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# TWISTORIAL EIGENVALUE ESTIMATES FOR GENERALIZED DIRAC OPERATORS WITH TORSION

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Abstract. We study the Dirac spectrum on compact Riemannian spin manifolds $M$ equipped with a metric connection $\nabla$ with skew torsion $T \in \Lambda^3 M$ by means of twistor theory. An optimal lower bound for the first eigenvalue of the Dirac operator with torsion is found that generalizes Friedrich's classical Riemannian estimate. We also determine a novel twistor and Killing equation with torsion and use it to discuss the case in which the minimum is attained in the bound.

## 1. Introduction & summary

This paper is devoted to a systematic investigation–via twistor theory–of the Dirac spectrum of compact Riemannian spin manifolds $(M^n, g)$ with a metric connection $\nabla$ with skew-symmetric torsion $T \in \Lambda^3(M^n)$. The manifolds we consider are non-integrable geometric structures endowed with the characteristic connection $\nabla = \nabla^c$ (see the survey [Ag06]). A. Gray was the first to investigate manifolds and connections of this kind using the notion of weak holonomy [Gra71]. Nowadays many different ways exist to tackle the issue of weak holonomy, and they can all be described in our setting: to name but a few, the intrinsic-torsion approach ([Sal89], [Sw00]), or that involving the critical points of some distinguished functional defined on differential forms [Hit00]. The Dirac operator that one should look at is, hence, not the one associated with $\nabla^c$, but rather

$$\not{D} = D^g + \frac{1}{4}T,$$

where $D^g$ is the Riemannian Dirac operator. This generalized Dirac operator with torsion corresponds to the torsion form $T/3$ (see [AF04a], [AF04b]). As a matter of fact, $\not{D}$ coincides with the so-called "cubic Dirac operator" studied by B. Kostant ([Ko99], [Ag03]) on naturally reductive spaces, and also with the Dolbeault operator of a Hermitian manifold ([Bi89], [Gau97]). More recently, theoretical physicists from superstring theory have begun to take interest in the operator $\not{D}$ and its symmetries [HKWY10]. To obtain spectral estimates it turns out crucial to require $\nabla^c T = 0$, for it is this conservation law that ensures the compatibility of the actions of $\nabla^c$ and $T$ (viewed as an endomorphism) on the spinor bundle. There are several manifolds that are classically known to admit parallel characteristic torsion, namely nearly Kähler manifolds, Sasakian manifolds, nearly parallel $G_2$-manifolds, and naturally reductive spaces; these classes have been considerably enlarged in more recent work (see [Va79], [GO98], [FrI02], [Al03], [AlFS04], [Fr07a], [Sch07]), eventually leading to a host of instances to which our results can be applied.

On a Riemannian manifold $(M^n, g)$ Penrose's twistor operator ($\nabla^g$ denotes the Levi-Civita connection)

$$(*) \qquad P := \sum_{k=1}^{n} e_k \otimes \left[\nabla^g_{e_k}\psi + \frac{1}{n} e_k \cdot D^g \psi\right]$$

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is known to encode much information on both the spinorial behaviour and the conformal geometry of the underlying manifold (see [Li87], [Li88], [HL88], [Fr89], [Ha90]). The key point to us is that the twistor operator can be used to prove Friedrich's estimate for the smallest eigenvalue $\lambda^g$ of $(D^g)^2$ ([Fr80], [Se98])

$$(**) \qquad \lambda^g \;\geq\; \frac{n}{4(n-1)} \min_{x\in M^n} \mathrm{Scal}^g,$$

and to discuss the case where equality holds. But whereas Dirac operators with torsion are by now well-established analytical tools in the study of special geometric structures, all attempts to develop a sort of twistor theory with torsion have failed, so far. The main problem is that the operator defined by $(*)$ has no straightforward generalization in presence of torsion; one might try to replace $\nabla^g$ by the canonical connection $\nabla^c$, or equally well substitute $D^g$ with $\not{D}$ or with the Dirac operator of $\nabla^c$. Yet one realises quickly that either possibility is unlikely to be very meaningful from the geometrical viewpoint.

In this article we derive a twistor operator with torsion by asking which generalization of $P$ yields, on a suitable class of geometries with torsion, a lower estimate for the smallest eigenvalue of $\not{D}$ that contains the optimal estimate $(**)$ in the limiting case of vanishing torsion.

Throughout the article we will assume $(M^n, g)$ is an oriented Riemannian manifold endowed with a metric connection $\nabla^c$ with skew-symmetric torsion $T \in \Lambda^3(M^n)$. The situation we have in mind is that of the characteristic connection of a $G$ structure; as described in [Ag06], this is–if existent–the unique $G$-invariant metric connection with skew-symmetric torsion, and is well understood in all standard geometries. It has to be stressed, however, that our results apply to *any* metric connection with parallel skew-symmetric torsion. It will be useful to consider the one-parameter family of connections

$$\nabla^s_X Y \;=\; \nabla^g_X Y + 2s\, T(X, Y, -),$$

with normalisation chosen so that $\nabla^s$ has torsion $T$ if $s = 1/4$, whence $\nabla^c = \nabla^{1/4}$. Obviously $\nabla^0 = \nabla^g$, so $\nabla^s$ can be thought of as a line in the space of connections joining the Levi-Civita to the characteristic connection. For each $s$ the respective scalar curvatures fulfill $\mathrm{Scal}^s = \mathrm{Scal}^g - 24s^2\|T\|^2$. The connection $\nabla^s$ may be lifted to the spin bundle $\Sigma M$, and will be denoted by the same symbol,

$$\nabla^s_X \psi \;=\; \nabla^g_X \psi + s(X \lrcorner T)\cdot \psi.$$

The spin connection $\nabla^s$ induces a twistor operator $P^s$. At the heart of the paper lies a twistorial integral equation, which is the content of

**Theorem 3.2**. *Suppose $\nabla^c T = 0$. For any spinor field $\psi$, the Dirac operator $\not{D}$ of the connection with torsion $\frac{1}{3}T$ satisfies the following integral formula:*

$$\begin{aligned}\int_M \langle \not{D}^2\psi, \psi\rangle dM \;&=\; \frac{n}{n-1}\int_M \|P^s\psi\|^2 dM + \frac{n}{4(n-1)}\int_M \mathrm{Scal}^g\|\psi\|^2 dM \\ &\quad + \frac{n(n-5)}{8(n-3)^2}\|T\|^2\int \|\psi\|^2 dM + \frac{n(4-n)}{4(n-3)^2}\int_M \langle T^2\psi, \psi\rangle dM.\end{aligned}$$

*The parameter $s$ appearing in the twistor operator $P^s$ has the value $s = \dfrac{n-1}{4(n-3)}$.*

This leads to a twistorial eigenvalue estimate for $\not{D}$ that improves all existing eigenvalue estimates known (Corollaries 3.1 and 3.2) and has a wider application range than these:

**Corollary 3.2**. *For $\nabla^c T = 0$ and $M$ compact, the smallest eigenvalue $\lambda$ of $\not{D}^2$ satisfies the inequality*

$$\lambda \;\geq\; \frac{n}{4(n-1)}\mathrm{Scal}^g_{\min} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 + \frac{n(4-n)}{4(n-3)^2}\max(\mu_1^2, \ldots, \mu_k^2) \;=:\; \beta_{\mathrm{tw}},$$

*where $\mu_1, \ldots, \mu_k$ denote the eigenvalues of the torsion endomorphism $T$ on the spin bundle $\Sigma M$.*

The limiting case is obtained precisely when the eigenspinor is a twistor spinor for the twistor operator with torsion $P^s$, whith $s = \dfrac{n-1}{4(n-3)}$; thus, its torsion is a multiple of the initial characteristic torsion *and depends on the dimension $n$ of $M$*, through the parameter $s$. This is a rather surprising fact, and it explains why it had not been possible to guess the 'right' twistor operator beforehand, although $(*)$ for the Riemannian case already indicates that any answer must involve $\dim M$. In Lemma 5.1, we show that the twistor equation $P^s\psi = 0$ is equivalent to the field equation

$$\nabla^c_X\psi + \frac{1}{n}X\cdot \not{D}\psi + \frac{1}{2(n-3)}(X\wedge T)\cdot\psi \;=\; 0 \;\; \forall X.$$

Friedrich's original proof of his estimate relies on a clever deformation of the Levi-Civita connection, not on twistor techniques. The same idea was later used for the operator $\not{D}$ as well, see [AFK08], [Ka10]. But in contrast to the Riemannian situation, the twistor approach (first described in the Riemannian setting in [Se98]) yields results that *differ* from the deformation ansatz in the presence of torsion. Section 4 thoroughly discusses the estimate obtained, and compares it to the other available estimates, if any.

In Section 5 we prove that twistor spinors with torsion generalize Killing spinors with torsion (as of Definition 5.1) in the most natural way, and then we discuss the basic geometric properties of both kinds. We compute the full integrability condition for the existence of Killing spinors with torsion (whose details are deferred to Appendix B). This constraint is then used to prove that Einstein-Sasaki manifolds cannot admit Killing spinors with torsion (Corollary A.1). This result, albeit obtained as a by-product of the aforementioned discussion, is remarkable in its own right. On the other hand, we show that non-trivial twistor, and even Killing, spinors with torsion do exist: noteworthy instances are certain 5- and 7-dimensional Stiefel manifolds endowed with their natural contact structures (Examples 5.1 and 5.2).

It emerges from the treatise that the case of dimension $n = 6$ stands out (Section 6). We prove that, in this distinguished situation, the Killing equation and the twistor equation *are equivalent* (Corollary 6.1). For nearly Kähler manifolds, we can even prove:

**Theorem 6.1**. *On a 6-dimensional nearly Kähler manifold $(M^6, g, J)$ with its characteristic connection $\nabla^c$, the following classes of spinors coincide:*

(1) *Riemannian Killing spinors,*
(2) *$\nabla^c$-parallel spinors,*
(3) *Killing spinors with torsion,*
(4) *Twistor spinors with torsion.*

In the last part of the paper the twistorial approach is applied to manifolds with reducible characteristic holonomy. It is a standard fact that the splitting of the tangent bundle of a Riemannian spin manifold $(M^n, g)$ under the action of the Riemannian holonomy group has important consequences for the spectrum of the Dirac operator $D^g$ ([Ki04], [Al07]); in the simplest one-dimensional case, that assumption just means that $M$ admits a $\nabla^g$-parallel vector field [AlGI98]. So in a similar fashion we can consider local products of manifolds with parallel characteristic torsion, called, for the present purposes, *geometries with reducible parallel torsion*; the precise formulation is found in Definition 7.2. We analyse in detail manifolds with reducible parallel torsion and their curvature properties, the study of which was lacking in the literature. We derive the necessary partial Schrödinger-Lichnerowicz formulas compatible with the splitting (Proposition 7.2), and from this obtain another interesting spectral estimate for $\not{D}$ (Theorem 7.1): roughly speaking, the estimate is the same as in Corollary 3.2, but now the dimension $n$ of the manifold is replaced by the largest dimension of a parallel distribution of the tangent bundle. This result is complemented by the ensuing discussion of the equality case. The section's closing result (Theorem 7.2) is:

**Theorem 7.2**. If $M$ is locally a product $M_1 \times \ldots \times M_k$, the eigenvalues $\lambda^g$ of $(D^g)^2$ on $M$ and $\lambda_i^g$ of $(D_i^g)^2$ on $M_i$ satisfy the inequality

$$\lambda^g - \sum_{i=1}^{k} \frac{\lambda_i^g}{\dim M_i} \;\geq\; \frac{\mathrm{Scal}^g_{\min}}{4},$$

and equality is obtained if and only if there exists a Riemannian Killing spinor on each factor of $M$.

The proof is based on twistor techniques and, alas, we show that is has no analogue for connections with torsion.

## 2. Review of the universal eigenvalue estimate

We recall here the generalized Schrödinger-Lichnerowicz identities for Dirac operators with torsion and the eigenvalue estimates one can derive from them. A crucial first order differential operator that will appear in several instances is

$$\mathcal{D}^s \;=\; \sum_{i=1}^{n} (e_i \lrcorner T) \cdot \nabla^s_{e_i} \psi.$$

Contrary to a Dirac operator with torsion, it has no Riemannian counter part. Furthermore, define (see Appendix C)

$$\sigma_T \;:=\; \frac{1}{2} \sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner T).$$

In [FrI02], the following identities are proved:

**Theorem 2.1.**

(1) *The square of the Dirac operator $D^s$ satisfies the relation*

$$(D^s)^2 \;=\; \Delta^s + 3s\, dT - 8s^2 \sigma_T + 2s\, \delta T - 4s\, \mathcal{D}^s + \frac{1}{4}\mathrm{Scal}^s.$$

(2) *The anticommutator of $D^s$ with $T$ is given by*

$$D^s T + T D^s \;=\; dT + \delta T - 8s\sigma_T - 2\mathcal{D}^s.$$

Since the spectrum of $\mathcal{D}^s$ is usually beyond control, these relations are hard to evaluate for, say, non parallel spinor fields. The main improvement of [AF04a, Thm 6.2] is the following rescaling result:

**Theorem 2.2** (Generalized Schrödinger-Lichnerowicz formula)**.** *For arbitrary torsion $T$, one has the identity*

$$(*) \qquad (D^{s/3})^2 \;=\; \Delta^s + s\, dT + \frac{1}{4}\mathrm{Scal}^g - 2s^2 \|T\|^2.$$

*If in addition $dT = 2\sigma_T$, this may be simplified to*

$$(D^{s/3})^2 \;=\; \Delta^s - sT^2 + \frac{1}{4}\mathrm{Scal}^g + (s - 2s^2)\, \|T\|^2.$$

This is in particular satisfied if the torsion is parallel for $s = 1/4$. In this case, the last relation has a remarquable consequence. For then $\Delta^s$ commutes with $T$, and this is trivially correct for the multiplication by $T^2$ and by scalars, hence (compare [AF04b, Prop. 3.4])

$$(D^{s/3})^2 \circ T \;=\; T \circ (D^{s/3})^2.$$

It is therefore possible to split the spin bundle in the orthogonal sum of its eigenbundles for the $T$ action,

$$\Sigma M \;=\; \bigoplus_{\mu} \Sigma_\mu,$$

and to consider $(D^{s/3})^2$ on each of them, since $\nabla^s$ and $(D^{s/3})^2$ both preserve this splitting. We shall henceforth denote the different eigenvalues of $T$ on $\Sigma M$ by $\mu_1, \ldots, \mu_k$. We therefore obtain the following *universal eigenvalue estimate* for the first eigenvalue $\lambda = \lambda(\not{D}^2)$ of $\not{D}^2$ for the connection with torsion $T/3$. We state it separately on $\Sigma_\mu$ and the whole spin bundle $\Sigma M$, for examples teach us that going over to $\Sigma M$ often means throwing away too much detail information.

**Theorem 2.3** (Universal eigenvalue estimate)**.** *For $\nabla^c T = 0$, the smallest eigenvalue $\lambda$ of $\not{D}^2$ on $\Sigma_\mu$ satisfies the inequality*

$$\lambda(\not{D}^2\big|_{\Sigma_\mu}) \;\geq\; \frac{1}{4}\mathrm{Scal}^g_{\min} + \frac{1}{8}\|T\|^2 - \frac{1}{4}\,\mu^2 \;=:\; \beta_{\mathrm{univ}}(\mu),$$

*and equality occurs if and only if $\mathrm{Scal}^g$ is constant and $\Sigma_\mu$ contains a $\nabla^c$-parallel spinor. For the smallest eigenvalue $\lambda$ of $\not{D}^2$ on the whole spin bundle $\Sigma$, one thus obtains the estimate*

$$\lambda \;\geq\; \frac{1}{4}\mathrm{Scal}^g + \frac{1}{8}\|T\|^2 - \frac{1}{4}\,\max(\mu_1^2, \ldots, \mu_k^2) \;=:\; \beta_{\mathrm{univ}}.$$

Equality occurs if and only if the eigenspinor is $\nabla^c$-parallel, which can indeed happen in some special geometries (compare Section 4). If $T = 0$, this is an estimate for the Riemannian Dirac operator that fails to be optimal. It was improved 1980 by Thomas Friedrich by a clever deformation trick for the Levi-Civita connection. This method, which we will (somehow vaguely) call the *deformation method* in this paper, was successfully applied to Dirac operators with torsion ([AFK08], [Ka10]. Nevertheless, an optimal estimate could not be derived in all cases of interest and many open questions remain.

An alternative approach to Friedrich's inequality is by twistorial techniques. One goal of this article is thus to work out this ansatz in detail for connections with torsion, and to improve the results obtaind by the deformation method. Contrary to the Riemannian case, the two approaches turn out not to be equivalent.

## 3. The twistorial eigenvalue estimate

If $m : TM \otimes \Sigma M \to \Sigma M$ denotes Clifford multiplication, the projection $p : TM \otimes \Sigma M \to \ker m \subset TM \otimes \Sigma M$ is locally given by

$$p(X \otimes \psi) \;=\; X \otimes \psi + \frac{1}{n}\sum_{k=1}^{n} e_k \otimes e_k \cdot X \cdot \psi.$$

The *Penrose-* or *Twistor operator* is the composition $P^s := p \circ \nabla^s$. Locally,

$$P^s\psi = \sum_{k=1}^{n} e_k \otimes \left\{\nabla^s_{e_k}\psi + \frac{1}{n} e_k \cdot D^s\psi\right\}.$$

A spinor $\psi$ is called a *twistor spinor* if it lies in the kernel of $P^s$: $P^s\psi = 0$. This is equivalent to the *twistor equation* (for still arbitrary parameter value $s$)

$$\nabla^s_X\psi + \frac{1}{n} X \cdot D^s\psi \;=\; 0,$$

which has to hold for any vector field $X$. One easily checks that some properties of Riemannian twistor spinors ([BFGK91, Section 1.4, Thms 2, 3]) carry over without modification to the case with torsion. We omit the proof.

**Theorem 3.1.**

(1) *$\psi$ is a twistor spinor if and only if the following condition holds for any vector fields $X, Y$:*

$$X\nabla^s_Y\psi + Y\nabla^s_X\psi \;=\; \frac{2}{n} g(X,Y) D^s\psi.$$

(2) *$\psi$ is a twistor spinor if and only if the expression $X \cdot \nabla^s_X\psi$ does not depend on the unit vector field $X$.*

(3) *Any twistor spinor $\psi$ satisfies:* $(D^s)^2\psi = n\,\Delta^s\psi$.

(4) *Any spinor field $\varphi$ satisfies:* $\|P^s\varphi\|^2 + \frac{1}{n}\|D^s\varphi\|^2 = \|\nabla^s\varphi\|^2$.

The following calculation is fundamental for the twistorial estimate.

**Lemma 3.1.**

$$\begin{aligned}(D^{s/3})^2 - \frac{1}{n}(D^s)^2 &= \frac{n-1}{n}\left[D^0 + \frac{s(n-3)}{n-1}T\right]^2 + \frac{4s^2}{1-n}T^2\\ &= \frac{n-1}{n}\left[D^s - \frac{2sn}{n-1}T\right]^2 + \frac{4s^2}{1-n}T^2.\end{aligned}$$

*Proof.* Consider the difference ($D^0 = D^g$):

$$\begin{aligned}(D^{s/3})^2 - \frac{1}{n}(D^s)^2 &= (D^0+sT)^2 - \frac{1}{n}(D^0+3sT)^2\\ &= (D^0)^2 + s^2T^2 + s(D^0T+TD^0) - \frac{1}{n}[(D^0)^2 + 9s^2T^2 + 3s(D^0T+TD^0)]\\ &= \left(1-\frac{1}{n}\right)(D^0)^2 + s^2\left(1-\frac{9}{n}\right)T^2 + s\left(1-\frac{3}{n}\right)(D^0T+TD^0)\\ &= \frac{n-1}{n}[(D^0)^2 + s^2\frac{n-9}{n-1}T^2 + s\frac{n-3}{n-1}(D^0T+TD^0)].\end{aligned}$$

The square of any Dirac operator $D^0 + \mu T$ can be expanded into

$$(D^0+\mu T)^2 = (D^0)^2 + \mu(D^0T+TD^0) + \mu^2T^2\,.$$

If we set $\mu = s\frac{n-3}{n-1}$, the difference above may be rewritten as

$$\begin{aligned}(D^{s/3})^2 - \frac{1}{n}(D^s)^2 &= \frac{n-1}{n}\left[\left(D^0+\mu T\right)^2 - s^2\frac{4n}{(n-1)^2}T^2\right]\\ &= \frac{n-1}{n}\left[\left(D^0 + s\frac{n-3}{n-1}T\right)^2 - s^2\frac{4n}{(n-1)^2}T^2\right]\\ &= \frac{n-1}{n}\left[D^0 + s\frac{n-3}{n-1}T\right]^2 - s^2\frac{4}{n-1}T^2 \qquad \square\end{aligned}$$

This calculation allows us to prove a crucial integral formula, which will yield the desired estimate as an easy corollary. The key idea is that the difference of squares of Dirac operators on the left hand side can again by expressed as the square of a suitably renormalized Dirac operator (and a multiple of the endomorphism $T^2$). We may assume without loss that $n \geq 4$, since the case $n = 3$ is not very interesting. Recall that we write $\nabla^c$ for the connection with parameter $s = 1/4$.

**Theorem 3.2** (Twistorial integral formula)**.** *Suppose $\nabla^c T = 0$. For any spinor field $\psi$, the Dirac operator $\not{D}$ of the connection with torsion $\frac{1}{3}T$ satisfies the following integral formula:*

$$\begin{aligned}\int_M \langle \not{D}^2\psi,\psi\rangle dM &= \frac{n}{n-1}\int_M \|P^s\psi\|^2 dM + \frac{n}{4(n-1)}\int_M \mathrm{Scal}^g\|\psi\|^2 dM\\ &\quad + \frac{n(n-5)}{8(n-3)^2}\|T\|^2\int \|\psi\|^2 dM + \frac{n(4-n)}{4(n-3)^2}\int_M \langle T^2\psi,\psi\rangle dM.\end{aligned}$$

*Here, the parameter $s$ appearing the the twistor operator $P^s$ has the value $s = \dfrac{n-1}{4(n-3)}$.*

*Proof.* Consider the operator $D^{s/3} = D^g + 3sT$. Integrating over the generalized Schrödinger-Lichnerowicz formula $(*)$ of Theorem 2.2, we obtain (we omit the volume form and the domain of integration in most integrals)

$$\int \langle (D^{s/3})^2\psi,\psi\rangle \;=\; \int \|\nabla^s\psi\|^2 + s\int \langle dT\psi,\psi\rangle + \frac{1}{4}\int \mathrm{Scal}^g\|\psi\|^2 - 2s^2\int \|T\|^2\|\psi\|^2.$$

By identity (4) from Theorem 3.1, the length $\|\nabla^s\psi\|^2$ can be expressed through the twistor and the Dirac operator, yielding

$$\int \langle [(D^{s/3})^2 - \frac{1}{n}(D^s)^2]\psi,\psi\rangle = \int \|P^s\psi\|^2 + s\int \langle dT\psi,\psi\rangle + \frac{1}{4}\int \mathrm{Scal}^g\,\|\psi\|^2 - 2s^2\int \|T\|^2\|\psi\|^2.$$

The main idea is now to view the difference on the left hand side as the square of a single Dirac operator by a clever choice of the parameter $s$. We rewrite the left hand side using the fundamental calculation from Lemma 3.1,

$$\begin{aligned}\frac{n-1}{n}\int \langle (D^0 + s\frac{n-3}{n-1}T)^2\psi,\psi\rangle \;&=\; \int \|P^s\psi\|^2 + \frac{1}{4}\int \mathrm{Scal}^g\|\psi\|^2 + \\ &\quad + s\int \langle dT\psi,\psi\rangle - 2s^2\int |T|^2|\psi|^2 + s^2\frac{4}{n-1}\int \langle T^2\psi,\psi\rangle.\end{aligned}$$

We now choose the parameter $s$ such that the operator on the left hand side becomes just $\not{D}$, i. e. the Dirac operator with torsion $\frac{1}{3}T$. Since $\not{D} = D^0 + \frac{1}{4}T$, this requires $s\frac{n-3}{n-1} = \frac{1}{4}$, hence we obtain $s = \frac{n-1}{4(n-3)}$, the value encountered in the statement of the result. Inserting this value of $s$ yields

$$\begin{aligned}\frac{n-1}{n}\int \langle \not{D}^2\psi,\psi\rangle \;&=\; \int \|P^s\psi\|^2 + \frac{1}{4}\int \mathrm{Scal}^g\|\psi\|^2 + \\ &\quad + \frac{n-1}{4(n-3)}\cdot\int \langle dT\cdot\psi,\psi\rangle - \frac{(n-1)^2}{8(n-3)^2}\int \|T\|^2\|\psi\|^2 + \frac{n-1}{4(n-3)^2}\int \langle T^2\psi,\psi\rangle.\end{aligned}$$

The assumption $\nabla^c T = 0$ implies $dT = 2\sigma_T$, and since $T^2 = -2\sigma_T + \|T\|^2$ always holds, we get $dT = -T^2 + \|T\|^2$. This means for the previous equation

$$\int \langle \not{D}^2\psi,\psi\rangle = \frac{n}{n-1}\int \|P^s\psi\|^2 + \frac{n}{4(n-1)}\int \mathrm{Scal}^g\|\psi\|^2 + \frac{n(n-5)}{8(n-3)^2}\int \|T\|^2\|\psi\|^2 - \frac{n(n-4)}{4(n-3)^2}\int \langle T^2\psi,\psi\rangle.$$

□

It is to be understood that all eigenvalue estimates based on this integral identity are meant on *compact* manifolds, even if this is not repeateds throughout. We can assume that an eigenspinor $\psi$ of $\not{D}^2$ with eigenvalue $\lambda$ lies in one subbundle $\Sigma_\mu$ (see the general comments on the universal estimate). Thus, we obtain:

**Corollary 3.1** (Twistorial eigenvalue estimate in $\Sigma_\mu$)**.** *For $\nabla^c T = 0$, the smallest eigenvalue $\lambda$ of $\not{D}^2$ on $\Sigma_\mu$ satisfies the inequality*

$$\lambda(\not{D}^2|_{\Sigma_\mu}) \;\geq\; \frac{n}{4(n-1)}\mathrm{Scal}^g_{\min} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 + \frac{n(4-n)}{4(n-3)^2}\mu^2 \;=:\; \beta_{\mathrm{tw}}(\mu),$$

*and equality holds if and only if the following conditions are satisfied:*

(1) *the Riemannian scalar curvature of $(M,g)$ is constant,*
(2) *the eigenspinor $\psi$ is a twistor spinor for $s = \dfrac{n-1}{4(n-3)}$.*

**Corollary 3.2** (Twistorial eigenvalue estimate)**.** *For $\nabla^c T = 0$, the smallest eigenvalue $\lambda$ of $\not{D}^2$ satisfies the inequality*

$$\lambda \;\geq\; \frac{n}{4(n-1)}\mathrm{Scal}^g_{\min} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 + \frac{n(4-n)}{4(n-3)^2}\max(\mu_1^2,\ldots,\mu_k^2) \;=:\; \beta_{\mathrm{tw}},$$

*and equality holds if and only if, in additon to the two conditions from the previous Corollary, the eigenspinor $\psi$ lies in the subbundle $\Sigma_\mu$ corresponding to the largest eigenvalue of $T^2$.*

We make some first pertinent comments on this result.

(1) The twistorial eigenvalue estimate reduces for $T = 0$ to Friedrich's inequality, thus showing its optimality at least in this situation. The quality of the estimate increases if the scalar curvature becomes large and dominates the terms in $\|T\|^2$ and $\max(\mu_1^2, \dots, \mu_k^2)$. But there also exist a few examples with negative scalar curvature (for example, Sasaki metrics on compact quotients of Heisenberg groups, see [FrI02], [FI03a], [AF04b]); for these manifolds, the twistorial eigenvalue estimate becomes rather bad.
(2) In some situations, the geometric data yield additional information in which subbundle $\Sigma_\mu$ the smallest eigenvalue can occur, or which bundles are of particular interest. In these cases, the global estimate from Corollary 3.2 is too coarse, and one should apply instead the estimate in one or several well-chosen subbundles as stated in Corollary 3.1. The case of parallel spinors discussed in Section 4 below is an example of such a situation.
(3) Strictly speaking, there is not one equality case, but one in every subbundle $\Sigma_\mu$. Thus, it may happen that the twistorial eigenvalue estimate on the whole spin bundle $\Sigma$ is not sharp (for the $\mu$ belonging to the maximum of the $T$ eigenvalues), but twistor spinors exist nevertheless (namely, for some other $\mu$).
(4) Unfortunately, one cannot construct a $\not{D}$ eigenspinor from a $\not{D}^2$ eigenspinor that would still lie in one of the subbundles $\Sigma_\mu$ – hence, a twistor spinor realizing the optimal eigenvalue estimate does not have to be some kind of Killing spinor. Nevertheless, a very reasonable Killing equation with torsion exists, and every Killing spinor with torsion is necessarily a twistor spinor with torsion. In dimension 6, the converse can be shown (see Section 6).

In the next Section, we will discuss applications of this estimate in different special geometries with torsion and compare it to the universal estimate. Section 5 will be devoted to the discussion of the equality case in the twistorial estimate, in particular to the description of Killing and twistor spinors with torsion and examples of manifolds where such spinors exist.

## 4. Discussion of the twistorial estimate

**The case $n = 4$.** The 4-dimensional case is special in many respects. For purely algebraic reasons, $\sigma_T = 0$, hence $\nabla^c T = 0$ implies $dT = 0$ and $T^2$ acts by scalar multiplication with $\|T\|^2$, i. e. the only $T$ eigenvalues are $\pm\|T\|$. Furthermore, $\nabla^c T = 0$ implies $\nabla^g * T = 0$, i. e. there exists a LC-parallel 1-form on $(M^4, g)$. Set $c = \mathrm{Scal}^g_{\min}/\|T\|^2$. In [AFK08] it was proved that

$$\lambda \;\geq\; \begin{cases} \frac{\|T\|^2}{4}\left[c - \frac{1}{2}\right] & \text{for } c \geq 3/2, \\ \frac{\|T\|^2}{16}[\sqrt{6c} - 1]^2 & \text{for } 1/6 \leq c \leq 3/2 \end{cases}$$

The first estimate is just the universal estimate given in Theorem 2.3. Indeed, the deformation method used in this paper has the typical property of yielding eigenvalue estimates that are valid only for some restricted parameter range. In particular, no improvement was possible for $c \geq 3/2$. In contrast, the twistorial eigenvalue estimate from Corollary 3.2 yields

$$\lambda \;\geq\; \frac{\|T\|^2}{3}\left[c - \frac{3}{2}\right] \quad \text{for } c \geq 3/2.$$

Hence, the parameter range for which the twistor ansatz yields an improvement is complementary to the results obtained via deformation techniques. One checks that the twistor estimate lies above the universal estimate for $c \geq 9/2$.

**The case $n = 5$.** For a 5-dimensional manifold, the twistorial eigenvalue estimate becomes

$$\lambda \;\geq\; \frac{5}{16}[\mathrm{Scal}^g_{\min} - \max(\mu_1^2, \dots, \mu_k^2)].$$

Thus, the quality of the estimate increases for large scalar curvatures. In Example 5.1, we show that the Stiefel manifold $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2)$ carries a metric for which this estimate becomes optimal. On the other hand, we can identify manifolds for which the twistorial estimate yields no improvement. This is for example the case for Sasaki manifolds $(M^5, g, \xi, \eta, \varphi)$: In this case, there exists a unique connection $\nabla$ with totally skew-symmetric torsion preserving the Sasakian structure by [FrI02]. The torsion form is given by the formula $T = \eta \wedge d\eta$, and $||T|| = 8$ holds. $T$ splits the spinor bundle into two 1-dimensional bundles and one 2-dimensional bundle,

$$\Sigma_{\pm 4} \;=\; \left\{\psi \in \Sigma M^5 \,:\, T\psi \;=\; \pm\, 4\,\psi\right\}, \quad \Sigma_0 \;=\; \left\{\psi \in \Sigma M^5 \,:\, T\psi \;=\; 0\right\}.$$

Thus, $\max(\mu_1^2, \ldots, \mu_k^2) = 16$ and the twistorial estimate becomes

$$\lambda \;\geq\; \frac{5}{16}\mathrm{Scal}^g_{\min} - 5.$$

On the other hand, it was proved by the deformation ansatz in [AFK08] that

$$\lambda \;\geq\; \begin{cases} \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2 & \text{for } -4 < \mathrm{Scal}^g_{\min} \leq 4(9+4\sqrt{5}) \\ \frac{5}{16}\,\mathrm{Scal}^g_{\min} & \text{for } \mathrm{Scal}^g_{\min} \geq 4(9+4\sqrt{5}) \simeq 71,78. \end{cases}$$

The estimates coincide for $\mathrm{Scal}^g_{\min} = 36$; for all other possible scalar curvatures, the deformation estimate is better. Thus, the main advantage of the twistorial estimates lies here in its universality: It makes a statement for non-Sasaki manifolds as well, a case that is not covered by [AFK08].

**The case $n = 6$.** Nearly Kähler manifolds will be discussed in Section 6. Hence, let us consider some of the other classes of manifolds with parallel characteristic torsion. Almost Hermitian 6-manifolds with parallel characteristic torsion were classified by Schoemann in [Sch07]; in particular, it was shown that there exist many almost Hermitian manifolds of Gray-Hervella type $W_3$ or $W_4$ with parallel characteristic torsion – nilpotent Lie groups, naturally reductive spaces, $S^1$-fibrations over Sasaki 5-manifolds etc. For both classes, the torsion has eigenvalues $\mu = 0, \pm\sqrt{2}\,\|T\|$, thus the twistorial eigenvalue estimate for $W_3$ or $W_4$ geometries with parallel torsion is given by

$$\lambda \;\geq\; \frac{3}{10}\mathrm{Scal}^g_{\min} - \frac{7}{16}\|T\|^2.$$

The right hand side is non-negative for $\mathrm{Scal}^g_{\min} \geq 35\|T\|^2/18$. For this curvature range, the deformation technique did not yield any improvement of the universal eigenvalue estimate [Ka10]. However, it was proved therein that there exist no $\nabla^c$-parallel spinors, hence the universal estimate could not be optimal; thus, the twistor estimate is better for these large scalar curvatures.

**Existence of $\nabla^c$-parallel spinors.** In general, the twistor and the universal estimate cannot be compared abstractly. But if there exists a $\nabla^c$-parallel spinor field $\psi \in \Sigma_\mu$, the universal eigenvalue estimate $\lambda$ (see Theorem 2.3) is sharp for some $T$ eigenvalue $\mu$, i. e.

$$\lambda \;=\; \frac{1}{4}\mathrm{Scal}^g_{\min} + \frac{1}{8}\|T\|^2 - \frac{1}{4}\mu^2 \;=:\; \beta_{\mathrm{univ}}(\mu).$$

Notice that $\mathrm{Scal}^g$ has to be constant in this situation: By identity (1) from Theorem 2.1 [FrI02, Cor. 3.2] for parallel torsion, such a spinor satisfies $\sigma_T\psi + \mathrm{Scal}^c\psi/4 = 0$, so the fact that $\nabla^c\sigma_T = 0$ implies $\mathrm{Scal}^c = \mathrm{const}$, and then the claim follows (this generalizes the well-known fact that the Riemannian scalar curvature vanishes in the presence of a $\nabla^g$-parallel spinor, see [Hit74]). Hence we can drop the minimum in the formula for $\lambda$. On the other hand, $\nabla^c\psi = 0$ implies $\not{D}\psi = -\frac{1}{2}T\psi = -\frac{\mu}{2}\psi$, hence $\lambda = \mu^2/4$. Thus, we have the relation

$$(*) \qquad \mathrm{Scal}^g \;=\; -\frac{1}{2}\|T\|^2 + 2\mu^2.$$

A priori, it is not so easy to compare this result with the twistorial eigenvalue estimate $\lambda \geq \beta_{\mathrm{tw}}(\mu)$. However, in the presence of $\nabla^c$-parallel spinor fields, the twistorial estimate cannot be larger

than the universal estimate, i. e. $\beta_{\rm tw}(\mu) \leq \beta_{\rm univ}(\mu)$ needs to hold. This observation leads to the following result, which is of interest on his own:

**Lemma 4.1.** *Suppose that $\nabla^c T = 0$, that there exists at least one $\nabla^c$-parallel spinor field $0 \neq \psi \in \Sigma_\mu$, and that $n \leq 8$. Then the following inequalities hold:*

$$(**) \qquad 0 \;\leq\; 2n\|T\|^2 + (n-9)\mu^2, \quad \mathrm{Scal}^g \;\leq\; \frac{9(n-1)}{2(9-n)}\|T\|^2.$$

*Furthermore, equality is attained if and only if $\beta_{\rm tw}(\mu) = \beta_{\rm univ}(\mu)$.*

*Proof.* We only sketch the argument, leaving out the routine computations. First, one checks that $\beta_{\rm tw}(\mu) \leq \beta_{\rm univ}(\mu)$ is equivalent to

$$\mathrm{Scal}^g \;\leq\; \frac{(n-1)(9-n)}{2(n-3)^2}\|T\|^2 + \frac{(n-1)(2n-9)}{(n-3)^2}\mu^2.$$

Rewriting relation $(*)$ as $-\mathrm{Scal}^g = \|T\|^2/2 - 2\mu^2$ and adding this to the previous inequality, one obtains the first of the two statements. It becomes trivial for $n \geq 9$. For $n \leq 8$, it can be rewritten as $\mu^2 \leq 2n\|T\|^2/(9-n)$. We then use this in order to eliminate $\mu^2$ from the identity $(*)$, yielding the second statement. □

We thus have an easy criterion for excluding the existence of parallel spinors and for checking whether the two eigenvalue estimates yield the same result.

For example, consider a 6-dimensional nearly Kähler manifold $(M^6, g, J)$ with its characteristic connection $\nabla^c$. These are Einstein spaces of positive scalar curvature, $\|T\|^2 = \frac{2}{15}\mathrm{Scal}^g$, and $T$ has the eigenvalues $\mu = 0$ (multiplicity 6) and $\mu = \pm 2\|T\|$ (each with multiplicity 1). It is well-known that the two Riemannian Killing spinors $\varphi_\pm$ are $\nabla^c$-parallel and lie in $\Sigma_{\pm 2\|T\|}$. One then checks by hand that $\beta_{\rm tw}(\mu) = \beta_{\rm univ}(\mu) = \frac{2}{15}\mathrm{Scal}^g$, and indeed one sees that the relations $(**)$ hold with an equality sign. Qualitatively, the same happens for nearly parallel $G_2$ manifolds.

## 5. Killing and twistor spinors with torsion

**Lemma 5.1.** *Suppose $\nabla^c T = 0$. The twistor equation $P^s\psi = 0$ corresponding to the parameter value $s = \dfrac{n-1}{4(n-3)}$ is equivalent to*

$$\nabla^c_X\psi + \frac{1}{n}X\cdot \not{D}\psi + \frac{1}{2(n-3)}(X\wedge T)\cdot\psi \;=\; 0,$$

*and each such twistor spinor satisfies*

$$\not{D}^2\psi \;=\; \left[\frac{n}{4(n-1)}\mathrm{Scal}^g + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 + \frac{n(4-n)}{4(n-3)^2}T^2\right]\psi.$$

*Proof.* For $s = \frac{n-1}{4(n-3)}$, one has

$$\nabla^s_X\psi = \nabla^c_X\psi + \frac{1}{2(n-3)}(X \lrcorner T)\cdot\psi$$

and

$$D^s = \not{D} + \frac{n}{2(n-3)}T\,.$$

Inserting these expressions into the twistor equation leads to

$$\nabla^c_X\psi + \frac{1}{n}X\cdot\not{D}\psi + \frac{1}{2(n-3)}(X\cdot T + X\lrcorner T)\cdot\psi = 0.$$

The claim then follows, since $X\cdot T = X\wedge T - X\lrcorner T$. To derive the identity for $\not{D}^2$ on twistor spinors, we proceed similarly as in the proof of the twistorial integral formula (Theorem 3.2), but

with one crucial change. Let $\psi$ be a twistor spinor, $s = \frac{n-1}{4(n-3)}$. We start with the generalized Schrödinger-Lichnerowicz formula $(*)$ of Theorem 2.2,

$$(D^{s/3})^2\psi \;=\; \Delta^s\psi + sdT\psi + \frac{1}{4}\mathrm{Scal}^g\psi - 2s^2\|T\|^2\psi.$$

Instead of the norm identity (4) for any spinor from Theorem 3.1, we can now use the operator identity (3), $(D^s)^2\psi = n\Delta^s\psi$ to rewrite this as

$$(D^{s/3})^2\psi - \frac{1}{n}(D^s)^2\psi \;=\; s\,dT\psi + \frac{1}{4}\mathrm{Scal}^g\psi - 2s^2\|T\|^2\psi.$$

By the fundamental Lemma 3.1, the left hand side can be expressed through $\not{D}^2$,

$$\frac{n-1}{n}\not{D}^2\psi - s^2\frac{4}{n-1}T^2\psi \;=\; s\,dT\psi + \frac{1}{4}\mathrm{Scal}^g\psi - 2s^2\|T\|^2\psi.$$

Now one finishes the calculation as in the proof of Theorem 3.2, and it comes as no surprise that the identity obtained is exactly the operator version of the twistorial integral formula without $\int \|P^s\psi\|\,dM$ term. □

Observe that being *only* a twistor spinor does not imply being a $\not{D}^2$ eigenspinor, hence we cannot conclude from the last identity that the scalar curvature has to be constant if such a spinor exists. We shall now define the *Killing equation with torsion* for any dimension $n$.

**Definition 5.1** (Killing spinor with torsion)**.** A spinor field $\psi$ is called a *Killing spinor with torsion* if the equation $\nabla^s_X\psi = \kappa X\cdot\psi$ for $s = \frac{n-1}{4(n-3)}$ holds. The spinor then satisfies $D^s\psi = -n\kappa\psi$, which is equivalent to

$$\not{D}\psi \;=\; -n\kappa\psi - \frac{n}{2(n-3)}T\psi.$$

Thus, contrary to a twistor spinor with torsion, *any* Killing spinor with torsion known to lie in some $T$-eigenspace $\Sigma_\mu$ is a $\not{D}$-eigenspinor.

Observe that the Killing vector fields of a metric connection with antisymmetric torsion are precisely those of the Levi-Civita connection, hence the vector field

$$X_\psi \;:=\; \sum_{i=1}^n i\langle\psi, e_i\cdot\psi\rangle e_i \tag{1}$$

is Killing as in the Riemannian situation (compare [BFGK91, p. 30]). The next result claims that all Killing spinors are twistor spinors with matching parameters. We omit the easy proof.

**Lemma 5.2.** *Suppose $\nabla^c T = 0$ and $\psi\in\Sigma_\mu$. The Killing equation $\nabla^s_X\psi = \kappa X\cdot\psi$ for $s = \frac{n-1}{4(n-3)}$ is then equivalent to*

$$\nabla^c_X\psi - \left[\kappa + \frac{\mu}{2(n-3)}\right]X\cdot\psi + \frac{1}{2(n-3)}(X\wedge T)\psi \;=\; 0.$$

*In particular, $\psi$ is a twistor spinor with torsion for the same value $s$, and the Killing number $\kappa$ satisfies the quadratic equation*

$$n\left[\kappa + \frac{\mu}{2(n-3)}\right]^2 \;=\; \frac{1}{4(n-1)}\mathrm{Scal}^g + \frac{n-5}{8(n-3)^2}\|T\|^2 - \frac{n-4}{4(n-3)^2}\mu^2.$$

*In particular, the scalar curvature has to be constant.*

**Remark 5.1.** If $T = 0$, and a fortiori $\mu = 0$, this quadratic equation reduces to the well-known relation $\mathrm{Scal}^g = 4n(n-1)\kappa^2$ for Riemannian Killing spinors [Fr80]. However, $\kappa = 0$ does *not* correspond to $\nabla^c$-parallel spinors because of the torsion shift hidden in the value of $s$, hence 0 is a priori an admissible Killing number (contrary to the Riemannian case). An easy formal calculation shows: *Any spinor field parallel for the connection with torsion $T$ is a Killing spinor with torsion with $\kappa = 0$ for the connection with torsion $\frac{n-3}{n-1}T$.* This can be a useful remark

when parallel spinors are known to exist for a connection with non-parallel torsion for which the rescaled torsion $\frac{n-3}{n-1}T$ becomes parallel.

The Killing equation with torsion can be used to express the curvature operator of a manifold admitting such spinor fields. Thus, we obtain an algebraic identity for the Ricci tensor; the rather lengthy proof is deferred to Appendix B.

**Theorem 5.1** (integrability condition)**.** *Let $\psi$ be a Killing spinor with torsion with Killing number $\kappa$, set $\lambda := \frac{1}{2(n-3)}$ for convenience, and recall that $s = \frac{n-1}{4(n-3)}$. Then the Ricci curvature of the characteristic connection satisfies the identity*

$$\begin{aligned}\mathrm{Ric}^c(X)\psi &= -16s\kappa(X \lrcorner T)\psi + 4(n-1)\kappa^2 X\psi + (1-12\lambda^2)(X \lrcorner \sigma_T)\psi + \\ &\quad +2(2\lambda^2+\lambda)\sum e_k(T(X,e_k) \lrcorner T)\psi\,.\end{aligned}$$

As a typical application of this result, it is shown in Corollary A.1:

**Corollary 5.1.** *A 5-dimensional Einstein-Sasaki manifold $(M, g, \xi, \eta, \varphi)$ endowed with its characteristic connection cannot admit Killing spinors with torsion.*

**Example 5.1** (A 5-dimensional manifold with Killing spinors with torsion)**.** The 5-dimensional Stiefel manifold $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2)$ carries a one-parameter family of metrics constructed by G. Jensen [Jen75] with many remarkable properties. Embed $H = \mathrm{SO}(2)$ into $G = \mathrm{SO}(4)$ as the lower diagonal $2\times 2$ block. Then the Lie algebra $\mathfrak{so}(4)$ splits into $\mathfrak{so}(2) \oplus \mathfrak{m}$, where $\mathfrak{m}$ is given by

$$\mathfrak{m} = \left\{ \left[\begin{array}{cc|c} 0 & -a & \multirow{2}{*}{$-X^t$} \\ a & 0 & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$X$}} & 0 \quad 0 \\ & & 0 \quad 0 \end{array}\right] =: (a, X) \;:\; a \in \mathbb{R},\; X \in \mathcal{M}_{2,2}(\mathbb{R}) \right\}.$$

Denote by $\beta(X,Y) := \mathrm{tr}(X^tY)$ the Killing form of $\mathfrak{so}(4)$. Then the Jensen metric on $\mathfrak{m}$ to the parameter $t \in \mathbb{R}$ is defined by

$$\langle (a,X),(b,Y)\rangle \;=\; \frac{1}{2}\beta(X,Y) + t\beta(a,b) \;=\; \frac{1}{2}\beta(X,Y) + 2t\cdot ab\,.$$

For $t = 2/3$, G. Jensen proved that this metric is Einstein, and Th. Friedrich showed that it carries a homogeneous spin structure and that it admits two Riemannian Killing spinors [Fr80] and thus realizes the equality case in his estimate for the first eigenvalue of the Dirac operator. A detailed investigation of this family of metrics from the point of view of metric connections with torsion may be found in [Ag03]; in particular, we refer to these two papers for all proofs of formulas given below (however, we will write down whatever is needed to follow our argument). Denote by $E_{ij}$ the standard basis of $\mathfrak{so}(4)$. Then the elements

$$Z_1 := E_{13},\; Z_2 := E_{14},\; Z_3 = E_{23},\; Z_4 = E_{24},\; Z_5 = \frac{1}{\sqrt{2s}}E_{12}$$

form an orthonormal basis of $\mathfrak{m}$. Identifying $\mathfrak{m}$ with $\mathbb{R}^5$ via the chosen basis, the isotropy representation of an element $g(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in H = \mathrm{SO}(2)$ and its lift to the 4-dimensional spinor representation $\kappa : \mathrm{Spin}(\mathbb{R}^5) \to \mathrm{GL}(\Delta_5)$ can be computed,

$$\mathrm{Ad}\, g(\theta) \;=\; \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \kappa\big(\widetilde{\mathrm{Ad}}\, g(\theta)\big) \;=\; \begin{bmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the basis elements $\psi_3$ and $\psi_4$ of $\Delta_5$ define sections of the spinor bundle $S = G \times_{\kappa(\widetilde{\mathrm{Ad}})} \Delta_5$ if viewed as constant maps $G \to \Delta_5$. In fact, for $t = 2/3$, $\psi^{\pm} := \pm i\psi_3 + \psi_4$ are exactly the Riemannian Killing spinors from [Fr80]. For the undeformed metric ($t = 1/2$), these two

spinors are parallel. In [Jen75, Prop. 3], the author computed the abstract formulas for the map $\Lambda^g_{\mathfrak{m}} : \mathfrak{m} \cong \mathbb{R}^5 \to \mathfrak{so}(5)$ defining the Levi-Civita connection in the sense of Wang's Theorem [KN69, Ch. X, Thm 2.1]. In our example, this yields (viewed as endomorphisms of $\mathbb{R}^5$):

$$\Lambda^g_{\mathfrak{m}}(Z_1) \,=\, \sqrt{\frac{t}{2}}E_{35}, \quad \Lambda^g_{\mathfrak{m}}(Z_2) \,=\, \sqrt{\frac{t}{2}}E_{45}, \quad \Lambda^g_{\mathfrak{m}}(Z_3) \,=\, -\sqrt{\frac{t}{2}}E_{15}, \quad \Lambda^g_{\mathfrak{m}}(Z_4) \,=\, -\sqrt{\frac{t}{2}}E_{25},$$

$$\Lambda^g_{\mathfrak{m}}(Z_5) \,=\, \frac{1-t}{\sqrt{2t}}(E_{13}+E_{24})\,.$$

The space $\mathfrak{m}$ has a preferred direction, namely $\xi = Z_5$, which is fixed under the isotropy representation. Denote its dual 1-form, $\eta(X) = \langle Z_5, X\rangle$ by $\eta$. As discussed in [Ag03], there exist three almost contact metric structures intertwining the isotropy representation, and their characteristic connections with torsion coincide (see [FrI02] for general results). To fix the ideas, choose for example the skew-symmetric endomorphism $\varphi : TM \to TM$ defined by $\varphi(Z_1) = -Z_3,\ \varphi(Z_2) = -Z_4, \varphi(Z_5) = 0$. The differential of its fundamental form $F(X,Y) := \langle X, \varphi(Y)\rangle$ and its Nijenhuis tensor vanish. Thus, the Stiefel manifold $V_{4,2}$ admits a characteristic connection $\nabla^c$ with torsion

$$T \,=\, \eta \wedge d\eta \,=\, -\sqrt{2t}\,(Z_1 \wedge Z_3 + Z_2 \wedge Z_4) \wedge Z_5\,.$$

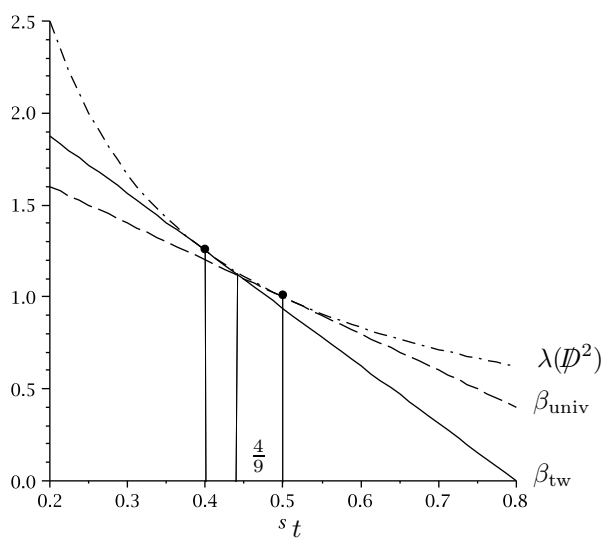
One checks that $T$ is parallel, hence the methods described here apply. Furthermore, one has the following geometric data,

$$\|T\|^2 \,=\, 4t, \quad \mu \in \{0, \pm 2\sqrt{2t}\}, \quad \mathrm{Scal}^g \,=\, 8-2t, \quad \mathrm{Ric}^g \,=\, \mathrm{diag}(2-t, 2-t, 2-t, 2-t, 2t).$$

Using the formulas for the Levi-Civita connection, one checks that $\psi^\pm$ is an $\not{D}$-eigenspinor to the eigenvalue $\pm 1/\sqrt{2t}$. Since $\psi^\pm$ lies in the bundle $\Sigma_\mu$ with $\mu = \mp 2\sqrt{2t}$, the universal eigenvalue estimate and the twistorial eigenvalue estimate take the numerical values

$$\beta_{\mathrm{univ}} \,=\, 2(1-t), \quad \beta_{\mathrm{tw}} \,=\, \frac{5}{2} - \frac{25}{8}t.$$

Thus, the two estimates coincide for $t = 4/9$; below this value, the twistorial estimate is better, while above, the universal estimate is to be preferred. For large $t$ – corresponding to highly negative curvature – both estimates are not applicable. In the figure below, these estimates and the known eigenvalue $\lambda(\not{D}^2) = 1/2t$ of $\not{D}^2$ are drawn.




A priori, $\lambda(\not{D}^2)$ has no reason to be the smallest eigenvalue. However, in the case $t = 1/2$ the spinor fields $\psi^\pm$ are $\nabla^c$-parallel, and thus they realize the equality case in the universal estimate. For the twistorial estimate, one sees that $\beta_{\mathrm{tw}}$ becomes optimal for $t = 2/5$, and thus $\psi^\pm$ are automatically twistor spinors with torsion. A more detailed computer computation reveals that

more is true in this case: $\psi^\pm$ is a Killing spinor with torsion to the Killing number[1] $\kappa = \pm\sqrt{5}/10$. These are two of the four solutions of the two quadratic equations for $\kappa$ from Lemma 5.2 (one equation for each value of $\mu$), the other two being $\pm 3\sqrt{5}/10$ (for $\mu = \mp 2\sqrt{2t}$). These would yield a larger $\not{D}$-eigenvalue, thus they cannot correspond to twistor spinors. One checks that the Killing vector fields associated to $\psi^\pm$ by equation (1) are non-vanishing multiples of $\xi = Z_5$.

The example also illustrates the eigenvalue estimate in single eigensubbundles $\Sigma_\mu$, as described in Corollary 3.1. The torsion $T$ has the eigenvalues $0, \pm 2\sqrt{2t}$, thus $\max(\mu_i^2) = 8t$, and as we saw, the twistor spinors with torsion lie in the eigensubbundles corresponding the the $\mu$ values for which the maximum is attained. On the other side, Corollary 3.1 allows an eigenvalue estimate on the remaining bundle with $\mu = 0$. Since $n = 5$, the twistorial eigenvalue estimate takes the form

$$\lambda(\not{D}^2|_{\Sigma_0}) \;\geq\; \beta_{\mathrm{tw}}(0) \;=\; \frac{5}{16}\mathrm{Scal}^g \;=\; \frac{5(4-t)}{8}.$$

This is exactly the Riemannian estimate. It is optimal if and only if there exists a Riemannian Killing spinor in $\Sigma_0$, which is never the case. We can again compare this estimate with the universal estimate from Theorem 2.3,

$$\lambda(\not{D}^2|_{\Sigma_0}) \;\geq\; \beta_{\mathrm{univ}}(0) \;=\; \frac{1}{4}\mathrm{Scal}^g + \frac{1}{8}\|T\|^2 \;=\; 2.$$

Hence, the twistorial estimate lies above the universal estimate for $t \leq 4/5$. Since equality in the universal estimate is obtained for $\nabla^c$-parallel spinors, it can presumably not be obtained as well, though we have no strict argument that excludes the existence of parallel spinors in $\Sigma_0$.

**Example 5.2** (A 7-dimensional manifold with Killing spinors with torsion)**.** We shortly describe a second example, relatively similar to the previous one, hence we will not give so many details. Consider the 7-dimensional Stiefel manifold $V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3)$, whith $\mathrm{SO}(3)$ embedded as upper diagonal $(3\times 3)$-block. The complement $\mathfrak{m}$ of $\mathfrak{so}(3)$ inside $\mathfrak{so}(5)$ splits under the isotropy representation into two copies of the defining representation and a one-dimensional trivial summand, $\mathfrak{m} = \mathfrak{m}_3 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_1$. We define a new metric on $V_{5,2}$ by deforming the Killing form in direction of $\mathfrak{m}_1$ by a factor $t > 0$, $g_t := \beta|_{\mathfrak{m}_3\oplus\mathfrak{m}_3} + t\beta|_{\mathfrak{m}_1}$. This manifold is known to be Einstein Sasaki and to have two Riemannian Killing spinors for $t = 3/2$, see [FKMS97], [Ka00]. $V_{5,2}$ carries an almost metric contact structure in direction $\mathfrak{m}_1 = \mathbb{R}\cdot Z_7$ with vanishing Nijenhuis tensor and $\varphi = E_{14} + E_{25} + E_{36}$. It admits a characteristic connection $\nabla^c$ with torsion

$$T \;=\; \eta\wedge d\eta \;=\; -\sqrt{t}\,(Z_1\wedge Z_4 + Z_2\wedge Z_5 + Z_3\wedge Z_6)\wedge Z_7\,.$$

One checks that $\nabla^c T = 0$ for all $t$ and that the $T$ eigenvalues are $3\sqrt{t}$ (multiplicity 2) and $-\sqrt{t}$ (multiplicity 6). The lift of the isotropy representation to the spin representation has two invariant spinors $\varphi_\pm \in \Sigma_{3\sqrt{t}}$, which will thus define global spinor fields on $V_{5,2}$. After computing the formulas for the Levi-Civita connection, one checks that $\varphi^\pm$ are $\not{D}$-eigenspinors to the eigenvalue $\pm -3/2\sqrt{t}$. Furthermore,

$$\|T\|^2 \;=\; 3t, \quad \mathrm{Scal}^g \;=\; 18 - 3t/2, \quad \mathrm{Ric}^g \;=\; \mathrm{diag}(3-t/2,\ldots,3-t/2,3t/2).$$

The universal eigenvalue estimate and the twistorial eigenvalue estimate take the numerical values

$$\beta_{\mathrm{univ}} \;=\; \frac{9}{4}(2-t), \quad \beta_{\mathrm{tw}} \;=\; \frac{21}{4} - \frac{49}{16}t.$$

Equality is reached for $t = 12/13$, hence the twistorial estimate is better for metrics with $t < 12/13$. Indeed, $\varphi_\pm$ are Killing spinors with torsion for $t = 42/49$ with the Killing number $\kappa = -\sqrt{42}/56$.

[1]The example thus shows that the Killing number of a Killing spinor with torsion on a compact manifold can be of either sign, in contrast to the Riemannian case.

## 6. The twistor equation in dimension 6

In dimension 6, the twistor equation can be further reduced to a Killing equation, thus leading to considerable simplifications. For convenience, recall that the twistorial eigenvalue estimate for $n = 6$ amounts to

$$\lambda \;\geq\; \frac{3}{10}\mathrm{Scal}^g + \frac{1}{12}\|T\|^2 - \frac{1}{3}\mu^2.$$

**Lemma 6.1.** *Assume $\nabla^c T = 0$ und let $\psi$ be a twistor spinor for $s = \frac{n-1}{4(n-3)}$. Then $\not{D}$ and $T$ satisfy the relation*

$$\left[\not{D}T + (1-\frac{6}{n})T\not{D}\right]\psi \;=\; \left[\frac{5-n}{n-3}T^2 - \frac{2}{n-3}||T||^2\right]\psi.$$

*Proof.* We start with the anticommutator relation from [FrI02], cited in Theorem 2.1, (2):

$$D^sT + TD^s = dT + \delta T - 8s\sigma_T - 2\mathcal{D}^s.$$

For $\nabla^c$-parallel torsion and $s = 1/4$, the three first terms on the right hand side vanish, hence

$$D^cT + TD^c = -2\mathcal{D}^c\,.$$

Since $D^c = \not{D} + \frac{1}{2}T$, this may be restated as

$$\not{D}T + T\not{D} + T^2 = -2\mathcal{D}^c\,.$$

The action of $\mathcal{D}^c$ on a twistor spinor $\psi$ may be computed from the twistor equation (Lemma 5.1) with $s = \frac{n-1}{4(n-3)}$ and $X = e_i$, multiplying by $e_i \lrcorner T$ and then summing over $i$,

$$\mathcal{D}^c\psi + \frac{3}{n}T\not{D}\psi + \frac{1}{n-3}T^2 - \frac{1}{n-3}||T||^2 \;=\; 0.$$

Now one obtains the desired result by inserting the expression for $\mathcal{D}^c\psi$ in the previous relation. □

This relation has particularly interesting consequences for $n = 6$.

**Corollary 6.1.** *Let $n = 6$ and $\nabla^c T = 0$. If $\psi$ is a twistor spinor for $s = \frac{n-1}{4(n-3)} = \frac{5}{12}$ in the $T$ eigenbundle $\Sigma_\mu$, exactly one of the two following cases holds:*

(1) *$\mu = 0$: either $T = 0$ or $\psi = 0$;*
(2) *$\mu \neq 0$: $\psi$ is a $\not{D}$ eigenspinor with eigenvalue*

$$\not{D}\psi \;=\; -\frac{1}{3}\left[\mu + 2\frac{\|T\|^2}{\mu}\right]\psi$$

*and the twistor equation for $\psi$ and $s = 5/12$ is equivalent to the Killing equation $\nabla^s\psi = \kappa X\cdot\psi$ for the same value of $s$ and Killing number $\kappa = \frac{1}{9}\left[\frac{\|T\|^2}{\mu} - \mu\right]$, thus leading to the Killing equation in its final form*

$$\nabla^c_X\psi - \frac{1}{18}\left[\mu + 2\frac{\|T\|^2}{\mu}\right]X\cdot\psi + \frac{1}{6}(X\wedge T)\psi \;=\; 0.$$

*In particular, the scalar curvature is constant and satisfies the relation*

$$\frac{3}{10}\mathrm{Scal}^g \;=\; \frac{4}{9}\mu^2 + \frac{13}{36}\|T\|^2 + \frac{4}{9}\frac{\|T\|^4}{\mu^2}\;.$$

*Proof.* Lemma 6.1 yields for $n = 6$ and a twistor spinor $\psi$

$$\not{D}T\psi \;=\; -\frac{1}{3}\left[T^2 + 2\|T\|^2\right]\psi.$$

By assumption, $T\psi = \mu\psi$, hence

$$\mu\not{D}\psi \;=\; -\frac{1}{3}\left[\mu^2 + 2\|T\|^2\right]\psi.$$

In case $\mu = 0$, we get $\|T\|^2\psi = 0$, hence the first claim. For $\mu \neq 0$, we may divide by $\mu$ and $\psi$ is an eigenspinor of $\not{D}$. We insert this eigenvalue relation into the twistor equation from Lemma 5.1,

$$\nabla^c_X\psi - \frac{1}{18}\left[\mu + 2\frac{\|T\|^2}{\mu}\right] X\cdot\psi + \frac{1}{6}(X\wedge T)\psi \;=\; 0.$$

On the other hand, suppose that $\varphi$ is a Killing spinor in $\Sigma_\mu$ for $s = \frac{5}{12}$. The Killing equation $\nabla^s_X\varphi = \kappa X\cdot\varphi$ implies that $\varphi$ is an eigenspinor of $D^s$ with eigenvalue $-n\kappa$. For us, this means

$$D^{\frac{5}{12}}\varphi \;=\; (\not{D} + T)\varphi \;=\; -6\kappa\varphi.$$

Since $\varphi$ is also an eigenspinor for $\not{D}$, this last equation yields for the Killing number $\kappa$ the value

$$\kappa \;=\; \frac{1}{9}\left[\frac{\|T\|^2}{\mu} - \mu\right].$$

With this value of $\kappa$, the Killing equation for $s = \frac{5}{12}$ becomes equivalent to

$$\nabla^c_X\varphi - \frac{1}{18}\left[\mu + 2\frac{\|T\|^2}{\mu}\right] X\cdot\varphi + \frac{1}{6}(X\wedge T)\varphi \;=\; 0.$$

Hence, every twistor spinor in $\Sigma_\mu$ is necessarily a Killing spinor.

The claim on the scalar curvature follows essentially from Lemma 5.1. For a twistor spinor $\psi$ in the $T$ eigenbundle $\Sigma_\mu$, it yields

$$\not{D}^2\psi = \frac{3}{10}\mathrm{Scal}^g\cdot\psi + \frac{1}{12}\|T\|^2\psi - \frac{1}{3}\mu^2\psi.$$

On the other side, the $\not{D}$ eigenvalue equation squared amounts to

$$\not{D}^2\psi = \frac{1}{9}(\mu^2 + 4\|T\|^2 + \frac{4}{\mu^2}\|T\|^4)\psi.$$

A direct comparison leads to the relation for $\mathrm{Scal}^g$. □

**Remark 6.1.** The value given for the scalar curvature in the previous Corollary allows to solve explicitly the general quadratic equation for the Killing number $\kappa$ stated in Lemma 5.2. One obtains the possible solutions

$$\kappa_1 \;=\; \frac{1}{9}\left[\frac{\|T\|^2}{\mu} - \mu\right] \quad\text{or}\quad \kappa_2 \;=\; -\frac{1}{9}\left[\frac{\|T\|^2}{\mu} + 2\mu\right]$$

The previous Lemma thus shows that $\kappa_2$ cannot occur.

**Example 6.1.** Consider a 6-dimensional nearly Kähler manifold $(M^6, g, J)$ with its characteristic connection $\nabla^c$ (see also the discussion at the end of Section 4). These are Einstein spaces of positive scalar curvature, $\|T\|^2 = \frac{2}{15}\mathrm{Scal}^g$, and $T$ has the eigenvalues $\mu = 0$ of multiplicity 6 and $\mu = \pm 2\|T\|$, each with multiplicity 1, and the torsion is always parallel [AlFS04]. It is well-known that it has two Riemannian Killing spinors $\varphi_\pm$ [FG85], that these coincide with the $\nabla^c$-parallel spinors [FrI02, Thm. 10.8], and that they lie in $\Sigma_{\pm 2\|T\|}$. As observed before, $\beta_{\mathrm{tw}}(\mu) = \beta_{\mathrm{univ}}(\mu) = \frac{2}{15}\mathrm{Scal}^g$: Thus, *any* $\nabla^c$-parallel spinor has to be a twistor spinor with torsion by Corollary 3.2. Corollary 6.1 then implies that it is already a Killing spinor with torsion. If $\mu = 0$, any twistor spinor with torsion would have to be a Riemannian twistor spinor, but it is kwown that these do not exist in $\Sigma_0$. Hence, we proved:

**Theorem 6.1.** *On a 6-dimensional nearly Kähler manifold $(M^6, g, J)$ with its characteristic connection $\nabla^c$, the following classes of spinors coincide:*

(1) *Riemannian Killing spinors,*
(2) *$\nabla^c$-parallel spinors,*
(3) *Killing spinors with torsion ($s = 5/12$),*
(4) *Twistor spinors with torsion ($s = 5/12$).*

*Furthermore, there is exactly one such spinor $\varphi_\pm$ in each of the subbundles $\Sigma_{\pm 2\|T\|}$, their Killing numbers (with torsion) are $\kappa = \mp\frac{\|T\|}{6} = \mp\frac{1}{3}\sqrt{\frac{\mathrm{Scal}^g}{30}}$ and their $\not{D}$ eigenvalues are $\mp\|T\| = \mp\sqrt{\frac{2\,\mathrm{Scal}^g}{15}}$.*

Further examples of 6-dimensional manifolds with Killing spinors with torsion well be discussed in a forthcoming paper.

## 7. Twistorial estimates for manifolds with reducible holonomy

Recall that we assume that $(M^n, g)$ is an oriented Riemannian manifold endowed with a metric connection $\nabla$ with skew-symmetric torsion $T \in \Lambda^3(M^n)$. The holonomy group $\mathrm{Hol}(M^n;\nabla)$ (sometimes just abbreviated $\mathrm{Hol}(\nabla)$ if no confusions are possible) is then a subgroup of $\mathrm{SO}(n)$, and we shall assume that it is a closed subgroup to avoid pathological cases. In order to distinguish it from the torsion, the tangent bundle and its subbundles will be denoted by $\mathcal{T}M^n$, $\mathcal{T}_1, \mathcal{T}_2 \ldots$.

**Definition 7.1** (parallel distribution)**.** Let $x \in M^n$ and $\mathcal{T}_x$ be a $\mathrm{Hol}(M^n;\nabla)$-invariant subspace of $\mathcal{T}M^n$. For any other point $y$, choose a curve $\gamma$ from $x$ to $y$ and denote by $\mathcal{T}_y$ the image of $\mathcal{T}_x$ in $\mathcal{T}_yM^n$ under parallel transport along $\gamma$. The subspace $\mathcal{T}_y$ does not depend on the choice of $\gamma$, for any other curve $\tilde\gamma$ defines a closed loop through $x$ by $\mu := \tilde\gamma^{-1}\gamma$ and $\mathcal{T}_x$ is by assumption invariant under parallel transport along $\mu$, meaning $\tilde\gamma^{-1}\gamma(T_x) = T_x$. This implies $\gamma(T_x) = \tilde\gamma(T_x)$ as stated. In particular, any $\mathrm{Hol}(M^n;\nabla)$-invariant subspace $\mathcal{T}_x \subset \mathcal{T}_xM^n$ defines a distribution $\mathcal{T} \subset \mathcal{T}M^n$. Any distribution occuring in this way will be called *parallel*.

The proof of the following basic lemma carries over from Riemannian geometry without modifications (see for example [KN63, Prop. 5.1]).

**Lemma 7.1.** *Let $\mathcal{T} \subset \mathcal{T}M^n$ be a parallel distribution and $Y \in \mathcal{T}$. For any $X \in \mathcal{T}M^n$, $\nabla_X Y$ is again in $\mathcal{T}$; in particular, $R(X_1, X_2)Y \in \mathcal{T}$ for any $X_1, X_2$.*

For a torsion free connection, this property implies of course that any parallel distribution is involutive; but for general metric connections this conclusion does not hold anymore.

Let $\mathcal{T}$ be a parallel distribution, $\mathcal{N}$ its orthogonal distribution defined by $\mathcal{N}_x := \mathcal{T}_x^\perp$ in every point $x \in M^n$. The fact that all elements of $\mathrm{Hol}(M^n;\nabla)$ are orthogonal transformations implies that $\mathcal{N}$ is again a parallel distribution. Thus, the tangent bundle splits into an orthogonal sum of parallel distributions ($n_i := \dim \mathcal{T}_i$)

$$\mathcal{T}M^n \;=\; \mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_k, \text{ and } \mathrm{Hol}(M^n;\nabla) \subset \mathrm{O}(n_1) \times \ldots \times \mathrm{O}(n_k) \subset \mathrm{SO}(n).$$

We assume that every distribution $\mathcal{T}_i$ is again orientable and that the holonomy preserves the orientation, i. e. *we assume*

$$\mathrm{Hol}(M^n;\nabla) \subset \mathrm{SO}(n_1) \times \ldots \times \mathrm{SO}(n_k).$$

In this case, every parallel distribution $\mathcal{T}_i$ defines a parallel $n_i$-form: if $X_1, \ldots, X_{n_i}$ is a generating frame for $\mathcal{T}_i$, then $\alpha_i := X_1 \wedge \ldots \wedge X_{n_i}$ is a differential form and spans a 1-dimensional $\mathrm{SO}(n_i)$-invariant subspace of $\Lambda^{n_i}(M^n)$. This is necessarily the trivial representation, meaning that $\alpha_i$ is invariant under parallel transport. We agree that any orthonormal frame $e_1, \ldots, e_m$ of $\mathcal{T}M^n$ shall respect the splitting in parallel distributions (i. e. no $e_k$ has parts in different distributions $\mathcal{T}_i$), and that $e_1^i, \ldots, e_{n_i}^i$ is to denote an orthonormal frame of $\mathcal{T}_i$, $i = 1, \ldots, k$.

We will now describe the 'block structure' of the curvature. All curvatures are meant to be those of the connection $\nabla$. Recall that the curvature tensor of a metric connection has the symmetry property

$$g(\mathcal{R}(X,Y)W_1, W_2) \;=\; -g(\mathcal{R}(X,Y)W_2, W_1).$$

Since the distributions $\mathcal{T}_i, \mathcal{T}_j$ are orthogonal, Lemma 7.1 implies for any vector fields $X, Y$ that

$$g(\mathcal{R}(X,Y)\mathcal{T}_i, \mathcal{T}_j) = 0 \text{ if } i \neq j. \tag{2}$$

Furthermore, the Ambrose-Singer theorem implies that the curvature operator $R(X,Y)$ vanishes if $X \in \mathcal{T}_i,\ Y \in \mathcal{T}_j,\ i \neq j$,

$$R(\mathcal{T}_i, \mathcal{T}_j) \ = \ 0 \ \text{ for } i \neq j, \tag{3}$$

since the holonomy group is generated by all curvature operators. We now consider the Ricci curvature.

**Proposition 7.1.** *The Ricci tensor has block structure,*

$$\mathrm{Ric} \ = \ \left[\begin{array}{c|c|c} \mathrm{Ric}_1 & 0 & \\ \hline 0 & \ddots & 0 \\ \hline & 0 & \mathrm{Ric}_k \end{array}\right],$$

*i. e.* $\mathrm{Ric}(X,Y) \neq 0$ *can only happen if* $X, Y \in \mathcal{T}_i$ *for some* $i$.

*The scalar curvature splits into 'partial scalar curvatures'* $\mathrm{Scal}_i := \mathrm{tr}\,\mathrm{Ric}_i$, *and* $\mathrm{Scal} = \sum_{i=1}^{k} \mathrm{Scal}_i$.

*Proof.* A summand of the Ricci tensor $\mathrm{Ric}(X,Y) = \sum_{m=1}^{n} R(e_m, X, Y, e_m)$ can only be non trivial if the vectors $(e_m, X)$ lie in the same $\mathcal{T}_i$ by equation (3), and if the vectors $(e_m, Y)$ lie in the same $\mathcal{T}_j$ by equation (2). But since the vector $e_m$ is the same in both cases (and thus cannot lie in two different distributions), we conclude that $e_m, X$, and $Y$ all have to lie in some $\mathcal{T}_i$. Thus we can define

$$\mathrm{Ric}_i(X,Y) \ := \ \sum_{m=1}^{n_i} R(e^i_m, X, Y, e^i_m).$$

Then, $\mathrm{Ric} = \sum_{i=1}^{k} \mathrm{Ric}_i$ and it has the stated block structure. The partial scalar curvatures are now just the traces of these partial Ricci tensors. □

**Remark 7.1.** Observe that the partial Ricci tensor vanishes in directions of parallel vector fields ($n_i = 1$).

**Remark 7.2.** Be cautious that despite of the block structure of the Ricci curvature, one has in general that $R(X,Y,U,V) \neq 0$ if $X, Y \in \mathcal{T}_i,\ U, V \in \mathcal{T}_j$ for $i \neq j$.

Let us now consider, as in the first part of this paper, the 1-parameter family of connections

$$\nabla^s_X Y \ = \ \nabla^g_X Y + 2s\, T(X, Y, -).$$

All quantities (curvature etc.) belonging to the connection $\nabla^s$ will carry an upper index $s$. We make the following crucial assumptions:

**Definition 7.2.** Assume that

(1) there exists a value $s_0$ such that $\nabla^{s_0} T = 0$; without loss of generality, we will assume that the torsion is normalized in such a way that $s_0 = 1/4$. Instead of $\nabla^{1/4}$, we will write $\nabla^c$,
(2) the tangent bundle $\mathcal{T}M^n = \bigoplus_{i=1}^{k} \mathcal{T}_i$ splits into $\nabla^s$-parallel distributions $\mathcal{T}_i$ for all parameters $s$ and $\mathrm{Hol}(M^n; \nabla^s) \subset \mathrm{SO}(n_1) \times \ldots \times \mathrm{SO}(n_k)$,
(3) the torsion splits into a sum $T = \sum_{i=1}^{k} T_i$, $T_i \in \Lambda^3(\mathcal{T}_i)$.

We observe that the conditions contain some redundancy: if the torsion splits as described, then it is sufficient to assume that the tangent bundle is a sum of parallel distributions *for one parameter* $s$.

By de Rham's Theorem, $(M^n, g)$ is then locally a product of Riemannian manifolds, i. e. the universal cover $\tilde{M}$ of $M$ splits into $\tilde{M} = M_1 \times \ldots \times M_k$ with $\dim M_i = n_i$. A result of Cleyton and Moroianu [CM11] implies that each $T_i$ satisfies $\nabla^c T_i = 0$, and in fact, each $T_i$ is the projection to $M$ of a 3-form in $\Lambda^3(M_i)$. In the sequel, we shall call a manifold satisfying these assumptions a *geometry with reducible parallel torsion.*

**Remark 7.3.** Since the forms $T_i$ live on disjoint distributions, $T_iT_j = -T_jT_i$ for $i \neq j$ in the Clifford algebra. This implies $T^2 = \sum_{i=1}^k T_i^2$ and hence every eigenvalue $\mu^2$ of $T^2$ is a sum of eigenvalues of the single $T_i^2$. By the orthogonality of the distributions $\mathcal{T}_i$, the identity $\|T\|^2 = \sum_{i=1}^k \|T_i\|^2$ holds.

We can deduce some further curvature properties from this assumption:

**Lemma 7.2.** *If $(M,g)$ carries a geometry with reducible parallel torsion, $\mathcal{R}^s(X,Y,Z,V)$ can only be non-zero if all vectors lie in the same subspace $\mathcal{T}_i$ for some $i$. Furthermore, the 4-form $\sigma_T$ splits in $\sigma_T = \displaystyle\sum_{i=1}^k \sigma_i$, where $\sigma_i := \sigma_{T^i}$.*

*Proof.* The splitting $T = \sum T_i$ implies $\sigma_T = \sum \sigma_i$, $\sigma_i := \sigma_{T^i}$, by definition of $\sigma_T$. From Theorem B.1, we know that the Bianchi identity in this case reads

$$\underset{\mathfrak{S}}{X,Y,Z} \mathcal{R}^s(X,Y,Z,V) \;=\; s\,[6-8s]\;\sigma_T(X,Y,Z,V).$$

Consider now vectors $X,Y,U,V$. In order for $\mathcal{R}^s(X,Y,U,V)$ to be possibly non zero, we need to assume that $X,Y \in \mathcal{T}_i,\ U,V \in T_j$ for some indices $i,j$. The elements $\mathcal{R}^s(Y,U,X,V)$ and $\mathcal{R}^s(U,X,Y,V)$, however, vanish from equations (3) and (2), thus we are left in this case with

$$\mathcal{R}^s(X,Y,U,V) \;=\; s(6-8s)\,\sigma_T(X,Y,Z,V).$$

But $\sigma_T(X,Y,Z,V)$ can only be different from zero if all vectors lie in the same $\mathcal{T}_i$. □

Assuming that $M^n$ is spin, the curvature $R_\Sigma$ of the spinor bundle $\Sigma M^n$ for the connection $\nabla^s$

$$R^s_\Sigma(X,Y)\psi \;:=\; \nabla^s_X\nabla^s_Y\psi - \nabla^s_Y\nabla^s_X\psi - \nabla^s_{[X,Y]}\psi$$

is related to the curvature $\mathcal{R}^s$ of the tangent bundle $\mathcal{T}M^n$ by

$$R^s_\Sigma(X,Y)\psi \;=\; \frac{1}{2}R^s(X\wedge Y)\cdot\psi,$$

where we interpret the curvature transformation $\mathcal{R}^s$ as an endomorphism on 2-forms: $\mathcal{R}^s(e_i\wedge e_j) := \sum_{k<l} R^s_{ijkl}e_k\wedge e_l$. This allows to draw conclusions on $R^s_\Sigma$ from the described splittings.

**Partial Schrödinger-Lichnerowicz formulas.** The splitting of the tangent bundle makes a certain amount of bookkeeping unavoidable. Let $p_i$ denote the orthogonal projection from $\mathcal{T}M^n$ onto $\mathcal{T}_i$ and define the 'partial connections'

$$\nabla^{s,i}_X \;:=\; \nabla^s_{p_i(X)}, \quad \text{hence } \nabla^s \;=\; \sum_{i=1}^k \nabla^{s,i}.$$

They induce the notions of 'partial Dirac operators' and 'partial spinor Laplacians' ($\mu$ is the usual Clifford multiplication) through

$$D^s_i \;:=\; \mu\circ\nabla^{s,i}, \quad D \;=\; \sum_{i=1}^k D^s_i, \quad \Delta^s_i \;:=\; (\nabla^{s,i})^*\nabla^{s,i}, \quad \Delta^s \;=\; \sum_{i=1}^k \Delta^s_i.$$

At a fixed point $p\in M^n$ we choose orthonormal bases $e^i_1,\dots,e^i_{n_i}$ of the distributions $\mathcal{T}_i$ ($i = 1,\dots,k$) such that $(\nabla^s_{e^i_m}e^j_l)_p = 0$ for all suitable indices $i,j,m,l$. Note that our chosen basis has the properties $[e^i_m,e^j_l] = -T(e^i_m,e^j_l)$ and $\nabla^g_{e^i_m}e^i_m = 0$. It is convenient (and consistent with the notation introduced above) to abbreviate $\nabla^s_{e^i_m}$ by $\nabla^{s,i}_m$. The partial Dirac and Laplace operators may then be expressed as

$$D^s_i \;:=\; \sum_{m=1}^{n_i} e^i_m\nabla^{s,i}_m, \quad \Delta^s_i \;:=\; -\sum_{m=1}^{n_i}\nabla^{s,i}_m\nabla^{s,i}_m.$$

The divergence term of the Laplacian vanishes because of $\nabla^g_{e^i_m} e^i_m = 0$. We compute the squares of the partial Dirac operators $D_i$ and their anticommutators. To formulate the statement, set

$$\mathcal{D}^s_i \;:=\; \sum_{m=1}^{n_i} (e^i_m \lrcorner\, T_i) \cdot \nabla^{s,i}_{e^i_m} \psi$$

in full analogy to the classical case without splitting of $\mathcal{T}M$ $(k=1)$.

**Proposition 7.2.** *Assume that $M$ carries a geometry with reducible parallel torsion. The partial Dirac operators $D^s_i$ then satisfy the identities*

(i) $(D^s_i)^2 \;=\; \Delta^s_i + s(6-8s)\,\sigma_i - 4s\mathcal{D}^s_i + \frac{1}{4}\mathrm{Scal}^s_i,$

(ii) $D^s_i D^s_j + D^s_j D^s_i = 0$ *for* $i \neq j$,

(iii) $(D^{s/3}_i)^2 \;=\; \Delta^s_i + 2s\,\sigma_i + \frac{1}{4}\mathrm{Scal}^g_i - 2s^2\|T_i\|^2.$

*Proof.* For the first identity, let $k$ and $l$ be indices running between 1 and $\dim \mathcal{T}_i = n_i$. We split the sum into terms with $k=l$ and $k \neq l$,

$$\begin{aligned}(D^s_i)^2\psi \;&=\; \sum_{k,l=1}^{n_i} e^i_k \nabla^{s,i}_k e^i_l \nabla^{s,i}_l \psi \;=\; -\sum_{k=1}^{n_i} \nabla^{s,i}_k \nabla^{s,i}_k \psi + \sum_{k\neq l} e^i_k e^i_l \nabla^{s,i}_k \nabla^i_l \psi \\ &=\; \Delta^s_i + \sum_{k<l} e^i_k e^i_l (\nabla^{s,i}_k \nabla^{s,i}_l - \nabla^{s,i}_l \nabla^{s,i}_k)\psi\end{aligned}$$

und express the second term through the curvature in the spinor bundle,

$$(D^s_i)^2\psi \;=\; \Delta^s_i\psi + \sum_{k<l} e^i_k e^i_l \left[\mathcal{R}^s_\Sigma(e^i_k, e^i_l) - \nabla^{s,i}_{T(e^i_k, e^i_l)}\right]\psi.$$

$\mathcal{R}^s_\Sigma$ in turn can be expressed through the curvature $R$, and by Lemma 7.2, only terms with all four vectors inside $\mathcal{T}_i$ can occur:

$$\sum_{k<l} e^i_k e^i_l \mathcal{R}^s_\Sigma(e^i_k, e^i_l) \;=\; \frac{1}{2}\sum_{k<l} e^i_k e^i_l \mathcal{R}^s(e^i_k \wedge e^i_l)\cdot\psi \;=\; \frac{1}{2}\sum_{k<l,p<q} \mathcal{R}^s(e^i_k, e^i_l, e^i_p, e^i_q) e^i_k e^i_l e^i_p e^i_q \psi.$$

The summands with same index pairs add up to half the partial scalar curvature, while totally different indices yield the Clifford multiplication by the 4-form $\sigma^i$. Index pairs with one common index add up to zero, because the Ricci tensor is symmetric (Theorem B.1). The third identity follows by a routine calculation.

For the second identity and $i \neq j$, we proceed similarly. However, there is no diagonal term resulting in an analogue of the Laplacian, hence

$$D^s_i D^s_j + D^s_j D^s_i \;=\; \sum_{k,l} e^i_k e^j_l \left[\mathcal{R}^s_\Sigma(e^i_k, e^j_l) - \nabla^s_{T(e^i_k, e^j_l)}\right].$$

But the mixed curvature operator vanishes as observed in equation (3), and $T(\mathcal{T}_i, \mathcal{T}_j)$ is zero as well by the assumption that $T$ does not contain any mixed terms. □

The second identity has a crucial consequence: all the operators $(D^s)^2, (D^s_1)^2, \ldots, (D^s_k)^2$ can be simultaneously diagonalized.

**Lemma 7.3.** *For all parameters $s$, $(D^s)^2 = \sum_{i=1}^{k} (D^s_i)^2$ and $(D^s_i)^2\,(D^s_j)^2 = (D^s_j)^2\,(D^s_i)^2\ \forall\, i \neq j$.*

In particular, any eigenvalue $\lambda$ of $(D^s)^2$ is the sum of eigenvalues $\lambda_i$ of $(D^s_i)^2$, $\lambda = \sum_{i=1}^{k} \lambda_i$.

**Adapted Twistor Operator.** We define a twistor operator $P^s : \Gamma(\Sigma M) \to \Gamma(\mathcal{T}M^* \otimes \Sigma M)$ adapted to the splitting $\mathcal{T}M^n = \bigoplus_{i=1}^k \mathcal{T}_i$ of the tangent bundle by

$$P^s\psi \;=\; \nabla^s\psi + \sum_{i=1}^k \frac{1}{n_i}\sum_{l=1}^{n_i} e_l^i \otimes e_l^i \cdot D_i^s\psi. \tag{4}$$

By a simple computation, one checks that

$$\|P^s\psi\|^2 \;=\; \langle(\Delta^s - \sum_{i=1}^k \frac{1}{n_i}(D_i^s)^2)\psi,\psi\rangle. \tag{5}$$

**Theorem 7.1** (Twistorial eigenvalue estimate for products)**.** *Assume that $M$ carries a geometry with reducible parallel torsion, and that the dimensions of the subbundles $\mathcal{T}_i$ are ordered by ascending dimensions, $n_1 \le n_2 \le \ldots \le n_k$. The smallest eigenvalue $\lambda$ of $\not{D}^2$ satisfies the inequality*

$$\lambda \;\ge\; \frac{n_k}{4(n_k-1)}\mathrm{Scal}^g_{\min} + \frac{n_k(n_k-5)}{8(n_k-3)^2}\|T\|^2 + \frac{n_k(4-n_k)}{4(n_k-3)^2}\max(\mu_1^2,\ldots,\mu_k^2). \tag{$*$}$$

*Let $\tilde{M} = M_1 \times \ldots \times M_k$ be the universal cover of $M$, $\dim M_i = n_i$, and $\tilde{s} := \frac{n_k-1}{4(n_k-3)}$. Equality holds in $(*)$ if and only if the following conditions are satisfied:*

1. *The Riemannian scalar curvature of $(M,g)$ is constant,*
2. *There exists a twistor spinor with torsion for $\tilde{s}$ on $M_k$,*
3. *For $i = 1,\ldots,k-1$:*
   - (a) *If $n_i < n_k$: there exists a $\nabla^{\tilde{s}}$-parallel spinor on $M_i$,*
   - (b) *If $n_i = n_k$: there exists a $\nabla^{\tilde{s}}$-parallel or a twistor spinor with torsion for $\tilde{s}$ on $M_i$,*
4. *these spinors lie in the subbundle $\Sigma_\mu(M_i)$ corresponding to the largest eigenvalue of $T_i^2$ $(i = 1\ldots,k)$.*

*Proof.* From Theorem 2.2, we know that

$$(D^{s/3})^2 \;=\; \Delta^s - sT^2 + \frac{1}{4}\mathrm{Scal}^g + (s-2s^2)\|T\|^2.$$

Now from Theorem 7 and equation (5)

$$\begin{aligned}\int\langle(D^{s/3})^2\psi,\psi\rangle\rangle \;&=\; \int\langle\Delta^s\psi,\psi\rangle - s\int\langle T^2\psi,\psi\rangle + \frac{1}{4}\int\langle\mathrm{Scal}^g\psi,\psi\rangle + (s-2s^2)\int\|T\|^2\|\psi\|^2\\ &=\; \int\|P^s\psi\|^2 + \sum_{i=1}^k\int\frac{1}{n_i}\|D_i^s\psi\|^2\\ &\qquad -s\int\langle T^2\psi,\psi\rangle + \frac{1}{4}\int\langle\mathrm{Scal}^g\psi,\psi\rangle + (s-2s^2)\int\|T\|^2\|\psi\|^2.\end{aligned}$$

Adding $-\int\langle\frac{1}{n_k}(D^s)^2\psi,\psi\rangle$ on the both sides

$$\begin{aligned}\int\langle\left[(D^{s/3})^2 - \frac{1}{n_k}(D^s)^2\right]\psi,\psi\rangle \;&=\; \int\|P^s\psi\|^2 + \int\langle\left[\sum_{i=1}^k\int\frac{1}{n_i}(D_i^s)^2 - \frac{1}{n_k}(D^s)^2\right]\psi,\psi\rangle\\ &\qquad -s\int\langle T^s\psi,\psi\rangle + \frac{1}{4}\int\langle\mathrm{Scal}^g\psi,\psi\rangle + (s-2s^2)\int\|T\|^2\|\psi\|^2.\end{aligned}$$

Using the same techniques as in Lemma 3.1 and Theorem 3.2 we have

$$(D^{s/3})^2 - \frac{1}{n_k}(D^s)^2 = \frac{n_k-1}{n_k}(\not{D})^2 + \frac{4s^2}{1-n_k}T^2 \quad \text{with } s = \frac{n_k-1}{4(n_k-3)}.$$

We also note from Lemma 7.3 that $\int (D^s)^2 = \sum_{i=1}^{k} \int (D_i^s)^2$; hence, if we set $\tilde{s} := \frac{n_k-1}{4(n_k-3)}$, the above formula implies

$$\frac{n_k-1}{n_k}\int \langle (\not{D})^2\psi,\psi\rangle = \int \|P^s\psi\|^2 + \int \langle \left[\sum_{i=1}^{k-1}\int \left(\frac{1}{n_i}-\frac{1}{n_k}\right)(D_i^s)^2\right]\psi,\psi\rangle$$
$$+\frac{(1-n_k)(n_k-4)}{4(n_k-3)^2}\int \langle T^2\psi,\psi\rangle + \frac{1}{4}\int \langle \mathrm{Scal}^g\psi,\psi\rangle + \frac{(n_k-1)(n_k-5)}{8(n_k-3)^2}\int \|T\|^2\|\psi\|^2.$$

Since the first two terms on the right hand side of the first line are clearly $\geq 0$, the estimate follows. The equality case is obtained if the scalar curvature is constant, the eigenvalue $\mu^2$ of $T^2$ is maximal and

$$P^{\tilde{s}}\psi \;=\; 0, \quad \text{and } (D_i^{\tilde{s}})^2\psi \;=\; 0 \text{ if } n_i < n_k.$$

By the splitting of the tangent bundle, $P^{\tilde{s}}\psi = 0$ is equivalent to

$$\nabla^{\tilde{s},i}\psi + \frac{1}{n_i}\sum_{l=1}^{n_i} e_l^i \otimes e_l^i \cdot D_i^{\tilde{s}}\psi \;=\; 0 \quad \forall i = 1,\dots,k.$$

If $D_i^{\tilde{s}}\psi = 0$ (for example, if $n_i < n_k$), this implies $\nabla^{\tilde{s},i}\psi = 0$. If not, $n_i = n_k$ has to hold and

$$\nabla_X^{\tilde{s},i}\psi + \frac{1}{n_i}p_i(X)\cdot D_i^{\tilde{s}}\psi \;=\; 0,$$

where we recall that $p_i$ denoted the projection $\mathcal{T}M \to \mathcal{T}_i$. In order to obtain the corresponding spinor fields on the factors $M_i$, one argues as in [Al07]: Let $f_i : M_i \to \tilde{M}$ be the inclusion, $q : \tilde{M} \to M$ the projection. The pullback $(q\circ f_i)^*\Sigma M$ is a Clifford bundle over $M_i$, hence it is a sum of finitely many copies of $\Sigma M_i$, and the connection $\nabla^s$ with torsion $4sT$ pulls back to the connection with torsion $4sT_i$. The pullbacks of the spinor fields then satisfy the corresponding field equations. For the last claim, observe that $T^2 = \sum_{i=1}^k T_i^2$ as observed in Remark 7.3, hence the maximum is reached if and only if the eivenvalue of each single $T_i^2$ is maximal. □

**Remark 7.4.** The coefficient of the scalar curvature is strictly decreasing with growing $n$, while the other two coefficients (of $\|T\|^2$ and $\max(\mu_1^2,\dots,\mu_k^2)$) are first increasing (for $n \geq 5$ resp. $n \geq 6$), then decreasing for larger $n$ ($n \gtrsim 15$) and reach quickly their limits $1/8$ and $-1/4$. Thus, this estimate becomes better for products where the scalar curvature dominates the other two terms. We believe that the estimate will be particularly useful if $M$ is locally a product of manifolds of the same dimension (though not necessarily carrying the same geometry), because it has a 'nicer' equality case.

**Remark 7.5.** Theorem 7.1 generalizes the main result by E. C. Kim and B. Alexandrov ([Ki04], [Al07]) for the Riemannian Dirac operator on locally reducible Riemannian manifolds. They proved that the first eigenvalue $\lambda^g$ of the square of the Riemannian Dirac operator $(D^g)^2$ satisfies

$$\lambda^g \;\geq\; \frac{n_k}{4(n_k-1)}\mathrm{Scal}^g_{\min}, \tag{6}$$

again for ascending dimensions of the single factors, and obtained the corresponding equality case.

**Example 7.1.** Consider a 10-dimensional manifold that is a product of two 5-dimensional manifolds with parallel characteristic torsion; this defines a geometry with reducible parallel torsion. Then the 10-dimensional twistorial estimate of the eigenvalue of $\not{D}^2$ reads (Corollary 3.2)

$$\lambda \;\geq\; \frac{5}{18}\mathrm{Scal}^g_{\min} + \frac{25}{196}\|T\|^2 - \frac{15}{49}\max(\mu^2),$$

where $\max(\mu^2)$ denotes the maximal eigenvalue of $T^2$. On the other side, the twistorial eigenvalue estimate for products (Theorem 7.1) yields

$$\lambda \ \geq \ \frac{5}{4}\mathrm{Scal}^g_{\min} - \frac{5}{16}\max(\mu^2).$$

One recognizes that one gets a truly different estimate that will be of interest for large scalar curvatures.

**More comments on eigenvalues for product manifolds.** We shall now explain how the proof of the main Theorem in [Al07] can easily be modified to obtain a different kind of Riemannian estimate, which we find of interest on its own. It is plain that their main result (equation (6)) and our Theorem 7.1 contain a built-in asymmetry: The top dimension $n_k$ plays a particular role in the estimate and the equality case predicts a twistor spinor on the $n_k$-dimensional factor and a parallel spinor on all other factors. We asked ourselves: Is there an estimate (in the Riemannian case resp. torsion case) that is symmetric in all dimensions $n_i$ and that reaches equality if and only if there exists a twistor spinor on each factor?

Let's recall the global setting for the Riemannian case: Consider a compact spin manifold $M$ whose tangent bundle $\mathcal{T}M$ splits into parallel, pairwise orthogonal distributions $\mathcal{T}_i, i = 1, \ldots, k$, and where we assume that the dimensions are ordered by $1 \leq n_1 \leq \ldots \leq n_k$. Then the partial Levi-Civita derivatives $\nabla^{g,i}$, the partial Riemannian Dirac operators $D^g_i$, partial Riemannian curvatures etc. are defined as we did before, but without any torsion appearing. In particular, it is true that a $D^g$ eigenspinor can be chosen in such a way that it is also a $D^g_i$ eigenspinor for all $i$, and the eigenvalues of the squares of these operators satisfy $\lambda^g = \lambda^g_1 + \ldots + \lambda^g_k$. Since the manifold is locally a product, the partial eigenvalues $\lambda^g_i$ have really a geometric meaning of their own.

**Theorem 7.2.** *The eigenvalues $\lambda^g$ of $(D^g)^2$ and $\lambda^g_i$ of $(D^g_i)^2$ satisfy the inequality*

$$\lambda^g - \sum_{i=1}^{k} \frac{\lambda^g_i}{n_i} \ \geq \ \frac{\mathrm{Scal}^g_{\min}}{4},$$

*and equality is obtained if and only if there exists a Riemannian Killing spinor on each factor of $M$.*

*Proof.* For the adapted Riemannian twistor operator $P^0 =: P$ (compare equation (4)) and an arbitrary spinor field $\psi$, B. Alexandrov proves in [Al07] the integral formula

$$\|P\psi\|^2 \ = \ \left[1 - \frac{1}{n_k}\right] \langle (D^g)^2\psi, \psi\rangle - \sum_{i=1}^{k-1} \left[\frac{1}{n_i} - \frac{1}{n_k}\right] \langle (D^g_i)^2\psi, \psi\rangle - \langle \frac{\mathrm{Scal}^g}{4}\psi, \psi\rangle.$$

In order to obtain a result (our equation (6)) in which the partial eigenvalues $\lambda_i$ do not appear anymore, he observes that $1/n_i - 1/n_k \geq 0$ by assumption and neglects these terms together with the twistor term $\|P\psi\|^2$. For proving our Theorem, the main point is to keep these terms! Choose an eigenspinor $\psi$ such that $(D^g)^2\psi = \lambda^g\psi,\ (D^g_i)^2\psi = \lambda^g_i\psi$ (this is always possible). Since $\|P\psi\|^2 \geq 0$, we obtain

$$0 \ \geq \ \left[1 - \frac{1}{n_k}\right] \lambda^g\|\psi\|^2 - \sum_{i=1}^{k-1} \left[\frac{1}{n_i} - \frac{1}{n_k}\right] \lambda^g_i\|\psi\|^2 - \langle \frac{\mathrm{Scal}^g}{4}\psi, \psi\rangle.$$

Estimating the scalar curvature as usual by its minimum and and dividing by the length $\|\psi\|^2 > 0$ yields then the result after a short computation, which we omit. The discussion of the the equality case follows the same line of arguments as in [Al07]. □

Let us discuss the value of this result. On every single factor, Friedrich's classical estimate

$$\lambda^g_i \ \geq \ \frac{n_i}{4(n_i - 1)}(\mathrm{Scal}^g_i)_{\min}$$

holds, so by summation (and a quick calculation) we get

$$\lambda^g - \sum_{i=1}^{k} \frac{\lambda_i^g}{n_i} \;\geq\; \frac{1}{4}\sum_{i=1}^{k} (\mathrm{Scal}_i^g)_{\min}.$$

Since in general $\sum(\mathrm{Scal}_i^g)_{\min} \leq (\sum \mathrm{Scal}_i^g)_{\min}$, our result is non trivial. This is particularly plain when only $\mathrm{Scal}^g_{\min}$ is strictly positive, but not all $(\mathrm{Scal}_i^g)_{\min}$. From an aesthetic point of view, Theorem 7.2 removes the asymmetry of equation (6) and Theorem 7.1, as desired.

Unfortunately, Theorem 7.2 has no reasonable analogue for Dirac operators with torsion. In oder to obtain a similar result, one is lead to allow the parameter $s$ of the adapted twistor operator to change in each summannd $\mathcal{T}_i$. The resulting inequality then links the eigenvalues of $\not{D}^2$ and $(D_i^{s_i})$, where now $s_i = (n_i - 1)/4(n_i - 3)$. However, these operators no not have an intrinsic geometric meaning, nor do they commute with $\not{D}^2$. Hence, the result is not of interest.

## Appendix A. Proof and application of the integrability condition

We begin with a remarkable identity that relates the curvature operator and the Ricci operator in the spin bundle. Recall that the curvature operator of any spin connection can be understood as a endomorphism-valued 2-form,

$$\mathcal{R}(X,Y)\psi \;=\; \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi.$$

One checks that it is related to the curvature operator on 2-forms defined through

$$\mathcal{R}(e_i \wedge e_j) \;:=\; \sum_{k<l} R_{ijkl}\, e_k \wedge e_l$$

by the relation

$$\mathcal{R}(X,Y)\psi \;=\; \frac{1}{2}\,\mathcal{R}(X\wedge Y)\cdot\psi.$$

Furthermore, we understand the Ricci tensor as an endomorphism on the tangent bundle. Then the identity stated in the following theorem is crucial for deriving integrability conditions. It generalizes a well-known result of Friedrich [Fr80] for the Levi-Civita connection ($T = 0$). A special case of the result–that is, applied to a *parallel* spinor–may be found in [FrI02]. The result appeared for the first time in the diploma thesis of Mario Kassuba [Ka06], which was written under the supervision of the first author and Thomas Friedrich at Humboldt University Berlin in 2006.

**Theorem A.1.** *Let $\nabla^c$ be a metric spin connection with parallel torsion $T$, $\nabla^c T = 0$. Then, the following identity holds for any spinor field $\psi$ and any vector field $X$*

$$\mathrm{Ric}^c(X)\cdot\psi \;=\; -2\sum_{k=1}^{n} e_k\mathcal{R}^c(X,e_k)\psi + \frac{1}{2}X \lrcorner\, dT\cdot\psi.$$

*Proof.* Rewrite the first term on the right hand side (without the numerical factor) as

$$\sum_{k=1}^{n} e_k\mathcal{R}^c(e_l,e_k) \;=\; \frac{1}{2}\sum_{k=1}^{n} e_k\cdot\mathcal{R}^c(e_l\wedge e_k) \;=\; \frac{1}{2}\sum_{k=1}^{n}\sum_{i<j} R^c_{lkij}e_ke_ie_j \;=:\; R_1 + R_2,$$

where $R_1$ denotes all terms with three different indices $k,i,j$, and $R_2$ all terms with at least one repeated index. We first discuss $R_1$:

$$R_1 \;=\; \frac{1}{2}\sum_{i<j}\left[\sum_{k<i} R^c_{lkij}e_ke_ie_j + \sum_{i<k<j} R^c_{lkij}e_ke_ie_j + \sum_{j<k} R^c_{lkij}e_ke_ie_j\right] \;=\; \frac{1}{2}\sum_{k<i<j} \overset{k,i,j}{\mathfrak{S}}\; R^c_{lkij}e_ke_ie_j,$$

where the symbol $\mathfrak{S}$ denotes the cyclic sum. The first Bianchi identity for a metric connection with parallel skew torsion [FrI02], [Ag06]

$$\overset{X,Y,Z}{\mathfrak{S}}\; \mathcal{R}(X,Y,Z,V) \;=\; \frac{1}{2}dT(X,Y,Z,V)$$

implies then $R_1 = -e_l \lrcorner dT/4$. We now consider $R_2$. Here, the argument does not depend on the detailed type of the connection, only the property of being metric is used (it implies that $R^c_{ikjl}$ is antisymmetric in the third and fourth argument, see [Ag06, Section 2.8]). One checks that

$$R_2 \;=\; -\frac{1}{2}\sum_{r=1}^{n}\left[\sum_{p=1}^{r-1} R^c_{lppr}e_e + \sum_{q=r+1}^{n} R^c_{lqqr}e_r\right].$$

But since the Ricci tensor is exactly the contraction of the curvature, $R_2 = -\mathrm{Ric}^c(e_l)/2$. This ends the proof. □

We use this result to formulate the necessary curvature integrability conditions for Killing spinors with torsion. For Riemannian Killing spinors ($T = 0$), this results just means that the underlying manifold has to be Einstein.

**Theorem A.2.** *Suppose $\nabla^c T = 0$. Let $\psi$ be a Killing spinor with torsion with Killing number $\kappa$, set $\lambda := \frac{1}{2(n-3)}$ for convenience, and recall that $s = \frac{n-1}{4(n-3)}$. Then the Ricci curvature of the characteristic connection satisfies for all vector fields $X$ the identity*

$$\begin{aligned}\mathrm{Ric}^c(X)\psi \;&=\; -16s\kappa(X\lrcorner T)\psi + 4(n-1)\kappa^2 X\psi + (1-12\lambda^2)(X\lrcorner\sigma_T)\psi + \\ &\quad +2(2\lambda^2+\lambda)\sum e_k(T(X,e_k)\lrcorner T)\psi\,.\end{aligned}$$

*Proof.* We will first establish a relation between the actions of $\mathcal{R}^c$ and $\mathcal{R}^s$ on the Killing spinor $\psi$. As an abbreviation, we set $\lambda := \frac{1}{2(n-3)}$. For the curvature endomorphism, we obtain by using the identity $\nabla^s_X\psi = \nabla^c_X\psi + \lambda(X\lrcorner T)\psi$ and the product rule for the covariant derivative of Clifford products:

$$\begin{aligned}\mathcal{R}^s(X,Y)\psi \;&=\; \nabla^s_X\nabla^s_Y\psi - \nabla^s_Y\nabla^s_X\psi - \nabla^s_{[X,Y]}\psi \\ &=\; \mathcal{R}^c(X,Y)\psi + \lambda\nabla^c_X((Y\lrcorner T)\psi) + \lambda(X\lrcorner T)\nabla^c_Y\psi + \lambda^2(X\lrcorner T)(Y\lrcorner T)\psi \\ &\quad -\lambda\nabla^c_Y((X\lrcorner T)\psi) - \lambda(Y\lrcorner T)\nabla^c_X\psi - \lambda^2(Y\lrcorner T)(X\lrcorner T)\psi - \lambda([X,Y]\lrcorner T)\psi \\ &=\; \mathcal{R}^c(X,Y)\psi + \lambda(\nabla^c_X((Y\lrcorner T))\psi + \lambda(Y\lrcorner T)\nabla^c_X\psi + \lambda(X\lrcorner T)\nabla^s_Y\psi \\ &\quad -\lambda(\nabla^c_Y(X\lrcorner T))\psi - \lambda(X\lrcorner T)\nabla^c_Y\psi - \lambda(Y\lrcorner T)\nabla^s_X\psi - \lambda([X,Y]\lrcorner T)\psi \\ &=\; \mathcal{R}^c(X,Y)\psi + \lambda(\nabla^c_X Y\lrcorner T)\psi + \lambda^2(X\lrcorner T)(Y\lrcorner T)\psi \\ &\quad -\lambda(\nabla^c_Y(X\lrcorner T))\psi - \lambda^2(Y\lrcorner T)(X\lrcorner T)\psi - \lambda([X,Y]\lrcorner T)\psi\,.\end{aligned}$$

From the general formula $\nabla_X(Y\lrcorner\omega) = (\nabla_X Y)\lrcorner\omega + Y\lrcorner(\nabla_X\omega)$ and the assumption $\nabla^c T = 0$, we conclude:

$$\begin{aligned}\mathcal{R}^s(X,Y)\psi \;&=\; \mathcal{R}^c(X,Y)\psi + \lambda^2(X\lrcorner T)(Y\lrcorner T)\psi - \lambda^2(Y\lrcorner T)(X\lrcorner T)\psi \\ &\quad +\lambda(T(X,Y)\lrcorner T)\psi\,.\end{aligned}$$

Together with the identities (2) and (4) from the compilation of important formulas (Lemma C.1), this allows us to compute the summand $\sum e_k\mathcal{R}^s(X,e_k)\psi$,

$$\begin{aligned}\sum e_k\mathcal{R}^s(X,e_k)\psi \;&=\; \sum e_k\mathcal{R}^c(X,e_k)\psi + 3\lambda^2((X\lrcorner T)T - T(X\lrcorner T))\psi \\ &\quad -2\lambda^2\sum T(X,e_k)(e_k\lrcorner T)\psi + \lambda\sum e_k(T(X,e_k)\lrcorner T)\psi\,.\end{aligned}$$

The second term can be simplified through formulas (2) and (6) of Lemma C.1,

$$(X\lrcorner T)T - T(X\lrcorner T) = (XT^2 + T^2X) = -\frac{1}{2}X\sigma_T - \sigma_T X = -2X\lrcorner\sigma_T\,.$$

The third term can be simplified as follows,

$$\begin{aligned}\sum_k T(X,e_k)(e_k \lrcorner T) &= \sum_{k,m} T(X,e_k,e_m)e_m(e_k \lrcorner T)\\ &= \sum_m e_m \sum_k T(X,e_k,e_m)(e_k \lrcorner T)\\ &= -\sum_m e_m \sum_k T(X,e_m,e_k(e_k \lrcorner T)\\ &= -\sum_m e_m(T(X,e_m) \lrcorner T)\,.\end{aligned}$$

Thus, we obtain alltogether:

$$\sum e_k \mathcal{R}^s(X,e_k)\psi = \sum e_k \mathcal{R}^c(X,e_k)\psi - 6\lambda^2(X \lrcorner \sigma_T)\psi + (2\lambda^2+\lambda)\sum e_k(T(X,e_k) \lrcorner T)\psi\,.$$

Now we specialize to the case that $\psi$ is a Killing spinor with $\nabla^s_X\psi = \kappa X\psi$. In this case, $\mathcal{R}^s(X,Y)$ acts on $\psi$ by

$$\begin{aligned}\mathcal{R}^s(X,Y)\psi &= \nabla^s_X\nabla^s_Y\psi - \nabla^s_Y\nabla^s_X\psi - \nabla^s_{[X,Y]}\psi\\ &= \kappa T^s(X,Y)\psi + \kappa^2(YX-XY)\psi\,,\end{aligned}$$

where $T^s = 4sT$. We form again the desired sum. The relation

$$\sum e_k T(X,e_k)\psi = -\sum T(X,e_k)e_k\psi = 2(X \lrcorner T)\psi$$

yields

$$\sum e_k \mathcal{R}^s(X,e_k)\psi = 8s\kappa(X \lrcorner T)\psi + 2(1-n)\kappa^2 X\psi\,.$$

The claim now follows from Theorem A.1 if one observes that $dT = 2\sigma_T$ holds for parallel torsion. □

By contracting the identity for the Ricci curvature once more, one obtains a formula for the scalar curvature of a metric admitting Killing spinors with torsion. However, one checks that this result coincides with the equation for the scalar curvature stated in Lemma 5.2.

We now give a typical example how the previous result can be used to prove non-existence results for Killing spinors with torsion.

**Corollary A.1.** *A 5-dimensional Einstein-Sasaki manifold $(M,g,\xi,\eta,\varphi)$ endowed with its characteristic connection cannot admit Killing spinors with torsion.*

*Proof.* It is known that a 5-dimensional Einstein-Sasaki manifold admits a local frame such that

$$\xi \cong \eta = e_5, \quad d\eta = 2(e_1\wedge e_2 + e_3\wedge e_4), \quad T^c = \eta\wedge d\eta = 2(e_1\wedge e_2+e_3\wedge e_4)\wedge e_5.$$

Furthermore, in this frame, $\mathrm{Scal}^g = 20$, $\|T\| = 8$ and the eigenvalues of $T$ are $0, \pm 4$. Hence, Lemma 5.2 allows us to compute all possible values of the Killing number $\kappa$, leading to the table

| $\mu$ | $0$ | $4$ | $-4$ |
|---|---|---|---|
| $\kappa$ | $\pm\frac{1}{2}$ | $-1\pm\frac{\sqrt{5}}{10}$ | $1\pm\frac{\sqrt{5}}{10}$ |

In this situation, $s = 1/2$ and $\lambda = 1/4$. For the Ricci curvature, observe that these are related by

$$\mathrm{Ric}^g(X,Y) = \mathrm{Ric}^c(X,Y) + \frac{1}{4}\sum_{i=1}^5 g(T^c(X,e_i),T^c(Y,e_i)).$$

Thus, the Einstein condition $\mathrm{Ric}^g(e_i) = 4\cdot\mathrm{Id}$ implies $\mathrm{Ric}^c(e_i) = 2\cdot\mathrm{Id}$ for $i = 1, \dots, 4$, $\mathrm{Ric}^c(e_5) = 0$. Now pick your favorite spin representation and evaluate with arbitrary $\kappa$ and for $X = e_1$ the expression (just a $(4 \times 4)$-matrix)

$$2 \cdot \mathrm{Id} - \left[ -8\kappa(e_1 \lrcorner T) + 16\kappa^2 e_1 + \frac{1}{4}(e_1 \lrcorner \sigma_T) + \frac{3}{4}\sum_{k=1}^{5} e_k(T(e_1, e_k) \lrcorner T) \right]$$

and check that for all possible $\kappa$ values above, the determinant is nonzero. Hence, this endomorphism on the spin bundle has no kernel, that is, there cannot exist a spinor field $\phi$ (not even in a point) satisfying the integrability condition from Theorem A.2 for $X = e_1$. □

## Appendix B. Curvature properties for families of connections

The curvature of a metric connection $\nabla$ with parallel torsion $T \in \Lambda^3(TM^n)$ is known to have some special properties. In this section, we show how some of these properties can be transferred to a 1-parameter family of connections in which only one connection has this property, matching of course exactly the situation encountered in this paper.

**Theorem B.1.** *Assume that $(M, g)$ carries a 1-parameter family of metric connections $\nabla^s$ with skew torsion $T \in \Lambda^3(M)$*

$$\nabla^s_X Y \;=\; \nabla^g_X Y + 2s\, T(X, Y, -),$$

*and that $\nabla^c T = 0$, where $\nabla^c$ is the connection corresponding to $s = 1/4$. Then, for all $s \in \mathbb{R}$, the covariant derivative of the torsion is given by*

$$\nabla^s_X T(U, V, W) \;=\; \left[2s - \frac{1}{2}\right] \sigma_T(U, V, W, X),$$

*and the first Bianchi identity reduces to*

$$\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}^s(X, Y, Z, V) \;=\; s\,[6 - 8s]\; \sigma_T(X, Y, Z, V).$$

*This implies, in particular, that the curvature is symmetric under intertwining of blocks,*

$$\mathcal{R}^s(X, Y, U, V) \;=\; \mathcal{R}^s(U, V, X, Y).$$

*Furthermore, the Ricci tensor $\mathrm{Ric}^s$ is symmetric.*

*Proof.* The connections are related by

$$\nabla^s_X Y \;=\; \nabla^c_X Y + \left[2s - \frac{1}{2}\right] T(X, Y, -),$$

thus the covariant derivatives satisfy

$$\begin{aligned}
\nabla^s_X T(U, V, W) \;&=\; \nabla^c_X T(U, V, W) \\
&\quad - \left[2s - \frac{1}{2}\right] [T(T(X, U), V, W) + T(U, T(X, V), W) + T(U, V, T(X, W))] \\
&=\; 0 + \left[\frac{1}{2} - 2s\right] [-T(W, V, T(X, U)) - T(U, W, T(X, V)) + T(U, V, T(X, W))] \\
&=\; \left[\frac{1}{2} - 2s\right] [-g(T(W, V), T(X, U)) - g(T(U, W), T(X, V)) + g(T(U, V), T(X, W))] \\
&=\; \left[\frac{1}{2} - 2s\right] [-g(T(V, W), T(U, X)) - g(T(W, U), T(V, X)) - g(T(U, V), T(W, X))] \\
&=\; \left[\frac{1}{2} - 2s\right] [-\sigma_T(U, V, W, X)] \;=\; \left[2s - \frac{1}{2}\right] \sigma_T(U, V, W, X)
\end{aligned}$$

by the definition of $\sigma_T$ [Ag06, Dfn A.1]. Consider the first Bianchi identity [Ag06, Thm 2.6]

$$\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}^s(X, Y, Z, V) \;=\; dT^s(X, Y, Z, V) + \nabla^s_V T^s(X, Y, Z) - \sigma_{T^s}(X, Y, Z, V),$$

where $T^s = 4sT$ and $\sigma_{T^s} = 16s^2\sigma_T$ are the corresponding quantities of the connection $\nabla^s$. Since $\nabla^c T = 0$, its torsion $T$ satisfies $dT = 2\sigma_{T^c} = 2\sigma_T$. A routine calculation yields then the claimed formula. The property of the curvature tensor follows by the same symmetrization argument as in [Ag06, Remark 2.3].

For the symmetry of the Ricci tensor, we argue as follows: for alle parameters $s$, the $\nabla^s$-divergences $\delta^s$ of $T$ coincide, $\delta^s T = \delta^g T$ (see [Ag06, Prop. A.2]). But $\nabla^c T = 0$ implies $\delta^c T = 0$, so $\delta^s T = 0$, and this is precisely the antisymmetric part of the Ricci tensor, so symmetry follows at once ([FrI02], [Ag06, Thm A.1]). □

## Appendix C. Compilation of important formulas

We compile some remarkable identities that are used throughout this article. All of them are routine exercises, so we abstain from giving proofs or detailed references for all. Some of the less obvious formulas (4-6) can be found in [Ag06] and [FrI02], though earlier publications are certainly possible. Recall the definition of the important 4-form $\sigma_T$ derived from any 3-form $T$:

$$\sigma_T \ := \ \frac{1}{2}\sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner T).$$

**Lemma C.1.** *For a 3-form $T$, a $k$-form $\omega$, a vector field $X$, an orthonormal frame $e_1, \dots, e_n$ and spinor fields $\psi, \varphi$, the following identities hold:*

(1) $X \cdot T = X \wedge T - X \lrcorner T, \quad T \cdot X = -X \wedge T - X \lrcorner T$

(2) $X \cdot T + T \cdot X = -2X \lrcorner T$, *more generally,* $X \cdot \omega - (-1)^k \omega \cdot X = -2X \lrcorner \omega$.

(3) $\langle T \cdot \psi, \varphi \rangle = \langle \psi, T \cdot \varphi \rangle$

(4) $\displaystyle\sum_{i=1}^n (e_i \lrcorner T) e_i = \sum_{i=1}^n e_i (e_i \lrcorner T) = 3T$

(5) $\displaystyle\sum_{i=1}^n (e_i \lrcorner T) \cdot (e_i \lrcorner T) = \sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T) - 3\|T\|^2 = 2\sigma_T - 3\|T\|^2$

(6) $\displaystyle T^2 = -\sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T) + \|T\|^2 = -2\sigma_T + \|T\|^2$

(7) $\displaystyle\sum_{i=1}^n e_i \cdot (e_i \wedge T) = (3-n)T$

(8) $\displaystyle\sum_{j=1}^n T(X, e_j) \cdot e_j = -2X \lrcorner T$

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