^2(M, \mathrm{d}\mu)$ for any compact submanifold (Legendre polynomials in case $M = [-1, 1]$).

Example 1. Consider a revolution surface in $\mathbf{R}^3$ defined by two polynomials f, h

$$\begin{cases} x = f(u_1)\cos u_2 \\ y = f(u_1)\sin u_2 \\ z = h(u_1) \end{cases}, \quad f(u_1) > 0\,, \quad (u_1, u_2) \in \mathbf{R} \times [0, 2\pi]\,.$$

Then we have $\mathrm{d}\mu = f\sqrt{f'^2 + h'^2}\,\mathrm{d}u_1\mathrm{d}u_2$ and $r^2 = f^2 + h^2$, and thus obtain

$$\mathbf{R}[f\cos u_2,\ f\sin u_2,\ h] \ \text{ is dense in } \ L^2(\mathbf{R} \times [0, 2\pi], e^{-(f^2+h^2)} f\sqrt{f'^2 + h'^2}\,\mathrm{d}u_1\mathrm{d}u_2)\,.$$

In the special case of a cylinder, i.e. $f = 1$, $h = u_1$, this reduces to the well known fact that the ring

$$\mathbf{R}[u_1,\ \cos u_2,\ \sin u_2,] = \mathbf{R}[u_1] \otimes \mathbf{R}[\cos u_2,\ \sin u_2]$$

is indeed dense in the Hilbert space

$$L^2(\mathbf{R} \times [0, 2\pi], e^{-u_1^2}\,\mathrm{d}u_1\mathrm{d}u_2) = L^2(\mathbf{R}, e^{-u_1^2}\,\mathrm{d}u_1) \otimes L^2([0, 2\pi], \mathrm{d}u_2)\,.$$

Example 2. Let $F : \mathbf{C} \to \mathbf{C}$ be a polynomial and consider the surface defined by

$$f:\ \mathbf{C} \longrightarrow \mathbf{R}^3, \quad f(z) = (x, y, |F(z)|), \quad z = x + iy\,.$$

Then one checks that $\mathrm{d}\mu = \sqrt{1 + |F'|^2}\,|\mathrm{d}z|^2$ and $r^2 = |z|^2 + |F(z)|^2$. Thus the following holds:

$$\overline{\mathbf{R}[x, y, |F(z)|]} \ = \ L^2(\mathbf{R}^2, e^{-(|z|^2+|F(z)|^2)}\sqrt{1 + |F'|^2}\,|\mathrm{d}z|^2)\,.$$

Let us study the polynomial $F = z^{2k}$ in more detail. Here the coordinate ring coincides with the usual polynomial ring $\mathbf{R}[x, y]$ in two variables, and thus we have proved that these are dense in

$$L^2(\mathbf{R}^2, e^{-(|z|^2+|z|^{4k})}\sqrt{1 + 4k^2|z|^{2(2k-1)}}\,|\mathrm{d}z|^2)\,.$$

Example 3. We finish with a one-dimensional example: the graph $M = \{(x, f(x)\}$ of a polynomial $f : \mathbf{R} \to \mathbf{R}^n$. Then $\mathrm{d}\mu = \sqrt{1 + \|f'\|^2}\,\mathrm{d}x$, and we obtain

$$\overline{\mathbf{R}[x]} \ = \ L^2(\mathbf{R}, e^{-(x^2+\|f(x)\|^2)}\sqrt{1 + \|f'\|^2}\,\mathrm{d}x)\,.$$

Remark. The main result raises an interesting analogous problem in complex analysis which, to our knowledge, is still open. It is well known that the polynomials on $\mathbf{C}^n$ are dense in the Fock- or Bergman space

$$\mathcal{F}(\mathbf{C}^n) \ := \ \{f \in L^2(\mathbf{C}^n, e^{-r^2}\mathrm{d}\mu) \mid f \text{ holomorphic }\}\,.$$

Furthermore, a theorem by Stoll (see [Sto1], [Sto2]) states that from all complex analytic submanifolds N of $\mathbf{C}^n$, those with polynomial growth are *exactly* the algebraic ones, and thus the only ones for which the elements of the coordinate ring are square-integrable with respect to the Gaussian measure. It is then common to study the space

$$\mathcal{F}(N) \ := \ \{f \in L^2(N, e^{-r^2}\mathrm{d}\mu) \mid f \text{ holomorphic }\}\,,$$

but we were not able to find any results on whether $\mathbf{C}[N]$ is dense herein.

More elaborate applications of the main result to the situation where M carries a reductive algebraic group action will be discussed by the authors in some forthcoming works (see e.g. [Agr]). In this case, one can decompose the ring $\mathbf{R}[M]$ into isotypic components and, via Theorem 1, one obtains a decomposition of $L^2(M, e^{-r^2}\mathrm{d}\mu)$ analogous to the classical Frobenius reciprocity.

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