THE CLASSIFICATION OF NATURALLY REDUCTIVE HOMOGENEOUS SPACES IN DIMENSIONS $n \leq 6$

ILKA AGRICOLA, ANA CRISTINA FERREIRA, AND THOMAS FRIEDRICH

ABSTRACT. We present a new method for classifying naturally reductive homogeneous spaces – i.e. homogeneous Riemannian manifolds admitting a metric connection with skew torsion that has parallel torsion *and* curvature. This method is based on a deeper understanding of the holonomy algebra of connections with parallel skew torsion on Riemannian manifolds and the interplay of such a connection with the geometric structure on the given Riemannian manifold. It allows to reproduce by easier arguments the known classifications in dimensions $3, 4$, and 5, and yields as a new result the classification in dimension 6. In each dimension, one obtains a 'hierarchy' of degeneracy for the torsion form, which we then treat case by case. For the completely degenerate cases, we obtain results that are independent of the dimension. In some situations, we are able to prove that any Riemannian manifold with parallel skew torsion has to be naturally reductive. We show that a 'generic' parallel torsion form defines a quasi-Sasakian structure in dimension 5 and an almost complex structure in dimension 6.

1. INTRODUCTION AND SUMMARY

The classification of Riemannian symmetric spaces by Élie Cartan in 1926 was one of his major contributions to mathematics, as it linked in a unique way the algebraic theory of Lie groups and the geometric notions of curvature, isometry, and holonomy. In contrast, a classification of all Riemannian manifolds that are only homogeneous is, without further assumptions, genuinely impossible. Substantial work has been devoted to certain situations that are of particular interest; for example, isotropy-irreducible homogeneous Riemannian spaces, homogeneous Riemannian spaces with positive curvature, and compact homogeneous spaces of (very) small dimension have been investigated and classified. The present paper is devoted to one such class of homogeneous Riemannian manifolds—*naturally reductive (homogeneous) spaces*. Traditionally, they are defined as Riemannian manifolds $M = G/K$ with a transitive action of a Lie subgroup G of the isometry group and with a reductive complement $\mathfrak{m}$ of the Lie subalgebra $\mathfrak{k}$ inside the Lie algebra $\mathfrak{g}$ such that

$$(1.1) \qquad \langle [X,Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X,Z]_{\mathfrak{m}} \rangle \ = \ 0 \ \text{ for all } X, Y, Z \in \mathfrak{m}.$$

Here, $\langle -, - \rangle$ denotes the inner product on $\mathfrak{m}$ induced from the Riemannian metric g. Classical examples of naturally reductive homogeneous spaces include: irreducible symmetric spaces, isotropy-irreducible homogeneous manifolds, Lie groups with a bi-invariant metric, and Riemannian 3-symmetric spaces. The memoir [D'AZ79] is devoted to the construction and, under some assumptions, the classification of left-invariant naturally reductive metrics on compact Lie groups. Unfortunately, it is not always easy to decide whether a given homogeneous Riemannian space is naturally reductive, since one has to consider *all* possible transitive groups of isometries. Our study of naturally reductive spaces uses an alternative description, and while doing so, enables us to obtain some general structure results for them and another interesting larger class of manifolds, namely *Riemannian manifolds with parallel skew torsion*. As is well known, condition (1.1) states that the canonical connection ∇^c of $M = G/K$ (see [KN69]) has skew torsion $T^c \in \Lambda^3(M)$ (see Section 2 for precise definitions), and a classical result of Ambrose and Singer allows to conclude that the torsion T^c and the curvature $\mathcal{R}^c$ of ∇^c are then ∇^c-parallel [AS58]. For this reason, we agree to call a Riemannian manifold (M, g) *naturally reductive* if it

is a homogeneous space $M = G/K$ endowed with a metric connection ∇ with skew torsion T such that its torsion and curvature $\mathcal{R}$ are ∇-parallel, i.e. $\nabla T = \nabla \mathcal{R} = 0$. If M is connected, complete, and simply connected, a result of Tricerri states that the space is indeed naturally reductive in the previous sense [TV83, Thm 6.1]. As a general reference for the relation between these notions, we recommend [Tr93] and [CS04]. Appendix B explains that any manifold M that is neither covered by a sphere nor a Lie group carries at most one connection that makes it naturally reductive [OR12b]. Where possible, we will first investigate Riemannian manifolds with *parallel skew torsion*, i.e. we do not assume, a priori, homogeneity or parallel curvature. Recall that for a Riemannian manifold with K-structure, a characteristic connection is a K-connection with skew torsion (see [FI02], [Ag06], and [AFH13] for uniqueness). There are several classes of manifolds with a K-structure that are known to admit *parallel* characteristic torsion, for example nearly Kähler manifolds, Sasakian manifolds, or nearly parallel G_2-manifolds; these classes have been considerably enlarged in more recent work (see [Va79], [FI02], [Ale06], [AFS05], [Fr07], [Sch07]), eventually leading to a host of such manifolds that are not homogeneous. Nevertheless, we will be able to prove that for some instances of manifolds with parallel skew torsion, natural reductivity (and in particular, homogeneity) follows automatically.

The outline of our work is as follows. We recall some facts about metric connections with torsion, their curvature, and their holonomy. We introduce the 4-form σ_T that captures many geometric properties of the original torsion form T; among others, it is a suitable measure for the 'degeneracy' of T. We prove splitting theorems for Riemannian manifolds with parallel skew torsion if $\sigma_T = 0$ (the 'completely degenerate' case) and if $\ker T \neq 0$; these will be used many times in the sequel. In fact, we are able to show that an *irreducible* manifold with parallel skew torsion $T \neq 0$ with $\sigma_T = 0$ and dimension ≥ 5 is a Lie group. Then we turn to our main goal, namely, the classification of naturally reductive spaces, and, sometimes, even of spaces with parallel skew torsion, in dimension ≤ 6. The classification of naturally reductive spaces was obtained previously in dimension 3 by Tricerri and Vanhecke [TV83], and in dimensions 4, 5 by Kowalski and Vanhecke [KV83], [KV85]. Their approach relied on deriving a normal form for the curvature tensor, and the torsion then followed. With that approach, a classification in higher dimensions was impossible, and the geometric structure of the manifolds was not very transparent. In contrast, our approach is by looking at the parallel torsion as the fundamental entity, and the general philosophy is that, for sufficiently 'non-degenerate' torsion, the connection defined by the torsion is in fact the characteristic connection of a suitable K-structure on M. We make intensive use of the fact that a 3-form that is parallel for the connection it defines is far from being generic, and that its holonomy has to be quite special too: this allows a serious simplification of the situation. Furthermore, we reformulate the first Bianchi identity as a single identity in the Clifford algebra, a formulation that is computationally more tractable than its classical version. This is explained, together with the closely related Nomizu construction, in Appendix A. We begin by rederiving the classifications by our alternative method in dimensions 3 and 4. This turns to out to be very efficient, in particular in dimension 4—there, any parallel torsion $T \neq 0$ defines a parallel vector field $*T$ that induces a local splitting of the 4-dimensional space. This generalizes and explains the classical result in a few lines.

In dimension 5, the normal form of T depends on two parameters, which induce a case distinction (either $\sigma_T = 0$ or $\sigma_T \neq 0$, with subcases $\mathrm{Iso}(T) = \mathrm{SO}(2) \times \mathrm{SO}(2)$ and $\mathrm{Iso}(T) = \mathrm{U}(2)$). While the first case is immediately dealt with by our splitting theorems, the other two are quite distinct geometrically. We prove that a 5-manifold with parallel skew torsion is quasi-Sasakian in the second case and α-Sasakian in the third case, with Reeb vector field $*\sigma_T$ (up to a constant). In the second case, any such manifold is automatically naturally reductive, whereas non-homogeneous Sasaki manifolds are counter-examples for the analogous statement in the last case. We finish the discussion with the full description of all naturally reductive 5-manifolds; as a special case, we recover the classification from [KV85].

Our approach yields a complete classification of naturally reductive spaces in dimension 6. The crucial observation is that $*\sigma_T$ is a 2-form, i.e. an antisymmetric endomorphism that can have

rank 0, 2, 4, or 6. The ideal case $\operatorname{rk}(*\sigma_T) = 6$ and with all eigenvalues equal would define a Kähler form and thus the connection would be the characteristic connection of an almost Hermitian structure. We prove that this is basically what happens: the degenerate case $\operatorname{rk}(*\sigma_T) = 0$ is again easy. A 6-manifold with parallel skew torsion and $\operatorname{rk}(*\sigma_T) = 2$ is automatically naturally reductive and can be parametrized explicitly. In particular, it is a product of two 3-dimensional non-commutative Lie groups equipped with a family of left-invariant metrics. The case $\operatorname{rk}(*\sigma_T) = 4$ cannot occur, and in case $\operatorname{rk}(*\sigma_T) = 6$, the eigenvalues have to be indeed equal. The space is either a nearly Kähler manifold or a family of naturally reductive spaces that can be described explicitly. The generic examples are left-invariant metrics on the groups $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{Spin}(4) = S^3 \times S^3$.

For reference our paper includes a detailed section that describes all homogeneous spaces appearing in the classification that are neither products nor degenerate; we expect this part to be as useful as the classification theorems themselves.

2. METRIC CONNECTIONS WITH SKEW TORSION

Consider a Riemannian manifold (M^n, g). The difference between its Levi-Civita connection ∇^g and any linear connection ∇ is a $(2,1)$-tensor field A,

$$\nabla_X Y = \nabla^g_X Y + A(X, Y), \quad X, Y \in TM^n.$$

The curvature of ∇ resp. ∇^g will always be denoted by $\mathcal{R}$ resp. $\mathcal{R}^g$. Following Cartan, we study the algebraic types of the torsion tensor for a metric connection. Denote by the same symbol the $(3,0)$-tensor derived from a $(2,1)$-tensor by contraction with the metric. We identify TM^n with $(TM^n)^*$ using g from now on. Let $\mathcal{T}$ be the $n^2(n-1)/2$-dimensional space of all possible torsion tensors,

$$\mathcal{T} = \{T \in \otimes^3 TM^n \mid T(X, Y, Z) = -T(Y, X, Z)\} \cong \Lambda^2(M^n) \otimes TM^n.$$

A connection ∇ is metric if and only if A belongs to the space

$$\mathcal{A} := TM^n \otimes \Lambda^2(M^n) = \{A \in \otimes^3 TM^n \mid A(X, V, W) + A(X, W, V) = 0\}.$$

The spaces $\mathcal{T}$ and $\mathcal{A}$ are isomorphic as $\mathrm{O}(n)$ representations, reflecting the fact that metric connections can be uniquely characterized by their torsion. For $n \geq 3$, they split under the action of $\mathrm{O}(n)$ into the sum of three irreducible representations,

$$\mathcal{T} \cong TM^n \oplus \Lambda^3(M^n) \oplus \mathcal{T}'.$$

The last module is equivalent to the Cartan product of the representations TM^n and $\Lambda^2(M^n)$ (see [Ca25]). The eight classes of linear connections are now defined by the possible components of their torsions T in these spaces. The main case of interest is the following:

Definition 2.1. The connection ∇ is said to have *skew-symmetric torsion* or just *skew torsion* if its torsion tensor lies in the second component of the decomposition, i. e. it is given by a 3-form,

$$(2.1) \qquad\qquad \nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y, -).$$

Whenever we shall speak of a metric connection with skew torsion $T \in \Lambda^3(M^n)$, the connection is defined with the normalization as in equation (2.1). By a *Riemannian manifold with parallel skew torsion*, we mean a Riemannian manifold (M^n, g) equipped with a metric connection ∇

with skew torsion T such that $\nabla T = 0$. For any 3-form $T \in \Lambda^3(M^n)$, one can define a 4-form through

$$\sigma_T := \frac{1}{2}\sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T), \text{ or equivalently } \sigma_T(X,Y,Z,V) = \overset{X,Y,Z}{\mathfrak{S}} g(T(X,Y),T(Z,V)),$$

where $e_1,\ldots,e_n$ is a local orthonormal frame. It clearly defines a covariant from $\Lambda^3(\mathbb{R}^n)$ to $\Lambda^4(\mathbb{R}^n)$ for the standard $\mathrm{SO}(n)$-action. Furthermore, this 4-form is an important measure for the 'degeneracy' of T and appears in many formulas, like the identities on curvature and derivatives below, or the Nomizu construction (Appendix A). It also describes the algebraic action of any 2-form $X \lrcorner T$, identified with an element of $\mathfrak{so}(n)$, on T (see also [AF10a, proof of Prop. 2.1]),

$$(2.2) \qquad\qquad (X \lrcorner T)(T)(Y_1,Y_2,Y_3) = \sigma_T(Y_1,Y_2,Y_3,X).$$

The condition that ∇ is metric implies that each curvature transformation $\mathcal{R}(X,Y)$ is skew-adjoint, i.e. $\mathcal{R}$ can be interpreted as an endomorphism $\mathcal{R}: \Lambda^2(M^n) \to \mathfrak{hol}^\nabla \subset \Lambda^2(M^n)$ where $\mathfrak{hol}^\nabla$ is the holonomy algebra of ∇. If, in addition, $\nabla T = 0$, then $\mathcal{R}$ is a symmetric endomorphism,

$$g(\mathcal{R}(X,Y)V,W) = g(\mathcal{R}(V,W)X,Y),$$

and the Ricci tensor Ric is symmetric too. Consequently, the curvature operator $\mathcal{R} = \psi \circ \pi$ is given by the projection π onto the holonomy algebra and a symmetric endomorphism $\psi: \mathfrak{hol}^\nabla \to \mathfrak{hol}^\nabla$. If moreover ψ is $\mathfrak{hol}^\nabla$-equivariant, then $\mathcal{R}$ is ∇-parallel, i.e. T and $\mathcal{R}$ define a naturally reductive structure. For parallel skew torsion, the first Bianchi identity may be written as [KN63]

$$(2.3) \qquad\qquad \overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(X,Y,Z,V) = \sigma_T(X,Y,Z,V),$$

and the following remarkable relations hold [FI02],

$$(2.4) \qquad\qquad \nabla^g T = \frac{1}{2}\sigma_T, \quad dT = 2\sigma_T.$$

If the torsion *and* the curvature $\mathcal{R}$ of ∇ are ∇-parallel, the second Bianchi identity is reduced to the algebraic relation ([KN63])

$$(2.5) \qquad\qquad \overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(T(X,Y),Z) = 0.$$

These identities can be nicely formulated in the Clifford algebra (see Theorem A.2) and we will use this approach several times in our dicussion. Last but not least, let us recall that any ∇-parallel vector field V or 2-form Ω satisfies the differential equations (see [AF04a])

$$(2.6) \quad \langle \nabla_X^g V, Y \rangle = -T(X,V,Y)/2, \quad \text{or} \quad \delta^g \Omega = \frac{1}{2}\Omega \lrcorner T, \quad d\Omega = \sum_{i=1}^n (e_i \lrcorner \Omega) \wedge (e_i \lrcorner T).$$

In particular, any ∇-parallel vector field is a Killing vector field of constant length. A parallel torsion form is of special algebraic type. Indeed, in this case the holonomy algebra $\mathfrak{hol}^\nabla$ is contained in the isotropy algebra $\mathfrak{iso}(T)$ of T. Here are some useful observations on $\mathfrak{hol}^\nabla$.

Proposition 2.2. *Let (M^n, g, T) be a Riemannian manifold with parallel skew torsion. Then the following properties hold:*

   (1) *Either $\sigma_T = 0$ or the algebra $\mathfrak{hol}^\nabla \subset \mathfrak{iso}(T)$ is non-trivial.*
   (2) *If $\dim \mathfrak{hol}^\nabla \leq 1$, then there exists a 2-form ω such that $\sigma_T = \pm\omega \wedge \omega$.*
   (3) *If $\mathfrak{hol}^\nabla$ is abelian, then $\nabla\mathcal{R} = 0$ and therefore M^n is naturally reductive.*

Proof. The first claim follows from the first Bianchi identity (2.3). For the second, let $\omega \in \mathfrak{hol}^\nabla \subset \Lambda^2$ be a generator. Since $\nabla T = 0$, the symmetric curvature operator $\mathcal{R}: \Lambda^2 \to \Lambda^2$ is the projection onto $\mathfrak{hol}^\nabla$, $\mathcal{R} = a\,\omega \odot \omega$. Inside the Clifford algebra $\mathcal{C}$ the Bianchi identity reads as (see Appendix A)

$$-2\,\sigma_T + \|T\|^2 + \mathcal{R} = T^2 + \mathcal{R} \in \mathbb{R} \subset \mathcal{C}.$$

We compare the parts of order four in $\mathcal{C}$ and obtain the result. For the last assertion, we write again the curvature operator as $\mathcal{R} = \psi \circ \pi$, where π denotes the projection onto the holonomy algebra $\mathfrak{hol}^\nabla$ and $\psi : \mathfrak{hol}^\nabla \to \mathfrak{hol}^\nabla$ is a symmetric endomorphism. As a projection, π is $\mathfrak{hol}^\nabla$-equivariant, and $\mathfrak{hol}^\nabla$ acts by the adjoint action on itself – and this action is trivial, since $\mathfrak{hol}^\nabla$ is abelian. Therefore, ψ is trivially $\mathfrak{hol}^\nabla$-equivariant, and so this holds for the composition $\mathcal{R}$. This shows $\nabla \mathcal{R} = 0$ by the general holonomy principle. $\qquad\square$

3. Splitting properties

Splitting theorems allow us to identify situations in which a naturally reductive space has to be a product. We will use the following well-known statement.

Theorem 3.1 ([KV83])**.** *The Riemannian product $(M_1, g_1) \times (M_2, g_2)$ is naturally reductive if and only if both factors (M_i, g_i), $i = 1, 2$ are naturally reductive.*

The following two notions turn out to be appropriate for analyzing 3-forms.

Definition 3.2. For any 3-form $T \in \Lambda^3(M^n)$, we define the kernel

$$\ker T \; := \; \{X \in TM^n \,|\, X \lrcorner \, T = 0\},$$

and the Lie algebra generated by the image,

$$\mathfrak{g}_T \; := \; \mathrm{Lie}\langle X \lrcorner \, T \,|\, X \in TM^n \rangle.$$

The Lie algebra $\mathfrak{g}_T$ was first considered in [AF04a] and is *not* related in any obvious way to the isotropy algebra of T. If $\mathfrak{g}_T$ does not act irreducibly on TM^n, it is known that it splits into an orthogonal sum $V_1 \oplus V_2$ of $\mathfrak{g}_T$-modules and $T = T_1 + T_2$ with $T_i \in \Lambda^3(V_i)$ [AF04a, Prop.3.2]; thus V_1 and V_2 are ∇-parallel subbundles of TM^n. If $\sigma_T = 0$, equation (2.4) and de Rham's Theorem imply:

Theorem 3.3. *Let (M^n, g, T) be a complete, simply connected Riemannian manifold with parallel skew torsion T such that $\sigma_T = 0$, and let $TM^n = \mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_q$ be the decomposition of TM^n into $\mathfrak{g}_T$-irreducible, ∇-parallel distributions. Then all distributions $\mathcal{T}_i$ are ∇^g-parallel integrable distributions, M^n is a Riemannian product, and the torsion T splits accordingly,*

$$(M^n, g, T) \; = \; (M_1, g_1, T_1) \times \ldots \times (M_q, g_q, T_q), \quad T \; = \; \sum_{i=1}^{q} T_i \,.$$

The following new splitting theorem for manifolds with parallel skew torsion justifies why we will consider in the following only connections with parallel skew torsion whose kernel is trivial. Let us emphasize that the vanishing of σ_T is not needed for it.

Theorem 3.4. *Let (M^n, g, T) be a complete, simply connected Riemannian manifold with parallel skew torsion T. Then $\ker T$ and $(\ker T)^\perp$ are ∇-parallel and ∇^g-parallel integrable distributions. Furthermore, M^n is a Riemannian product such that T vanishes on one factor and has trivial kernel on the other,*

$$(M^n, g, T) \; = \; (M_1, g_1, T_1 = 0) \times (M_2, g_2, T_2), \quad \ker T_2 = \{0\}.$$

Proof. If T is parallel for the connection ∇ it defines, $\ker T$ and $(\ker T)^\perp$ are clearly ∇-parallel distributions. Suppose X is a vector field in $\ker T$. Then, for any vector field Y,

$$\nabla_Y^g X \; = \; \nabla_Y X - \frac{1}{2} T(Y, X) \; = \; \nabla_Y X$$

belongs again to $\ker T$. Consequently, $\ker T$ is ∇^g-parallel as well. Since ∇ and ∇^g are metric, the same holds for $(\ker T)^\perp$, and both distributions are integrable. The last assertion follows then from de Rham's Theorem. $\qquad\square$

In particular, this applies to the following situation: given a simply connected naturally reductive space with torsion T that does not depend on all directions, the space is a product of lower dimensional naturally reductive homogeneous spaces.

4. Vanishing of σ_T

Let (M^n, g) be a Riemannian manifold with parallel skew torsion T: we are interested in describing those manifolds for which $\sigma_T = 0$. Proposition 2.2 and Theorem 3.3 suggest that the vanishing of σ_T is a strong indication that the geometry is degenerate. Let us give some examples where this happens. In dimension 3, the volume form of any metric connection ∇ is ∇-parallel, and $\sigma_T = 0$ for trivial reasons: hence, any Riemannian 3-manifold is an example of such a geometry. In dimension 4, again $\sigma_T = 0$ for algebraic reasons, so the condition is vacuous. Finally, compact Lie groups with a bi-invariant metric and any connection from the Cartan-Schouten family of connections are well-known examples [KN69, Proposition 2.12]. We shall now prove:

Theorem 4.1. *Let (M^n, g) be an irreducible, complete, and simply connected Riemannian manifold with parallel skew torsion $T \neq 0$ such that $\sigma_T = 0$, $n \geq 5$. Then M^n is a simple compact Lie group with bi-invariant metric or its dual noncompact symmetric space.*

Proof. Let $\mathfrak{hol}^g$, $\mathfrak{hol}^\nabla$, and $\mathfrak{iso}(T)$ be the Lie algebras of the holonomy groups of ∇^g, ∇ and the Lie algebra of the isotropy group of T, respectively. By equation (2.4), $\nabla^g T = \sigma_T/2 = 0$, hence both $\mathfrak{hol}^\nabla$ and $\mathfrak{hol}^g$ are subalgebras of $\mathfrak{iso}(T)$. This implies also that any multiple of T defines again a connection with parallel skew torsion and vanishing 4-form. Fix some point $p \in M^n$, $V := T_p M$. The Nomizu construction (Lemma A.1) yields for $\sigma_T = 0$ that T defines a Lie algebra structure on V by $g_p([X, Y], Z) = \alpha\, T_p(X, Y, Z)$ for any $\alpha \in \mathbb{R}$. By assumption, $\mathfrak{hol}^g$ acts irreducibly on V. Consider the Lie algebra $\mathfrak{g}_T$ generated by all elements of the form $X \lrcorner T$, see Definition 3.2. Since $\mathfrak{so}(n) \cong \Lambda^2(V)$ acts on V, it also acts on $\Lambda^3(V)$, and for an element $X \lrcorner T \in \Lambda^2(V)$ this action on T is given by (see equation (2.2))

$$(X \lrcorner T)(T)(Y_1, Y_2, Y_3) \;=\; \sigma_T(Y_1, Y_2, Y_3, X) \;=\; 0.$$

Hence, $\mathfrak{g}_T \subset \mathfrak{iso}(T)$ as well. We can assume that $\mathfrak{g}_T$ acts irreducibly on V by Theorem 3.3. We conclude that (G_T, V, T) defines an irreducible skew torsion holonomy system in the sense of Definition B.1, where G_T denotes the Lie subgroup of $\mathrm{SO}(n)$ with Lie algebra $\mathfrak{g}_T$. By the Skew Holonomy Theorem of Olmos and Reggiani (Theorem B.2), (G_T, V, T) is either transitive or symmetric. If it is transitive, $G_T = \mathrm{SO}(n)$, hence $\mathfrak{g}_T = \mathfrak{so}(n) = \mathfrak{iso}(T)$. But for $n \geq 5$, this implies $T = 0$, a contradiction. Thus, the skew holonomy system is symmetric, i.e. G_T is simple, acts on V by its adjoint action, and T is, up to multiples, unique.
Next, we observe that the algebras $\mathfrak{g}_T$ and $\mathfrak{iso}(T)$ coincide. Indeed, any element of $\mathfrak{iso}(T)$ is a derivation of the simple Lie algebra $\mathfrak{g}_T$. But all derivations of $\mathfrak{g}_T$ are inner, i.e. defined by elements of $\mathfrak{g}_T$. As a consequence, the holonomy algebra $\mathfrak{hol}^g$ is also contained in $\mathfrak{g}_T$. If they were not equal, then we could decompose the tangent space into $V = \mathfrak{g}_T = \mathfrak{hol}^g \oplus (\mathfrak{hol}^g)^\perp$, which contradicts the assumption that M is an irreducible Riemannian manifold. To summarize, $\mathfrak{iso}(T) = \mathfrak{g}_T = \mathfrak{hol}^g$. By Berger's holonomy theorem, M^n is an irreducible Riemannian symmetric space, and its isotropy representation of $\mathfrak{hol}^g$ is its adjoint representation. Certainly, the compact simple Lie group G_T itself, viewed as a symmetric space (or its dual noncompact space), is such a manifold. Since a symmetric space is uniquely characterised by its isotropy representation, [He78, p.427, Theorem 5.11], other symmetric spaces cannot occur. $\qquad\square$

Example 4.2. Let T be a 3-form with constant coefficients on $\mathbb{R}^n$ satisfying $\sigma_T = 0$. Then the flat space $(\mathbb{R}^n, g, T)$ is a reducible Riemannian manifold with parallel skew torsion and $\sigma_T = 0$. This shows that the assumption that M be irreducible is crucial in this result. Observe that in this example, the Riemannian manifold is decomposable, but the torsion is not.

5. The classification in dimension 3

Let (M^3, g) be a complete Riemannian 3-manifold admitting a metric connection ∇ with parallel torsion $T \neq 0$. Since $\|T\| = $ const, T defines a nowhere vanishing 3-form and hence M^3 is orientable. Denoting the volume form by dM, we conclude $T = \lambda dM$ for some constant $\lambda \neq 0$.

We begin by giving a proof of the classification of 3-dimensional naturally reductive spaces that illustrates the methods that we will use later. This result is due to Tricerri and Vanhecke [TV83], but with another approach. A detailed description of naturally reductive metrics on $\mathrm{SL}(2, \mathbb{R})$ (case 2b in the theorem below) may be found in [HI96].

Remark 5.1. For ease of notation, we will often omit the wedge product between 1-forms $e_i, e_j \ldots$, i.e. $e_{ijk} := e_i \wedge e_j \wedge e_k$, like in the following proof. Furthermore, we will freely identify $\mathfrak{so}(n)$ and $\Lambda^2(\mathbb{R}^n)$ where necessary, i.e. there is no difference between the 2-form e_{ij} and the endomorphism E_{ij} mapping e_i to e_j, e_j to $-e_i$ and eveything else to zero.

Theorem 5.2 ([TV83, Thm 6.5, p. 63]). *A three-dimensional complete, simply connected naturally reductive space (M^3, g) is either*

(1) *a space form: $\mathbb{R}^3, S^3$ or $\mathbb{H}^3$, or*
(2) *isometric to one of the following Lie groups with a suitable left-invariant metric:*
    (a) *the special unitary group $\mathrm{SU}(2)$,*
    (b) *$\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the universal cover of $\mathrm{SL}(2, \mathbb{R})$,*
    (c) *the 3-dimensional Heisenberg group H^3 (see section 9.3).*

Proof. Indeed, the Ambrose-Singer Theorem tells us that (M^3, g) is homogeneous and the Lie algebra of the group G which acts transitively and effectively on M^3 is isomorphic to the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the holonomy algebra of ∇ and $\mathfrak{m}$ is the tangent space at some point. If M^3 is Einstein, it is isometric to $\mathbb{R}^3, S^3$, or H^3. If the Riemannian Ricci tensor Ric^g has distinct eigenvalues, $\mathfrak{k}$ is at most one-dimensional. Indeed, Ric is ∇-parallel, and the difference $\mathrm{Ric}^g - \mathrm{Ric}$ is given by T, which is also ∇-parallel. Hence $\nabla \mathrm{Ric}^g = 0$ and Ric^g is, by assumption, not a multiple of the idendity. Therefore $\mathfrak{k}$ cannot coincide with $\mathfrak{so}(3)$. Let us apply the Nomizu construction in this situation (Appendix A). The one-dimensional Lie algebra $\mathfrak{k}$ is generated by the 2-form $\Omega =: e_{12}$. The curvature operator is symmetric (see Section 2) and consequently a multiple of the projection onto the subalgebra $\mathfrak{k}$, i.e.

$$T = \lambda\, e_{123} \quad \text{and} \quad \mathcal{R} = \alpha\, e_{12} \odot e_{12}.$$

By the Nomizu construction, e_1, e_2, e_3, and Ω are a basis of $\mathfrak{g}$ with non-trivial Lie brackets

$$[e_1, e_2] = -\alpha\Omega - \lambda e_3 =: \tilde{\Omega}, \quad [e_1, e_3] = \lambda e_2, \quad [e_2, e_3] = -\lambda e_1, \quad [\Omega, e_1] = e_2, \quad [\Omega, e_2] = -e_1.$$

The 3-dimensional subspace $\mathfrak{h}$ spanned by e_1, e_2, and $\tilde{\Omega}$ is a Lie subalgebra of $\mathfrak{g}$ that is transversal to the isotropy algebra $\mathfrak{k}$ (since $\lambda \neq 0$). Consequently, the corresponding simply connected Lie group H acts transitively on M^3, i.e. M^3 is a Lie group with a left-invariant metric. One checks that $\mathfrak{h}$ has the commutator relations

$$[e_1, e_2] = \tilde{\Omega}, \quad [\tilde{\Omega}, e_1] = (\lambda^2 - \alpha)e_2, \quad [e_2, \tilde{\Omega}] = (\lambda^2 - \alpha)e_1.$$

For $\alpha = \lambda^2$, this is the 3-dimensional Heisenberg Lie algebra, otherwise it is $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{R})$ depending on the sign of $\lambda^2 - \alpha$. $\qquad\square$

6. The classification in dimension 4

The classification of naturally reductive spaces below appeared first in [KV83]. The original proof used a much more involved curvature calculation. Here we present a much simpler proof. An easy algebraic argument shows that $\sigma_T = 0$ for any 3-form T in dimension 4. Therefore:

Theorem 6.1. *Let (M^4, g, T) be a complete, simply connected Riemannian 4-manifold with parallel skew torsion $T \neq 0$. Then*

(1) *$V := *T$ is a ∇^g-parallel vector field.*
(2) *The Riemannian holonomy algebra $\mathfrak{hol}^g$ is contained in $\mathfrak{so}(3)$, and hence M^4 is isometric to a product $N^3 \times \mathbb{R}$, where (N^3, g) is a 3-manifold with a parallel 3-form T.*

Since any 3-form in dimension 4 has a 1-dimensional kernel, the result follows directly from Theorem 3.4. However, we can also give a direct geometric proof.

Proof. Any vector field V satisfies $V \lrcorner (*V) = 0$. But $*V = T$, so $V \lrcorner T = 0$. $\nabla V = 0$ is equivalent to $\nabla^g_X V = -T(X, V, -)/2$, and by the previous observation, the right hand side vanishes, so $\nabla^g_X V = 0$. The vector field V is complete, because it is a Killing vector field on a complete Riemannian manifold. Since the stabilizer of a vector field is $SO(3) \times \{1\}$ (acting on the orthogonal complement of V), the holonomy claim follows at once from the general holonomy principle. The isometric splitting is then a consequence of de Rham's Theorem. $\qquad \square$

Thus, Theorem 3.1 implies at once:

Corollary 6.2 ([KV83]). *A 4-dimensional simply connected naturally reductive Riemannian manifold with $T \neq 0$ is isometric to a Riemannian product $N^3 \times \mathbb{R}$, where N^3 is a 3-dimensional naturally reductive Riemannian manifold.*

7. The classification in dimension 5

The following lemma follows immediately from the well-known normal forms for 3-forms in five dimensions.

Lemma 7.1. *Let (M^5, g, T) be an orientable Riemannian 5-manifold with parallel skew torsion $T \neq 0$. Then there exists a local orthonormal frame $e_1, \ldots, e_5$ such that*

$$(7.1) \qquad T = -(\varrho e_{125} + \lambda e_{345}), \quad *T = -(\varrho e_{34} + \lambda e_{12}), \quad \sigma_T = \varrho \lambda e_{1234}$$

for two real constants ϱ, λ. The isotropy group and its action on the tangent space at any $x \in M$ is:

Case A	$\sigma_T = 0$	$\varrho\lambda = 0$	$\mathrm{Iso}(T) = SO(3) \times SO(2)$
Case B.1	$\sigma_T \neq 0$	$\varrho\lambda \neq 0, \varrho \neq \lambda$	$\mathrm{Iso}(T) = SO(2) \times SO(2) \times \{1\}$
Case B.2	$\sigma_T \neq 0$	$\varrho\lambda \neq 0, \varrho = \lambda$	$\mathrm{Iso}(T) = U(2) \times \{1\}$

The degenerate case A ($\sigma_T = 0$) follows from Theorem 3.4:

Proposition 7.2 (case A – with parallel skew torsion). *Let (M^5, g, T) be a complete, simply connected Riemannian 5-manifold with parallel skew torsion $T \neq 0$ such that $\sigma_T = 0$. Then M^5 is isometric to a product $N^3 \times N^2$, where N^3 is a Riemannian 3-manifold with torsion $T = c\,dN^3$, a multiple of the volume form, and N^2 is any Riemannian 2-manifold.*

Observing that the curvature of a surface is its Gaussian curvature, Theorem 3.1 implies:

Corollary 7.3 (case A – the naturally reductive case). *A simply connected naturally reductive Riemannian 5-manifold with $T \neq 0$, $\sigma_T = 0$ is isometric to a Riemannian product $N^3 \times N^2$, where N^3 is a 3-dimensional naturally reductive Riemannian manifold and N^2 a surface of constant Gaussian curvature.*

Recall that a Sasakian manifold always has parallel characteristic torsion, and that many non-homogeneous Sasakian manifolds are known (in all odd dimensions). Consequently, a classification of 5-manifolds with parallel skew torsion is not possible for $\lambda = \varrho$ (case B.2). In contrast to this, our next result shows that 5-manifolds with parallel skew torsion and $\lambda \neq \varrho$ (case B.1) are necessarily naturally reductive and can be completely described:

Theorem 7.4 (case B.1 – with parallel skew torsion). *Let (M^5, g, T) be an orientable Riemannian 5-manifold with parallel skew torsion and such that T has the normal form*

$$T = -(\varrho e_{125} + \lambda e_{345}), \quad \varrho\lambda \neq 0 \text{ and } \varrho \neq \lambda.$$

Then $\nabla \mathcal{R} = 0$. The admissible torsion forms and curvature operators depend on 4 parameters. Moreover, if M^5 is complete, then it is locally naturally reductive.

Proof. The curvature transformation is a symmetric endomorphism $\mathcal{R} : \Lambda^2(M^5) \to \Lambda^2(M^5)$ (see Section 2). We consider the 2-forms $\Omega_1 := e_{12}$ and $\Omega_2 := e_{34}$, of which we know that they are closed and ∇-parallel (see the proof of Theorem 7.10). We split

$$\mathfrak{so}(5) \cong \Lambda^2(M^5) = \mathrm{span}(\Omega_1, \Omega_2) \oplus \mathrm{span}(\Omega_1, \Omega_2)^\perp =: \mathfrak{h} \oplus \mathfrak{h}^\perp,$$

and observe that the first summand is not only a subspace, but a two-dimensional abelian subalgebra $\mathfrak{h} \cong \mathfrak{so}(2) \oplus \mathfrak{so}(2)$ – in fact, by Lemma 7.1, $\mathfrak{hol}^\nabla \subset \mathfrak{h}$. This last property implies that the curvature $\mathcal{R}$ has image inside $\mathfrak{h}$. In particular, it means that $\mathfrak{hol}^\nabla$ is abelian too and therefore $\nabla \mathcal{R} = 0$ by Proposition 2.2 (3). Hence, there exist constants a, b, c such that

$$\mathcal{R} = a \, \Omega_1 \odot \Omega_1 + b \, \Omega_1 \odot \Omega_2 + c \, \Omega_2 \odot \Omega_2.$$

One sees that inside the Clifford algebra, $(\mathcal{R} + T^2)_4 = (b - 2\lambda\varrho) \, e_1 e_2 e_3 e_4$, hence the first Bianchi identity in the Clifford version (Theorem A.2) yields $b = 2\lambda\varrho$. $\qquad\square$

Remark 7.5. A routine computation shows that the Ricci tensor of ∇ is then given by

$$\mathrm{Ric} = \mathrm{diag}(-a, -a, -c, -c, 0).$$

In particular, $a = c = 0$ yields a ∇-Ricci-flat space.

We shall now describe explicitly the naturally reductive spaces and the group G acting transitively on them covered by the previous theorem. As a vector space, its Lie algebra $\mathfrak{g}$ is given by the Nomizu construction as $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^5$ with basis $\Omega_1, \Omega_2, e_1, \ldots, e_5$. Using the explicit formulas for T and $\mathcal{R}$, one computes the commutator relations

$$[e_1, e_2] = -a\Omega_1 - \lambda\varrho\,\Omega_2 + \varrho\,e_5 =: \tilde{\Omega}_1, \quad [e_3, e_4] = -\lambda\varrho\,\Omega_1 - c\,\Omega_2 + \lambda\,e_5 =: \tilde{\Omega}_2,$$

$$[e_1, e_5] = -\varrho\,e_2, \quad [e_2, e_5] = \varrho\,e_1, \quad [e_3, e_5] = -\lambda\,e_4, \quad [e_4, e_5] = \lambda\,e_3,$$

$$[\Omega_1, e_1] = +e_2, \quad [\Omega_1, e_2] = -e_1, \quad [\Omega_2, e_3] = +e_4, \quad [\Omega_2, e_4] = -e_3.$$

The crucial observation is that the linear space spanned by $e_1, e_2, e_3, e_4, \tilde{\Omega}_1, \tilde{\Omega}_2$ is a Lie subalgebra that we will denote by $\mathfrak{g}_1$. Its non-vanishing Lie brackets are

$$[e_1, e_2] = \tilde{\Omega}_1, \quad [\tilde{\Omega}_1, e_1] = (\varrho^2 - a)\,e_2, \quad [e_2, \tilde{\Omega}_1] = (\varrho^2 - a)\,e_1,$$

$$[e_3, e_4] = \tilde{\Omega}_2, \quad [\tilde{\Omega}_2, e_3] = (\lambda^2 - c)\,e_4, \quad [e_4, \tilde{\Omega}_2] = (\lambda^2 - c)\,e_3.$$

Define

$$h_1 := \lambda\,\tilde{\Omega}_1 - \varrho\,\tilde{\Omega}_2 = \lambda(\varrho^2 - a)\,\Omega_1 + \varrho(c - \lambda^2)\,\Omega_2.$$

There are two cases to consider. If $a = \varrho^2$ and $c = \lambda^2$, $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are proportional, therefore $\mathfrak{g}_1$ is 5-dimensional, isomorphic to the 5-dimensional Heisenberg algebra, and $\mathfrak{g}_1 \cap \mathfrak{h}$ is trivial (since $\lambda \neq 0$). We conclude that the corresponding Heisenberg group H^5 acts transitively on the 5-manifold M. Consequently, (M, g) is isometric to a left-invariant metric on H^5, and the metric depends on two parameters.

In the second case, either $a \neq \varrho^2$ or $c \neq \lambda^2$. The intersection $\mathfrak{g}_1 \cap \mathfrak{h} =: \mathfrak{h}_1$ is 1-dimensional and generated by the element $h_1 \neq 0$. By the commutator equations above, the Lie algebra $\mathfrak{g}_1$ is a direct sum of two 3-dimensional ideals, and the last formula describes how $\mathfrak{h}_1$ is embedded in this direct sum decomposition of $\mathfrak{g}_1$. If we assume that the coefficients are chosen in such a way that h_1 generates a *closed* subgroup of G_1, we can assert M is isometric to $G/H \cong G_1/H_1$. Furthermore, each of these ideals is isomorphic to $\mathfrak{so}(3), \mathfrak{sl}(2, \mathbb{R})$, or to the 3-dimensional Heisenberg algebra, depending on the value of the constants $\varrho^2 - a$ and $c - \lambda^2$. We emphasize that the case of two 3-dimensional Heisenberg algebras is excluded: it would correspond to the vanishing of both constants, which was discussed earlier. We summarize this discussion in the following theorem:

Theorem 7.6 (case B.1 – classification)**.** *A complete, simply connected Riemannian 5-manifold (M^5, g, T) with parallel skew torsion T such that $T = -(\varrho e_{125} + \lambda e_{345})$ with $\varrho\lambda \neq 0$, $\varrho \neq \lambda$ is one of the following homogeneous spaces:*

(1) *The 5-dimensional Heisenberg group H^5 with a two-parameter family of left-invariant metrics (described in Section 9.3),*

(2) *A manifold of type $(G_1 \times G_2)/\mathrm{SO}(2)$ where G_1 and G_2 are either $\mathrm{SU}(2)$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, or H^3, but not both equal to H^3 with one parameter $r \in \mathbb{Q}$ classifying the embedding of $\mathrm{SO}(2)$ and a three-parameter family of homogeneous metrics (described in Section 9.1).*

Remark 7.7. Theorem 7.6 gives an isometric description with a one-dimensional isotropy group of the homogeneous spaces under consideration. The description as naturally reductive spaces uses a two-dimensional isotropy group.

Let us now discuss the case B.2, i.e. $\lambda = \varrho$. As observed before, there exist manifolds of type B.2 that have parallel skew torsion but that are not naturally reductive – hence, these cannot be classified.

Theorem 7.8 (case B.2 – classification). *Let (M^5, g, T) be a complete, simply connected naturally reductive homogeneous 5-space such that $T = -(\varrho e_{125} + \lambda e_{345})$ with $\varrho \lambda \neq 0, \varrho = \lambda$. Then M^5 is either isometric to one of the spaces discussed in Theorem 7.6, or to $\mathrm{SU}(3)/\mathrm{SU}(2)$ or $\mathrm{SU}(2,1)/\mathrm{SU}(2)$; in the last two cases, the family of metrics depends on two parameters (described in Section 9.2).*

Proof. By assumption, $\mathfrak{iso}(T) = \mathfrak{u}(2)$, hence the holonomy algebra $\mathfrak{hol}^\nabla$ of ∇ is a subalgebra of $\mathfrak{u}(2)$. If $\mathfrak{hol}^\nabla \subset \mathfrak{so}(2) \oplus \mathfrak{so}(2)$, we are in the situation B.1 discussed previously. If not, it has to contain $\mathfrak{su}(2)$, i.e. $\mathfrak{su}(2) \subset \mathfrak{hol}^\nabla \subset \mathfrak{u}(2)$. The curvature operator is a symmetric, $\mathfrak{hol}^\nabla$-invariant operator, hence $\mathcal{R} = \psi \circ \pi_{\mathfrak{u}(2)}$ with a $\mathfrak{hol}^\nabla$-equivariant map $\psi : \mathfrak{u}(2) \to \mathfrak{u}(2)$. Since $\mathfrak{u}(2)$ decomposes into $\mathfrak{su}(2) \oplus \mathbb{R} \cdot \omega$ (ω the central element), $\psi = a \, \mathrm{Id}_{\mathfrak{su}(2)} \oplus b \, \mathrm{Id}_\mathbb{R}$ for some real constants a, b, which in an explicit choice of basis means (the squares denote symmetric tensor powers)

$$(7.2) \qquad \mathcal{R} = a[(e_{13} + e_{24})^2 + (e_{14} - e_{23})^2 + (e_{12} - e_{34})^2] + b\,(e_{12} + e_{34})^2.$$

The first Bianchi identity in the Clifford version (Theorem A.2) yields then $b - 3a = \varrho^2$. A routine computation shows then that M is isometric to $\mathrm{SU}(3)/\mathrm{SU}(2)$ or its non-compact dual $\mathrm{SU}(2,1)/\mathrm{SU}(2)$. $\qquad\square$

Remark 7.9. The classification of 5-dimensional naturally reductive homogeneous spaces was obtained by O. Kowalski and L. Vanhecke in 1985 using other methods, see [KV85].

Let us look from another point of view at 5-dimensional Riemannian manifolds with parallel skew torsion and $\sigma_T \neq 0$. The torsion induces a canonical metric almost contact structure, i.e. a $(1,1)$-tensor $\varphi : TM^5 \to TM^5$, and 1-form η with dual vector field ξ of length one such that

$$(7.3) \qquad \varphi^2 = -\mathrm{Id} + \eta \otimes \xi \qquad \text{and} \qquad g(\varphi V, \varphi W) = g(V, W) - \eta(V)\eta(W).$$

The fundamental form is then defined by $F(X, Y) := g(X, \varphi Y)$. Moreover, a Nijenhuis tensor N is defined by an expression very similar to the one known from almost complex structures. A manifold with a metric almost contact structure is called

(1) a *quasi-Sasakian manifold* if $N = 0$ and $dF = 0$,

(2) an *α-Sasakian manifold* if $N = 0$ and $d\eta = \alpha F$. The Sasaki case corresponds to $\alpha = 2$.

Theorem 7.10 (case B – induced contact structure). *Let (M^5, g, T) be an orientable Riemannian 5-manifold with parallel skew torsion T such that $\sigma_T \neq 0$. Then M is a quasi-Sasakian manifold and ∇ is its characteristic connection. The quasi-Sasakian structure is α-Sasakian if and only if $\lambda = \varrho$ (case B.2), and it is Sasakian if $\lambda = \varrho = 2$.*

Proof. The vector field $V := *\sigma_T \neq 0$ is ∇-parallel, and therefore Killing and of constant length. It satisfies $V \lrcorner \sigma_T = 0$ because $V \lrcorner (*V) = 0$. It defines a ∇-parallel Killing vector field ξ of unit length,

$$\xi = \frac{1}{\|\sigma_T\|} * \sigma_T = \frac{1}{\varrho\lambda} * \sigma_T = e_5.$$

Denote by η its dual 1-form; in general, we identify vectors and 1-forms. Clearly $\nabla \eta = 0$ as well, and the formulas for computing the differential from the covariant derivative yield then

$$d\eta(X,Y) = \eta(T(X,Y)) = g(\xi, T(X,Y)) = T(X,Y,\xi),$$

hence, in the normal form of T, $d\eta = de_5 = -(\varrho e_{12} + \lambda e_{34})$. In particular, both expressions together show that $T = \eta \wedge d\eta$. Furthermore, $\nabla \xi = 0$ is equivalent to $\nabla^g_X \xi = -T(X,\xi)/2$, hence

$$\nabla^g \xi = -(\frac{\varrho}{2} e_{12} + \frac{\lambda}{2} e_{34}).$$

We define an endomorphism φ and its corresponding 2-form $F(X,Y) = g(X, \varphi Y)$ by

$$F = -(e_{12} + e_{34}), \quad \varphi = e_{12} + e_{34}.$$

Let us remark that F and φ are globally well-defined. If $\varrho \neq \lambda$, $\mathfrak{m}_1^2 = \langle e_1, e_2 \rangle$, $\mathfrak{m}_2^2 = \langle e_3, e_4 \rangle$ are 2-dimensional ∇-parallel distributions, the 2-forms $\Omega_1 := e_{12}$ and $\Omega_2 := e_{34}$ are ∇-parallel and globally defined. If $\varrho = \lambda$, F is proportional to $d\eta$, and again well-defined. For later, we observe that $\nabla^g \xi = -\varphi$ if and only if $\varrho = \lambda = 2$, and this is equivalent to $2F = d\eta$. One checks that φ, η and ξ satisfy conditions (7.3), which means that (ξ, η, φ) defines a metric almost contact structure. We now prove that this structure is quasi-Sasakian. First, consider F. For $\varrho = \lambda$, we have $\varrho dF = dd\eta = 0$. Furthermore, F is proportional to $\xi \lrcorner T$, and this is a parallel 2-form, so $\nabla F = 0$ as well, which is equivalent to $\nabla \varphi = 0$. If $\varrho \neq \lambda$, we just observed that Ω_1 and Ω_2 are ∇-parallel, hence $\nabla F = 0$, and we can conclude with formula (2.6) that $d\Omega_1 = d\Omega_2 = 0$, so $dF = 0$. To sum up, we obtain $\nabla F = 0$ and $dF = 0$ for all coefficients $\varrho \lambda \neq 0$. This is the first of the two conditions for a quasi-Sasakian structure. If a metric almost contact structure admits a connection with skew symmetric torsion, then $dF = 0$ implies that the Nijenhuis tensor vanishes, $N = 0$, see [FI02, Theorem 8.4]. Thus, the structure is quasi-Sasakian as claimed. $\square$

8. The classification in dimension 6

A naturally reductive 6-space whose torsion has non-trivial kernel is locally a product of lower-dimensional spaces of the same kind by Theorem 3.2. This applies, for example, to a torsion that has the normal form (7.1) and that depends on a 5-dimensional subspace only. Consequently, we will assume henceforth that $\ker T = 0$. We will discuss the different cases according to the algebraic properties of σ_T. Observe that $*\sigma_T$ is a 2-form and can hence be identified with a skew-symmetric endomorphism. Thus, $\mathrm{rk}(*\sigma_T)$ is either $0, 2, 4$, or 6; in analogy to the discussion in five dimensions, we will label these cases A, B, C, and D.

First, we treat the degenerate case $\sigma_T = 0$ (case A): this covers, for example, torsions with local normal form $\mu\, e_{123} + \nu\, e_{456}$. Our result shows that this is basically the only possibility:

Theorem 8.1 (case A – with parallel skew torsion)**.** *Let (M^6, g, T) be a complete, simply connected Riemannian 6-manifold with parallel skew torsion T such that $\sigma_T = 0$ and $\ker T = 0$. Then M^6 splits into two 3-dimensional manifolds with parallel skew torsion,*

$$(M^6, g, T) = (N_1^3, g_1, T_1) \times (N_2^3, g_2, T_2).$$

Proof. First, we argue that the Lie algebra $\mathfrak{g}_T$ cannot act irreducibly on the tangent space $V = T_p M^6$. If so, it would define an irreducible skew holonomy system, so $\mathfrak{g}_T = \mathfrak{so}(6)$ by Theorem B.2, since there are no 6-dimensional compact simple Lie algebras. But equation (2.2) and the condition $\sigma_T = 0$ imply

$$\mathfrak{g}_T = \mathfrak{so}(6) \subset \mathfrak{iso}(T),$$

hence $T = 0$, a contradiction. The assumption $\ker T = 0$ yields that V can only split into two 3-dimensional $\mathfrak{g}_T$-invariant subspaces. Now the claim follows from the splitting Theorem 3.3. $\square$

Corollary 8.2 (case A – the naturally reductive case)**.** *Any 6-dimensional simply connected naturally reductive space with $\sigma_T = 0$ and $\ker T = 0$ is isometric to a product of two 3-dimensional naturally reductive spaces.*

Next we classify the case where $\ker T = 0$ and the 2-form $*\sigma_T$ has rank 2,

$$*\sigma_T \;=\; \rho\, e_{56}, \quad \sigma_T \;=\; \rho\, e_{1234}.$$

The condition $\nabla T = 0$ implies $\nabla \sigma_T = 0$, i.e. $\mathfrak{hol}^\nabla \subset \mathfrak{iso}(T) \subset \mathfrak{so}(4) \oplus \mathfrak{so}(2)$. We use again equation (2.2),

$$(e_5 \,\lrcorner\, T)(T) \;=\; e_5 \,\lrcorner\, \sigma_T \;=\; 0 \;=\; e_6 \,\lrcorner\, \sigma_T \;=\; (e_6 \,\lrcorner\, T)(T)$$

and conclude that the 2-forms $e_5 \,\lrcorner\, T, e_6 \,\lrcorner\, T$ belong to the Lie algebra $\mathfrak{iso}(T) \subset \mathfrak{so}(4) \oplus \mathfrak{so}(2)$. Consequently, T does not contain a term of type $v \wedge e_5 \wedge e_6$, $v \in \mathbb{R}^4$. Let us write the 3-form T as

$$T \;=\; T_1 \,+\, \Omega_1 \wedge e_5 \,+\, \Omega_2 \wedge e_6$$

where $T_1 \in \Lambda^3(\mathbb{R}^4)$ and $\Omega_1, \Omega_2 \in \Lambda^2(\mathbb{R}^4)$ are forms in $\mathbb{R}^4$. Then $e_5 \,\lrcorner\, T = \Omega_1$ and $e_6 \,\lrcorner\, T = \Omega_2$, i.e. $\Omega_1, \Omega_2 \in \mathfrak{iso}(T)$ are linearly independent 2-forms preserving T. In particular, they preserve T_1 and commute,

$$\Omega_1 \wedge \Omega_2 \;=\; \Omega_2 \wedge \Omega_1.$$

Denote by $\mathfrak{so}(2) \oplus \mathfrak{so}(2) \subset \mathfrak{so}(4)$ the maximal abelian subalgebra of $\mathfrak{so}(4)$ generated by Ω_1 and Ω_2. The form $T_1 \in \Lambda^3(\mathbb{R}^4) \cong \mathbb{R}^4$ is invariant under $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$, but this subalgebra has no fixed vectors in $\mathbb{R}^4$. We conclude that T_1 vanishes. Moreover, we obtain

$$\mathfrak{hol}^\nabla \;\subset\; \mathfrak{iso}(T) \;=\; \mathfrak{so}(2) \oplus \mathfrak{so}(2) \;\subset\; \mathfrak{so}(4) \;\subset\; \mathfrak{so}(6).$$

Up to a conjugation in $\mathfrak{so}(4)$ we may assume that

$$\Omega_1 \;=\; \alpha\,(e_{12} + e_{34}), \quad \Omega_2 \;=\; \beta\,(e_{12} - e_{34})$$

and

(8.1) $$T \;=\; \alpha\,(e_{12} + e_{34}) \wedge e_5 + \beta\,(e_{12} - e_{34}) \wedge e_6.$$

We compute the square σ_T directly,

$$\sigma_T \;=\; (\alpha^2 - \beta^2)\, e_{1234} \;=\; \rho\, e_{1234}.$$

In particular, the length ρ of σ_T is given by α and β. The curvature operator $\mathcal{R}$ is the composition of the projection onto $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ with an invariant linear map. Since $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ is abelian, $\mathcal{R}$ depends on three parameters,

(8.2) $$\mathcal{R} \;=\; \frac{a}{\alpha^2}\, \Omega_1 \odot \Omega_1 + \frac{2\,c}{\alpha\beta}\, \Omega_1 \odot \Omega_2 + \frac{b}{\beta^2}\, \Omega_2 \odot \Omega_2.$$

The Bianchi identity, i.e. that $T^2 + \mathcal{R}$ should be a scalar in the Clifford algebra, yields the relation

$$\alpha^2 - \beta^2 - a + b \;=\; 0\,.$$

Finally, the curvature argument from Proposition 2.2 yields the following result:

Theorem 8.3 (case B – with parallel skew torsion)**.** *Let (M^6, g, T) be a complete, simply connected Riemannian 6-manifold with parallel skew torsion and such that $\ker T = 0$ and $\mathrm{rk}\,(*\sigma_T) = 2$. Then $\nabla \mathcal{R} = 0$, i.e. M is naturally reductive, and the family of admissible torsion forms and curvature operators depends on 5 parameters α, β, a, b, c satisfying the equation $\alpha^2 - \beta^2 - a + b = 0$. The torsion and the curvature are given by the explicit formulas (8.1) and (8.2).*

In this situation, too, we describe explicitly the naturally reductive space M^6 and the 8-dimensional group G acting transitively on it. Its Lie algebra $\mathfrak{g}$ is an 8-dimensional vector space with basis $e_1, \ldots, e_6$ and

$$\tilde{\Omega}_1 \;:=\; -\Big[\frac{a+c}{\alpha}\,\Omega_1 + \frac{b+c}{\beta}\,\Omega_2 + \alpha\, e_5 + \beta\, e_6\Big], \quad \tilde{\Omega}_2 \;:=\; -\Big[\frac{a-c}{\alpha}\,\Omega_1 + \frac{c-b}{\beta}\,\Omega_2 + \alpha\, e_5 - \beta\, e_6\Big]$$

We compute all commutator relations and observe that the linear space $\mathfrak{g}_1$ spanned by e_1, e_2, e_3, e_4 and $\tilde{\Omega}_1, \tilde{\Omega}_2$ is a Lie subalgebra such that $\mathfrak{g}_1 \cap \mathrm{span}(\Omega_1, \Omega_2) = 0$. Consequently, the simply

connected manifold M^6 is isometric to the 6-dimensional Lie group G_1 defined by $\mathfrak{g}_1$ and equipped with a family of left-invariant metrics. The non-vanishing commutator relations in $\mathfrak{g}_1$ are

$$[e_1, e_2] = \tilde{\Omega}_1, \quad [\tilde{\Omega}_1, e_1] = -(a + b + 2c - \alpha^2 - \beta^2)\, e_2, \quad [\tilde{\Omega}_1, e_2] = (a + b + 2c - \alpha^2 - \beta^2)\, e_1,$$

$$[e_3, e_4] = \tilde{\Omega}_2, \quad [\tilde{\Omega}_2, e_3] = -(a + b - 2c - \alpha^2 - \beta^2)\, e_4, \quad [\tilde{\Omega}_2, e_4] = (a + b - 2c - \alpha^2 - \beta^2)\, e_3.$$

The Lie algebra $\mathfrak{g}_1$ splits into two 3-dimensional ideals and any of them is isomorphic to $\mathfrak{so}(3)$, $\mathfrak{sl}(2, \mathbb{R})$ or to the Heisenberg algebra $\mathfrak{h}^3$. We have thus described completely the naturally reductive spaces classified in the previous Theorem. Remark that in the flat case $(a = b = c = 0)$ we obtain the result of Cartan and Schouten, see [AF10a], whereby a ∇-flat space with skew symmetric torsion is a product of compact simple Lie groups and the 7-dimensional sphere.

Theorem 8.4 (case B – classification). *A complete, simply connected Riemannian* 6-*manifold with parallel skew torsion T and* $\mathrm{rk}\,(*\sigma_T) = 2$ *is the product $G_1 \times G_2$ of two Lie groups equipped with a family of left-invariant metrics. G_1 and G_2 are either $S^3 = \mathrm{SU}(2)$, $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, or H^3.*

Remark 8.5. An explicit description of naturally reductive structures with $\mathrm{rk}\,(*\sigma_T) = 2$ on $S^3 \times S^3$ may be found in [Sch07, Exa 4.8]. They are disjoint to the examples with $\mathrm{rk}\,(*\sigma_T) = 6$ that will be described in Section 9.4.

Example 8.6. There are 3-forms $T \in \Lambda^3(\mathbb{R}^6)$ such that the corresponding 2-form $*\, \sigma_T \in \Lambda^2(\mathbb{R}^6)$ has rank 4. For example, consider the family

$$\begin{aligned} T \;&=\; e_5 \wedge (ae_{12} + be_{13} + ce_{14} + de_{23} + fe_{24} + he_{34}) \\ &+\; e_6 \wedge (se_{12} + te_{13} + ue_{14} + ve_{23} + we_{24} + xe_{34}) \end{aligned}$$

depending on 12 parameters. It turns out that the vectors e_5, e_6 are in the kernel of $*\,\sigma_T$ if and only if the equation

$$cd - bf + ah + uv - tw + sx \;=\; 0$$

is satisfied. Consequently, a generic choice of parameters constrained by the latter equation yields 3-forms with $\mathrm{rk}\,(*\sigma_T) = 4$. For example, the parameter values $a = b = c = d = h = -f = 1$, $s = \frac{1}{2}$, $t = u = w = -v = 1$, $x = -2$ are one out of many solutions with $\mathrm{rk}\,(*\sigma_T) = 4$. Surprisingly, our next result states that such 3-forms cannot occur as torsions forms of metric connections with parallel skew torsion, thus showing that $\nabla T = 0$ is a severe restriction on the algebraic type of a 3-form.

Almost Hermitian 6-manifolds have structure group $\mathrm{U}(3) \subset \mathrm{SO}(6)$. If we decompose $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}$, they are classified by their intrinsic torsion, which is an element of $\mathfrak{m} \otimes \mathbb{R}^6$. This was first done by Gray and Hervella in [GH80], who introduced the by-now standard notation

$$\mathfrak{m} \otimes \mathbb{R}^6 \;=\; \mathcal{W}_1^{(2)} \oplus \mathcal{W}_2^{(16)} \oplus \mathcal{W}_3^{(12)} \oplus \mathcal{W}_4^{(6)}$$

for its decomposition into $\mathrm{U}(3)$-modules (the upper index indicates the real dimension and will be dropped later; see also [AFS05] for a modern account). An almost Hermitian 6-manifold admits a characteristic connection if and only if the intrinsic torsion has no $\mathcal{W}_2$-component [FI02]. More precisely, $\Lambda^3(\mathbb{R}^6) = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ as a $\mathrm{U}(3)$-module, the torsion of the characteristic connection is a linear combination $T = T_1 + T_3 + T_4$ with $T_i \in \mathcal{W}_i$ and the Gray-Hervella class of the almost Hermitian structure corresponds to the non-vanishing contributions of T. For example, almost Hermitian manifolds of class $\mathcal{W}_1$ go under the name nearly Kähler manifolds, while Hermitian manifolds (vanishing Nijenhuis tensor) are of class $\mathcal{W}_3 \oplus \mathcal{W}_4$.

Theorem 8.7 (cases C and D – eigenvalues of σ_T). *If T is the torsion form of a metric connection ∇ in dimension 6 with parallel skew torsion, $\ker T = 0$ and $\mathrm{rk}\,(*\sigma_T) \geq 4$, then all eigenvalues of $*\sigma_T$ are equal. The* 2-*form defines a ∇-parallel almost Hermitian structure J of Gray-Hervella class $\mathcal{W}_1 \oplus \mathcal{W}_3$. In particular, the case $\mathrm{rk}\,(*\sigma_T) = 4$ cannot occur.*

Proof. If the eigenvalues of $*\sigma_T$ are equal, this 2-form can be viewed, up to a factor, as the Kähler form of an almost Hermitian structure. The metric as well as the almost complex structure are preserved by a connection with totally skew symmetric torsion. This implies that the $\mathcal{W}_2$-part in the Gray-Hervella classification vanishes. Moreover, $dT = 2\,\sigma_T$ yields that $*\sigma_T$ is coclosed, $\delta(*\sigma_T) = d(\sigma_T) = 0$, i.e. the $\mathcal{W}_4$-part vanishes too (see [GH80], [FI02] or [AFS05]).

By contradiction, let use assume that $*\sigma_T$ has two different eigenvalues. At least one of them defines a ∇-parallel 2-dimensional subbundle E^2, i.e. the tangent bundle splits into $TM^6 = E^2 \oplus F^4$ and the holonomy algebra of ∇ is contained in $\mathfrak{so}(E^2) \oplus \mathfrak{so}(F^4)$. In case $\mathrm{rk}\,(*\sigma_T) = 4$, we put $E^2 := \ker(*\sigma_T)$. Thus, the 2-form $*\sigma_T$ is non-degenerate on the subbundle F^4 in all cases. We fix a local frame e_5, e_6 in E^2. First we prove that the subbundle F^4 does not admit non-trivial ∇-parallel vector fields. In fact, if V is one, then $V \lrcorner (*\sigma_T)$ is ∇-parallel too. This restricts the holonomy algebra,

$$\mathfrak{hol}^\nabla \;\subset\; \mathfrak{so}(E^2) \oplus \mathfrak{u},$$

where $\mathfrak{u} \subset \mathfrak{so}(F^4)$ is a 1-dimensional subalgebra. We fix a basis $e_1, e_2, e_3 = V, e_4 = V \lrcorner (*\sigma_T)$ in F^4 as well as a generator Ω of the subalgebra $\mathfrak{u} \subset \mathfrak{so}(F^4)$. Since $e_3 \lrcorner \Omega = e_4 \lrcorner \Omega = 0$, Ω is given by $\Omega = a\,e_{12}$. Consider the curvature operator of ∇,

$$\mathcal{R} \;=\; \alpha\,e_{56} \odot e_{56} \,+\, \beta\,e_{56} \odot \Omega \,+\, \gamma\,\Omega \odot \Omega \,.$$

In the Clifford algebra the Bianchi identity reads as $-2\,\sigma_T + \mathcal{R} = T^2 + \mathcal{R} = 0$ modulo scalars. This implies the following formula for σ_T,

$$\sigma_T \;=\; \frac{\alpha\,\beta}{2}\,e_{1256}\,,$$

i.e. $2 * \sigma_T = \alpha\beta\,e_{34}$ has rank 2, a contradiction. This proves that F^4 has no ∇-parallel vector fields. We apply this fact to $e_{56} \lrcorner T$ and to the restriction $T_{|F^4} \in \Lambda^3(F^4) \cong F^4$. They are ∇-parallel vector fields in F^4, hence they should vanish. Therefore, the torsion form can be written locally as

$$T \;=\; e_5 \wedge \Omega_1 \,+\, e_6 \wedge \Omega_2$$

where $\Omega_1, \Omega_2 \in \mathfrak{so}(F^4)$ are 2-forms in F^4. Since the kernel of T is trivial and $\Omega_1 = e_5 \lrcorner T, \Omega_2 = e_6 \lrcorner T$, the forms Ω_1, Ω_2 are linearly independent. Using the well-known formulas for multiplication inside the Clifford algebra

$$T^2 \;=\; -2\,\sigma_T + \|T\|^2, \quad \Omega_i^2 \;=\; -\|\Omega_i\|^2 + \Omega_i \wedge \Omega_i, \quad i = 1, 2$$

we can compute σ_T :

$$2\,\sigma_T \;=\; (\Omega_1 \wedge \Omega_1 + \Omega_2 \wedge \Omega_2) - \, [\Omega_1, \Omega_2] \wedge e_{56}.$$

The forms Ω_1, Ω_2 and $[\Omega_1, \Omega_2]$ are linearly independent, otherwise the rank of σ_T would not be six or the Lie algebra $\mathfrak{so}(F^4)$ would contain a non-abelian 2-dimensional subalgebra. Since σ_T and e_{56} are ∇-parallel, the 2-form $[\Omega_1, \Omega_2]$ is ∇-parallel too. The holonomy algebra $\mathfrak{hol}^\nabla$ is at least one-dimensional (see Proposition 2.2) and preserves the 2-dimensional spaces $\mathrm{span}(e_5, e_6)$ and $\mathrm{span}(\Omega_1, \Omega_2)$. Consider an arbitrary element $\Sigma = A\,e_{56} + \Sigma_1 \in \mathfrak{so}(E^2) \oplus \mathfrak{so}(F^4)$. The element Σ preserves T if and only if

$$\Sigma(e_5) \;=\; A\,e_6, \quad \Sigma(e_6) \;=\; -A\,e_5, \quad [\Sigma_1, \Omega_1] \;=\; A\,\Omega_2, \quad [\Sigma_1, \Omega_2] \;=\; -A\,\Omega_1.$$

The holonomy algebra cannot lie completely inside $\mathfrak{so}(F^4)$. Indeed, the existence of a curvature operator $\mathcal{R} : \Lambda^2 \to \mathfrak{hol}^\nabla$ as well as the Bianchi identity implies in this case that $2\sigma_T = \alpha\,e_{1234}$, i.e. $*\sigma_T$ has rank ≤ 2, a contradiction. Consequently, there exists at least one element $\Sigma = A\,e_{56} + \Sigma_1 \in \mathfrak{hol}^\nabla$ with $A \neq 0$. The commutator relations imply that the 2-forms Ω_1, Ω_2 are orthogonal with equal lengths. Then Ω_1, Ω_2 generate a Lie algebra $\mathfrak{h}$ such that $\dim \mathfrak{h} \geq 3$.

If $\dim \mathfrak{h} = 3$, the vector space $\mathrm{span}(\Omega_1, \Omega_2, [\Omega_1, \Omega_2])$ is a Lie subalgebra. Since the kernel of T is trivial, the representation of this algebra in $\mathbb{R}^4$ is irreducible. Therefore this algebra can be

conjugated into $\mathfrak{su}(2) \cong \Lambda_-^2(F^4)$ or $\Lambda_+^2(F^4)$. We know already that Ω_1, Ω_2 are orthonormal and have the same length, hence (in an appropriate basis)

$$\Omega_1 \;=\; e_{12} - e_{34}, \; \Omega_2 \;=\; e_{13} + e_{24}, \quad 2\,\sigma_T \;=\; -4\,e_{1234} + 4\,(e_{23} - e_{14}) \wedge e_{56},$$

i. e. the eigenvalues of σ_T are equal, a contradiction. Thus, $\dim \mathfrak{h} = 3$ cannot occur. In the next step we prove that the holonomy algebra is one-dimensional, $\dim \mathfrak{hol}^\nabla = 1$. Indeed, suppose that $\dim \mathfrak{hol}^\nabla \geq 2$. Then there exists an element $0 \neq \Sigma^* \in \mathfrak{hol}^\nabla \cap \mathfrak{so}(F^4)$ and

$$\Sigma^*(e_5) \;=\; \Sigma^*(e_6) \;=\; 0, \quad [\Sigma^*, \Omega_1] \;=\; [\Sigma^*, \Omega_2] \;=\; 0$$

hold. Then Σ^* commutes with the whole Lie algebra $\mathfrak{h}$ generated by Ω_1, Ω_2, $[\Sigma^*, \mathfrak{h}] = 0$. Since the dimension of $\mathfrak{h}$ is at least 4, the existence of Σ^* implies automatically that $\dim \mathfrak{h} = 4, \Sigma^* \in \mathfrak{h}$. Moreover, Σ^* is the central element of $\mathfrak{h}$. Up to a conjugation inside $\mathfrak{so}(F^4)$, we know the Lie algebra has to be

$$\mathfrak{h} \;=\; \Lambda_-^2(F^4) \oplus \mathbb{R} \cdot (\Sigma^*)$$

and

$$\Sigma^* \;=\; e_{12} + e_{34}, \; \Omega_1 \;=\; \Omega_1^- + a\,\Sigma^*, \; \Omega_2 \;=\; \Omega_2^- + b\,\Sigma^*, \; \Omega_1^-, \Omega_2^- \in \Lambda_-^2(F^4)$$

for some constants $a, b \in \mathbb{R}$. We use once again the existing element with $A \neq 0$, namely $\Sigma = A e_{56} + \Sigma_1 = A\,e_{56} + \Sigma_1^+ + \Sigma_1^- \in \mathfrak{hol}^\nabla$ with $\Sigma_1^\pm \in \Lambda_\pm^2(F^4)$. Comparing the $\Lambda_\pm^2$-parts in $[\Sigma_1, \Omega_1] = A\,\Omega_2, [\Sigma_1, \Omega_2] = -A\,\Omega_1$ we obtain

$$A\,\Omega_2^- \;=\; [\Sigma_1^-, \Omega_1^-], \quad Ab\,\Sigma^* \;=\; a\,[\Sigma_1^+, \Sigma^*], \quad -A\,\Omega_1^- \;=\; [\Sigma_1^-, \Omega_2^-], \quad -Aa\,\Sigma^* \;=\; b\,[\Sigma_1^+, \Sigma^*].$$

The elements Σ^* and $[\Sigma_1^+, \Sigma^*]$ are orthogonal in Λ_+^2. Since $A \neq 0$ we conclude that $a = b = 0$. Then $\Omega_1, \Omega_2 \in \Lambda_-^2$ and the algebra $\mathfrak{h}$ is 3-dimensional; but we know already that this case is impossible. Consequently, the element $0 \neq \Sigma^*$ cannot exist, i. e. $\dim \mathfrak{hol}^\nabla = 1$. This algebra is generated by one element $\Sigma = A e_{56} + \Sigma_1, A \neq 0, \Sigma_1 \in \Lambda^2(F^4)$. Using the curvature operator as well as the Bianchi identity again, we obtain relations between Σ_1 and Ω_1, Ω_2,

$$2A\,\Sigma_1 \;=\; -[\Omega_1, \Omega_2], \; [\Sigma_1, \Omega_1] \;=\; A\,\Omega_2, \; [\Sigma_1, \Omega_2] \;=\; -A\,\Omega_2.$$

The vector space $\mathrm{span}(\Omega_1, \Omega_2, [\Omega_1, \Omega_2])$ becomes a Lie algebra and coincides with $\mathfrak{h}$, again a contradiction. This finishes the proof. $\square$

Remark 8.8. Any 2-form $*\sigma_T$ of rank ≥ 4 defines (locally) in a natural way a ∇-parallel almost Hermitian structure J on (M^6, g). N. Schoemann classified in the paper [Sch07] all admissible torsion forms of almost Hermitian structures with characteristic torsion, up to conjugation under U(3). Applying these explicit formulas for T, one can compute $*\sigma_T$ in all cases. This yields an alternative, more computational proof of Theorem 8.7.

Consider a complete, simply connected Riemannian 6-manifold (M^6, g, T) with parallel skew torsion T, $\mathrm{rk}\,(*\sigma_T) = 6$ and $\ker T = 0$. If the holonomy representation is $\mathbb{C}$-reducible, we obtain again a ∇-parallel decomposition $T(M^6) = E^2 \oplus F^4$ and the torsion form is given by $T = e_5 \wedge \Omega_1 + e_6 \wedge \Omega_2$. The proof of Theorem 8.7 says that the linear space $\mathfrak{h} := \mathrm{span}(\Omega_1, \Omega_2, [\Omega_1, \Omega_2])$ is a 3-dimensional Lie subalgebra of $\mathfrak{so}(F^4)$ and that it can be conjugated into $\Lambda_\pm^2(F^4)$. The two possible cases $\mathfrak{h} = \Lambda_\pm^2(F^4)$ yield torsion forms of pure type $\mathcal{W}_1$ or $\mathcal{W}_3$, respectively. We conclude that (M^6, g) is isometric to $\mathbb{CP}^3$ or $F(1,2)$, the twistor spaces of the 4-dimensional spaces S^4 or $\mathbb{CP}^2$ equipped with their nearly Kähler metric, see [BM01] and [AFS05, Thm 4.5] (the two different almost complex structures correspond to different orientations of J in the fibre direction). If the holonomy representation is $\mathbb{C}$-irreducible, there are four possibilities:

$$\mathrm{Hol}^\nabla \;=\; \mathrm{U}(3), \; \mathrm{SU}(3), \; \mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3) \subset \mathrm{SU}(3), \; \mathrm{U}(2)/\{\pm 1\} \subset \mathrm{U}(3).$$

The first case gives $T = 0$. If $\mathfrak{hol}^\nabla = \mathfrak{su}(3)$, the almost Hermitian structure is of pure type $\mathcal{W}_1$ and (M^6, g) is isometric to a nearly Kähler 6-manifold with irreducible holonomy of ∇. The discussion of the other cases provides the following final result:

Theorem 8.9 (case D – with parallel skew torsion)**.** *Let (M^6, g, T) be a complete, simply connected Riemannian 6-manifold with parallel skew torsion T, $\mathrm{rk}\,(\ast\sigma_T) = 6$ and $\ker T = 0$. Then one of the following cases occurs:*

> *Case D.1: (M^6, g) is isometric to a nearly Kähler 6-manifold.*
> *Case D.2: $\mathfrak{hol}^\nabla = \mathfrak{so}(3) \subset \mathfrak{su}(3)$ and (M^6, g) is naturally reductive. The family depends on three parameters and is of type $\mathcal{W}_1 \oplus \mathcal{W}_3$ in the Gray-Hervella classification of almost Hermitian structures.*

Furthermore, the case $\mathfrak{hol}^\nabla = \mathfrak{u}(2) \subset \mathfrak{u}(3)$ is not possible.

Proof. Let us fix a Lie subgroup $\mathrm{SO}(3) \subset \mathrm{SU}(3) \subset \mathrm{SO}(6)$; for computations, we choose the Lie subalgebra $\mathfrak{so}(3)$ given by

$$H_1 := -2(e_{35} + e_{46}), \quad H_3 := 2(e_{15} + e_{26}), \quad H_5 := -2(e_{13} + e_{24})$$

with commutators $[H_1, H_3] = -2H_5$, $[H_1, H_5] = 2H_3$, $[H_3, H_5] = -2H_1$. The group $\mathrm{SU}(3)$ acts trivially on $\mathcal{W}_1 \subset \Lambda^3(\mathbb{R}^6)$, which is spanned by the two elements

$$-e_{135} + e_{245} + e_{236} + e_{146} =: -e_{135} + \eta_1, \quad -e_{246} + e_{136} + e_{145} + e_{235} =: -e_{246} + \eta_2.$$

The space of $\mathrm{SO}(3)$-invariant 3-forms in $\mathcal{W}_3 \subset \Lambda^3(\mathbb{R}^6)$ has complex dimension one, and, for our choice of $\mathfrak{so}(3)$, it is spanned by $3\,e_{135} + \eta_1$ and $3\,e_{246} + \eta_2$ [AFS05, p.7 and Thm 4.4]. Hence the general $\mathrm{SO}(3)$-invariant 3-form in $\mathcal{W}_1 \oplus \mathcal{W}_3$ lies in $\mathrm{span}(e_{135}, e_{246}, \eta_1, \eta_2)$ and depends a priori on 4 real parameters. The one-dimensional center of $\mathrm{U}(3)$ leaves the Lie algebra $\mathfrak{so}(3) \subset \mathfrak{su}(3)$ invariant and its generator acts on $\mathrm{span}(e_{135}, e_{246}, \eta_1, \eta_2)$ by transforming e_{135} into e_{246} and η_1 into η_2. Consequently, it is possible to conjugate by a central element of $\mathrm{U}(3)$ in such a way that η_2 disappears. With this choice, the general $\mathrm{SO}(3)$-invariant 3-form in $\mathcal{W}_1 \oplus \mathcal{W}_3$ can be parametrized as

$$(8.3) \quad T = \alpha\,e_{135} + \alpha'\,e_{246} + \beta\,(e_{245} + e_{236} + e_{146}), \quad \text{hence } \sigma_T = \beta(\beta - \alpha)\,(e_{1256} + e_{1234} + e_{3456}).$$

We see that we need to require $\beta \neq 0$ and $\alpha \neq \beta$ to ensure that $\mathrm{rk}\,(\ast\sigma_T) = 6$. The curvature operator $\mathcal{R} : \Lambda^2(\mathbb{R}^6) \to \mathfrak{so}(3)$ depends a priori on nine parameters x_{ij},

$$\mathcal{R} = \sum_{i,j=1}^3 x_{ij}\, H_i \odot H_j\,.$$

The Bianchi identity $2\sigma_T = \mathcal{R}$ modulo scalars in the Clifford algebra constrains the coefficients, namely $x_{11} = x_{22} = x_{33}$ and $x_{ij} = 0$ for $i \neq j$. In particular, the curvature operator is the projection onto the subalgebra $\mathfrak{so}(3)$. Consequently, $\mathcal{R}$ is $\mathfrak{hol}^\nabla$-invariant and the space is naturally reductive. The explicit formula for the curvature is

$$(8.4) \qquad\qquad \mathcal{R} = \beta(\alpha - \beta)[(e_{35} + e_{46})^2 + (e_{15} + e_{26})^2 + (e_{13} + e_{24})^2]\,.$$

The torsion and the curvature depend on three parameters and they describe the naturally reductive spaces completely. The underlying almost Hermitian manifold is of pure type $\mathcal{W}_1$ or $\mathcal{W}_3$ if $\alpha + \beta = 0$, $\alpha' = 0$ or resp. $\alpha = 3\beta$, $\alpha' = 0$. In the next paragraph we will describe these spaces as well as the transitive automorphism groups in a more explicit way. Finally, the case of the irreducible representation $\mathrm{Hol}^\nabla = \mathrm{U}(2)/\{\pm 1\} \subset \mathrm{U}(3)$ cannot occur. Indeed, there is no admissible torsion in $\mathcal{W}_3 \subset \Lambda^3(\mathbb{R}^6)$, see [AFS05, Thm 4.4]. $\qquad\square$

Remark 8.10. Homogeneous nearly Kähler 6-manifolds have been classified in [Bu05]. The list is short: $S^6, S^3 \times S^3, \mathbb{CP}^3$, and the flag manifold $F(1, 2) = \mathrm{U}(3)/\mathrm{U}(1)^3$, all equipped with their unique strict nearly Kähler metrics. It is well known that they are naturally reductive, and that they are precisely the Riemannian 3-symmetric spaces. Using the result of Butruille and the previous Theorem, we immediately obtain the classification of all naturally reductive spaces in case D.1.

Remark 8.11. All naturally reductive spaces described in Theorem 8.9 are Einstein with parallel skew torsion (see [AF14] for details on this notion). Indeed, the expression (8.4) for the curvature implies that the Ricci tensor of the connection ∇ is given by

$$\mathrm{Ric} = -2\beta(\alpha - \beta)\,\mathrm{Id}.$$

The difference $\mathrm{Ric} - \mathrm{Ric}^g$ depends only on the torsion tensor; one checks that it is a multiple of the identity if and only if $\alpha' = 0$ and $\alpha = \pm\beta$. Hence, these are the only situations where the space will be Riemannian Einstein, and these are well known: $\alpha' = 0, \alpha = \beta$ is the bi-invariant metric on $S^3 \times S^3$ (recall that $\mathcal{R} = 0$ in this case), while $\alpha' = 0, \alpha = -\beta$ corresponds to nearly Kähler manifolds.

We shall now carry out the Nomizu construction (see Appendix A) for the naturally reductive spaces covered by case D.2 of the previous theorem; we keep the Ansatz for the torsion (equation (8.3)) and the notation of the proof. The only non-vanishing commutators on $\mathrm{span}(H_1, H_3, H_5) \oplus \mathrm{span}(e_1, \ldots, e_6)$ are then (besides the commutators of the elements H_i already mentioned in the above proof)

$$[e_1, e_3] = \frac{\beta(\alpha - \beta)}{2} H_5 - \alpha e_5, \quad [e_2, e_4] = \frac{\beta(\alpha - \beta)}{2} H_5 - \beta e_5 - \alpha' e_6, \quad [e_1, e_4] = [e_2, e_3] = -\beta e_6,$$

$$[e_1, e_5] = \frac{\beta(\beta - \alpha)}{2} H_3 + \alpha e_3, \quad [e_2, e_6] = \frac{\beta(\beta - \alpha)}{2} H_3 + \beta e_3 + \alpha' e_4, \quad [e_1, e_6] = [e_2, e_5] = +\beta e_4,$$

$$[e_3, e_5] = \frac{\beta(\alpha - \beta)}{2} H_1 - \alpha e_1, \quad [e_4, e_6] = \frac{\beta(\alpha - \beta)}{2} H_1 - \beta e_1 - \alpha' e_2, \quad [e_3, e_6] = [e_4, e_5] = -\beta e_2,$$

as well as

$$[e_5, H_3] = [H_5, e_3] = 2e_1, \quad [e_1, H_5] = [H_1, e_5] = 2e_3, \quad [e_3, H_1] = [H_3, e_1] = 2e_5,$$
$$[e_6, H_3] = [H_5, e_4] = 2e_2, \quad [e_2, H_5] = [H_1, e_6] = 2e_4, \quad [e_4, H_1] = [H_3, e_2] = 2e_6.$$

In the next step, we find a 6-dimensional Lie subalgebra. Define $\Omega_i := e_i + \frac{\beta - \alpha}{2} H_i$ for $i = 1, 3, 5$, $\mathfrak{h} := \mathrm{span}(\Omega_1, \Omega_3, \Omega_5)$, and $\mathfrak{m} := \mathrm{span}(e_2, e_4, e_6)$. Then $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$ is indeed 6-dimensional and, as the following non-vanishing commutators show, closed under the Lie bracket. More precisely, $\mathfrak{h}$ is a 3-dimensional Lie subalgebra,

$$[\Omega_1, \Omega_3] = (\alpha - 2\beta)\Omega_5, \quad [\Omega_1, \Omega_5] = (2\beta - \alpha)\Omega_3, \quad [\Omega_3, \Omega_5] = (\alpha - 2\beta)\Omega_1,$$

that is either abelian ($\alpha = 2\beta$) or isomorphic to $\mathfrak{so}(3)$ ($\alpha \neq 2\beta$). The space $\mathfrak{m}$ is a reductive complement of $\mathfrak{h}$ inside $\mathfrak{g}$,

$$[\Omega_1, e_4] = [e_2, \Omega_3] = (\alpha - 2\beta)e_6, \quad [\Omega_1, e_6] = [e_2, \Omega_5] = (2\beta - \alpha)e_4, \quad [\Omega_3, e_6] = [e_4, \Omega_5] = (\alpha - 2\beta)e_2.$$

The remaining commutators of elements from $\mathfrak{m}$ are

$$[e_2, e_4] = -\beta\Omega_5 - \alpha' e_6, \quad [e_2, e_6] = \beta\Omega_3 + \alpha' e_4, \quad [e_4, e_6] = -\beta\Omega_1 - \alpha' e_2.$$

Since $\mathfrak{g} \cap \mathrm{span}(H_1, H_3, H_5) = 0$, we can conclude that M^6 is isometric to the 6-dimensional simply connected Lie group G with Lie algebra $\mathfrak{g}$ equipped with a left-invariant metric. To determine G, we compute the Killing form $\tilde{\beta}(x, y) := \mathrm{tr}(\mathrm{ad}\,x \circ \mathrm{ad}\,y)$ in the ordered basis $\Omega_1, \Omega_3, \Omega_5, e_2, e_4, e_6$ of $\mathfrak{g}$,

$$\tilde{\beta} = \left[\begin{array}{c|c} -4(\alpha - 2\beta)^2 \cdot 1_3 & 2\alpha'(\alpha - 2\beta) \cdot 1_3 \\ \hline 2\alpha'(\alpha - 2\beta) \cdot 1_3 & \left(4\beta(\alpha - 2\beta) - 2\alpha'^2\right) \cdot 1_3 \end{array}\right].$$

The matrix $\tilde{\beta}$ always has two eigenvalues of multiplicity 3 each, hence $\mathrm{rk}\,\tilde{\beta} = 0, 3$, or 6. We can immediatly identify G if $\mathrm{rk}\,\tilde{\beta} = 6$, for then it has to be a semisimple Lie algebra with either negative definite or split Killing form. These are exactly $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ or $\mathfrak{so}(3, 1) \cong \mathfrak{sl}(2, \mathbb{C})$. Later, we will give an alternative argument for this result. Let us look at the determinant to identify the cases when $\tilde{\beta}$ does not have full rank, i.e. when G is not semisimple,

$$\det\tilde{\beta} = -64(\alpha - 2\beta)^6(4\beta(\alpha - 2\beta) - \alpha'^2)^3.$$

Therefore, there are three singular cases to be discussed separately: $\alpha - 2\beta = 0$, $4\beta(\alpha - 2\beta) - \alpha'^2 = 0$, or both conditions simultaneously. Let us first treat the last case, i.e. $\alpha = 2\beta$ and $\alpha' = 0$. Then $\mathfrak{g}$ is a 2-step nilpotent Lie algebra. With respect to the new basis $a_1 = e_2, a_2 = e_4, a_3 = e_6, a_4 = \beta\Omega_5, a_5 = -\beta\Omega_3, a_6 = \beta\Omega_1$, it can alternatively be described by

$$da_1 = da_2 = da_3 = 0, \quad da_4 = a_{12}, \quad da_5 = a_{13}, \quad da_6 = a_{23}.$$

Thus, it corresponds in standard notation to the nilpotent Lie algebra $(0, 0, 0, 12, 13, 23)$. Alternatively, it can be described as $\mathbb{R}^3 \times \mathbb{R}^3$ with the commutator

$$[(v_1, w_1), (v_2, w_2)] = (0, v_1 \times v_2).$$

In this realisation, it may be found as an example in Schoemanns' work [Sch07, Example 4.13, p.2208]. Up to a constant, the metric is unique.

Let us now consider one of the two cases where $\mathrm{rk}\,\tilde{\beta} = 3$, namely, $\alpha = 2\beta$, but $\alpha' \neq 0$. Then three blocks of $\tilde{\beta}$ vanish, and the lower right (3×3) block is $-2\alpha'^2 \cdot 1_3$. In the new basis $\Omega_1, \Omega_3, \Omega_5, f_2 := \beta\Omega_1 + \alpha' e_2, f_4 = \beta\Omega_3 + \alpha' e_4, f_6 := \beta\Omega_5 + \alpha' e_6$, the only non-vanishing Lie brackets are

$$[f_2, f_4] = -\alpha'^2 f_6, \quad [f_2, f_6] = \alpha'^2 f_4, \quad [f_4, f_6] = -\alpha'^2 f_2.$$

We conclude that $\mathfrak{g} = \mathbb{R}^3 \times \mathfrak{so}(3)$, hence G is the direct product $\mathbb{R}^3 \times S^3$.

We shall now discuss the case $\alpha \neq 2\beta$; as it turns out, the value of α' will only matter at the very end. Instead of looking at $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ itself, we look at the homogeneous space it defines: we observed before that $\mathfrak{m}$ is a reductive complement of the Lie subalgebra $\mathfrak{h}$, and by the assumption $\alpha \neq 2\beta$, the representation of $\mathfrak{h}$ on $\mathfrak{m}$ is irreducible and $\mathfrak{h}$ is isomorphic to $\mathfrak{so}(3)$. Hence, $\mathfrak{h}$ defines a *compact* (in particular, closed) subgroup of G, and $G/H =: P^3$ is a 3-dimensional manifold, of course homogeneous. Recall that the action of G on G/H is effective if and only if $\mathfrak{h}$ contains no nontrivial ideal of $\mathfrak{g}$. This is obviously the case if $\alpha \neq 2\beta$, making G a subgroup of the isometry group of P^3. However, the isometry group of a 3-dimensional manifold is at most 6-dimensional. We obtain the remarkable result that $\widetilde{\mathrm{Iso}(P^3)} = G$ has maximal dimension. Then P^3 has to be a space of constant curvature, i.e. $P^3 = \mathbb{R}^3, S^3$, or $\mathbb{H}^3$. Their isometry groups are well-known, and their universal coverings are

$$G = S^3 \ltimes \mathbb{R}^3, \quad \widetilde{\mathrm{SO}(4)} = S^3 \times S^3, \quad \text{or} \quad \mathrm{SL}(2, \mathbb{C}).$$

The semidirect product corresponds to the singular case $4\beta(\alpha - 2\beta) - \alpha'^2 = 0$ that we had postponed, while the other two correspond to non-singular choices of α, α', and β as stated before. To see explicitly how the Lie algebra satisfying $4\beta(\alpha - 2\beta) = \alpha'^2 \neq 0$ is isomophic to the semidirect product $S^3 \ltimes \mathbb{R}^3$, one performs the basis change

$$g_i := e_i + \frac{\alpha'}{2(\alpha - 2\beta)} \Omega_{i-1}, \quad i = 2, 4, 6, \quad \Omega_1, \ \Omega_3, \ \Omega_5 \ \text{unchanged}$$

and checks that $[g_i, g_j] = 0$ for $i, j = 2, 4, 6$ and that the elements of $\mathrm{span}(\Omega_1, \Omega_3, \Omega_5) \cong \mathfrak{so}(3)$ act on g_j by the adjoint representation. We summarize the result:

Theorem 8.12 (case D.2 – classification). *A complete, simply connected Riemannian 6-manifold (M^6, g, T) with parallel skew torsion T, $\mathrm{rk}\,(*\sigma_T) = 6$ and $\ker T = 0$ that is* not *isometric to a nearly Kähler manifold is one of the following Lie groups with a suitable family of left-invariant metrics:*

(1) *The nilpotent Lie group with Lie algebra $\mathbb{R}^3 \times \mathbb{R}^3$ with commutator $[(v_1, w_1), (v_2, w_2)] = (0, v_1 \times v_2)$ (see [Sch07]),*

(2) *the direct or the semidirect product of S^3 with $\mathbb{R}^3$,*

(3) *the product $S^3 \times S^3$ (described in Section 9.4),*

(4) *the Lie group $\mathrm{SL}(2, \mathbb{C})$ viewed as a 6-dimensional real manifold (described in Section 9.5).*

Remark 8.13. Large families of half-flat almost complex structures on $S^3 \times S^3$ were constructed in [SchH10] and [MS13]; they overlap those described in Section 9.4, but it is not evident how to test which have parallel torsion or curvature.

9. Explicit realisation of the occurring naturally reductive spaces

This section compiles all naturally reductive homogeneous spaces of dimension five and six (with the exception of a few degenerate, not-so-interesting cases like most products etc.); some of them occur in families that generalise to higher dimensions.

9.1. The Stiefel 5-manifold.
In what follows, we will study the homogeneous space

$$M^5 \;=\; \frac{\mathrm{SO}(3) \times \mathrm{SO}(3)}{\mathrm{SO}(2)_r}.$$

It appears in the 5-dimensional classification as the compact representative of case (2) in the Classification Theorem 7.6. Here, we denote by $\mathrm{SO}(2)_r$ the subgroup of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ consisting of products of matrices of the form

$$g(t) \;=\; \left(\begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos rt & -\sin rt & 0 \\ \sin rt & \cos rt & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$

where r is a rational parameter. The condition $r \in \mathbb{Q}$ guarantees that $\mathrm{SO}(2)_r$ is closed inside $\mathrm{SO}(3) \times \mathrm{SO}(3)$. Let $\{e_{12}, e_{13}, e_{23}\}$ be the standard basis of $\mathfrak{so}(3)$. The Lie algebra $\mathfrak{so}(2)_r =: \mathfrak{h}$ embeds inside $\mathfrak{so}(3) \oplus \mathfrak{so}(3) =: \mathfrak{g}$ as the one-dimensional Lie algebra generated by $h := (e_{12}, r e_{12})$. As invariant scalar product, we choose the multiple of the Killing form $K(X, Y) := \frac{1}{2}\mathrm{tr}(X^t Y)$, so that the e_{ij} have norm 1. Denote by $\mathfrak{m}$ the linear span of the elements

$$x_1 = (e_{13}, 0), \;\; x_2 = (e_{23}, 0), \;\; y_1 = (0, e_{13}), \;\; y_2 = (0, e_{23}), \;\; z = \frac{1}{\sqrt{r^2+1}}(-r e_{12}, e_{12}).$$

They are an orthornomal basis of $\mathfrak{m}$ and one checks that $\mathfrak{m}$ is a reductive orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let us now introduce two parameters in our metric, which we shall call $g_{a,b}$, $a, b > 0$, or simply g if the parameters are understood, by prescribing that the following is an orthonormal basis

$$u_1 = \sqrt{a}\, x_1, \quad u_2 = \sqrt{a}\, x_2, \quad v_1 = \sqrt{b}\, y_1, \quad v_2 = \sqrt{b}\, y_2, \quad \xi = z$$

for a fixed pair of parameters $a, b > 0$. Of course, we could introduce a third parameter $c > 0$ of the metric in direction z, but we normalize it to 1. As for the projection of the commutators of elements of $\mathfrak{m}$ to both $\mathfrak{h}$ and $\mathfrak{m}$ we have the following non-vanishing terms

$$[u_1, u_2] = -\tfrac{a\, r}{\sqrt{r^2+1}} \xi + \tfrac{a}{r^2+1} h \quad [u_1, \xi] = \tfrac{r}{\sqrt{r^2+1}} u_2, \quad [u_2, \xi] = -\tfrac{r}{\sqrt{r^2+1}} u_1,$$

$$[v_1, v_2] = \tfrac{b}{\sqrt{r^2+1}} \xi + \tfrac{b\, r}{r^2+1} h \quad\; [v_1, \xi] = -\tfrac{1}{\sqrt{r^2+1}} v_2, \quad [v_2, \xi] = \tfrac{1}{\sqrt{r^2+1}} v_1.$$

Comparing the example with the Classification Theorem 7.6, we see that the parameters are related by the formulas $\lambda = \frac{ra}{\sqrt{r^2+1}}$ and $\varrho = -\frac{b}{\sqrt{r^2+1}}$. Observe that, as it stands, we can only conclude that we have a naturally reductive metric for $a = b = 1$ (see Remark 7.7). To describe the isotropy representation $\mathrm{SO}(2)_r \to \mathrm{SO}(\mathfrak{m})$, we identify $\mathfrak{m}$ with $\mathbb{R}^5$, and choose the ordering $\{u_1, u_2, v_1, v_2, \xi\}$. Denote by $E_{i,j}$, $(i < j,\; i, j = 1, \ldots, 5)$ the standard basis of $\mathfrak{so}(5)$, see Remark 5.1. The linear isotropy representation $\lambda : \mathfrak{h} \longrightarrow \mathfrak{so}(\mathfrak{m})$ may then be expressed as

$$\lambda(h) = E_{12} + r\, E_{34}.$$

Let α_i, β_i and η be the dual forms of u_i, v_i and ξ, respectively $(i = 1, 2)$. The vector ξ is fixed under the isotropy representation, so it defines a global vector field which gives a preferred

direction in the tangent bundle. The forms η, $\alpha_1 \wedge \alpha_2$ and $\beta_1 \wedge \beta_2$ are also globally defined. Hence the tensor

$$\varphi = \alpha_1 \otimes u_2 - \alpha_2 \otimes u_1 + \beta_1 \otimes v_2 - \beta_2 \otimes v_1$$

is a well-defined almost contact structure which can be readily checked to be compatible with the metric $g_{a,b}$. The fundamental form F is given by $F = -(\alpha_1 \wedge \alpha_2 + \beta_1 \wedge \beta_2)$ which is always closed. The exterior derivative of the contact form η is given by

$$d\eta = \frac{ar}{\sqrt{r^2+1}} \alpha_1 \wedge \alpha_2 - \frac{b}{\sqrt{r^2+1}} \beta_1 \wedge \beta_2.$$

Hence, F and $d\eta$ are proportional if and only if $ar + b = 0$, whereas the Nijenhuis tensor vanishes for all values. Therefore, M^5 is a quasi-Sasaki manifold, and it is α-Sasaki if $ar + b = 0$. We can calculate the map $\Lambda^g : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ describing the Levi-Civita connection by means of the expression [KN69, Ch.X]

$$\Lambda^g(X)Y = \frac{1}{2}[X,Y]_\mathfrak{m} + \frac{1}{2}U(X,Y),$$

where U is such that $g(U(X,Y),Z) = g([Z,X],Y) + g([Z,Y],X)$. Then we have the following connection forms

$$\Lambda^g(u_1) = -\frac{ar}{2\sqrt{r^2+1}}E_{25}, \quad \Lambda^g(u_2) = \frac{ar}{2\sqrt{r^2+1}}E_{15}, \qquad \Lambda^g(v_1) = \frac{b}{2\sqrt{r^2+1}}E_{45},$$

$$\Lambda^g(v_2) = -\frac{b}{2\sqrt{r^2+1}}E_{35}, \quad \Lambda^g(\xi) = \frac{r(a-2)}{2\sqrt{r^2+1}}E_{12} + \frac{2-b}{2\sqrt{r^2+1}}E_{34},$$

and we see that ξ is a Killing field. Consequently, the quasi-Sasaki structure admits a characteristic connection ∇ whose torsion is given by [FI02]

$$T = \eta \wedge d\eta = \frac{ar}{\sqrt{r^2+1}} \eta \wedge \alpha_1 \wedge \alpha_2 - \frac{b}{\sqrt{r^2+1}} \eta \wedge \beta_1 \wedge \beta_2,$$

and it is described by the linear map Λ

$$\Lambda(u_1) = \Lambda(u_2) = \Lambda(v_1) = \Lambda(v_2) = 0, \quad \Lambda(\xi) = \frac{r(a-1)}{\sqrt{r^2+1}}E_{12} + \frac{1-b}{\sqrt{r^2+1}}E_{34}.$$

We immediately see (as expected) that T is indeed parallel. For the curvature tensor $\mathcal{R} : \Lambda^2 \to \Lambda^2$ we use the following formula

$$\mathcal{R}(X,Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X,Y]_\mathfrak{m}) - \lambda([X,Y]_\mathfrak{h}) .$$

Its image is contained in the 2-dimensional abelian Lie algebra generated by $\alpha_1 \wedge \alpha_2$ and $\beta_1 \wedge \beta_2$

$$\mathcal{R} = \frac{a(a-1)r^2 - a}{r^2+1}(\alpha_1 \wedge \alpha_2)^2 + \frac{b(b-1) - br^2}{r^2+1}(\beta_1 \wedge \beta_2)^2 - \frac{abr}{r^2+1}(\alpha_1 \wedge \alpha_2 \odot \beta_1 \wedge \beta_2).$$

We have $\nabla \mathcal{R} = 0 = \nabla T$, i.e. a 2-parameter family of naturally reductive metrics on the manifold M^5 (a third parameter c is allowed, but yields only a global rescaling of the metric). We can easily compute the Ricci tensor for ∇^g, it yields

$$\mathrm{Ric}^g = \mathrm{diag}\left(a - \frac{a^2r^2}{2(r^2+1)}, a - \frac{a^2r^2}{2(r^2+1)}, b - \frac{b^2}{2(r^2+1)}, b - \frac{b^2}{2(r^2+1)}, \frac{a^2r^2}{2(r^2+1)} + \frac{b^2}{2(r^2+1)}\right).$$

If we look for Einstein metrics, we get a system of two equations. A detailed discussion tells that a can be expressed through b, and that b is the unique real solution of a cubic equation

$$a = b\left[2 - \frac{3}{2(r^2+1)}b\right], \quad r^2b\left[2 - \frac{3}{2(r^2+1)}b\right]^2 = 2(r^2+1) - 2b.$$

Hence there is exactly one pair of values (a, b) for any parameter value of r^2. For example, we check that $r^2 = 1$ yields $a = b = 4/3$, and this is the Einstein-Sasaki metric of [Je75] and [Fr80].

9.2. **The Berger sphere** $S^5 = \mathrm{U}(3)/\mathrm{U}(2)$**.** The Berger sphere (and its non-compact sibling $\mathrm{U}(2,1)/\mathrm{U}(2)$) is the only 5-dimensional naturally reductive space that is α-Sasakian (case B.2), but not part of a quasi-Sasakian family (case B.1), see Theorem 7.8. We sketch two alternative ways to view the Berger sphere; since this is a relatively well-known example, we shall be brief. Consider the pair of Lie algebras

$$\mathfrak{u}(3) \cong \left\{ A \in \mathcal{M}_3(\mathbb{C}) : A + \bar{A}^t = 0 \right\}, \quad \mathfrak{u}(2) \cong \left\{ \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} : B \in \mathcal{M}_2(\mathbb{C}), B + \bar{B}^t = 0 \right\}.$$

First, we show that the realisation of S^5 as the quotient $\mathrm{U}(3)/\mathrm{U}(2)$ is already naturally reductive (in the traditional sense). Indeed, there are two elements commuting with $\mathfrak{u}(2)$ and a complementary 4-dimensional subspace,

$$Z_1 = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}, \quad \mathfrak{m}_0 = \left\{ \begin{bmatrix} 0 & -\bar{v}^t \\ v & 0 \end{bmatrix} : v \in \mathbb{C}^2 \right\}.$$

Any subspace of the family

$$\mathfrak{m}_{\alpha,\beta} = \mathfrak{m}_0 \oplus \mathbb{R} \cdot (\alpha Z_1 + \beta Z_2), \quad \alpha \neq 0$$

decomposes the Lie algebra $\mathfrak{u}(3) = \mathfrak{u}(2) \oplus \mathfrak{m}_{\alpha,\beta}$, and $[\mathfrak{u}(2), \mathfrak{m}_{\alpha,\beta}] \subset \mathfrak{m}_{\alpha,\beta}$ holds. The scalar product on the complement $\mathfrak{m}_{\alpha,\beta}$ is defined by a bi-invariant scalar product on $\mathfrak{u}(3)$. This is a 2-parameter family of naturally reductive structures, see Theorem 7.8. Alternatively, one can realize the 5-sphere as $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ with the same embeddings as above, now with vanishing trace. Then one chooses the reductive decomposition

$$\mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{m}_0 \oplus \langle \eta \rangle \ \text{ with } \eta = \frac{1}{\sqrt{3}} \operatorname{diag}(-2i, i, i) = \frac{1}{\sqrt{3}} (-2Z_1 + 3Z_2).$$

As basis of $\mathfrak{m}_0$, we choose the elements $e_1, \ldots, e_4$ corresponding to the vectors $v = (1,0)$, $(i,0)$, $(0,1)$, $(0,i) \in \mathbb{C}^2$. With respect to the Killing form $\beta(X,Y) = -\operatorname{tr}(XY)/2$ of $\mathfrak{su}(3)$, the vectors $e_1, \ldots, e_3, \eta$ are orthonormal. Thus, we can define a deformation of the scalar product by

$$g_\gamma := \beta\big|_{\mathfrak{m}_0} \oplus \frac{1}{\gamma} \beta\big|_{\langle \eta \rangle}, \quad \gamma > 0.$$

Again, a second parameter could be introduced by allowing a rescaling on $\mathfrak{m}_0$, but this has no intrinsic geometrical meaning. Now one checks that $F := e_{12} + e_{34}$ with Reeb vector field $\tilde{\eta} = \eta/\sqrt{\gamma} =: e_5$ defines an α-Sasaki structure with characteristic connection ∇; as an invariant connection, ∇ is described by the map $\Lambda : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ (see [KN69, Ch.X]) given by $\Lambda(e_i) = 0$ for $i = 1, \ldots, 4$ and $\Lambda(e_5) = (\sqrt{3/\gamma} - \sqrt{3\gamma})(E_{12} + E_{34})$. The torsion and curvature of the connection ∇ are given by

$$T = \tilde{\eta} \wedge d\tilde{\eta} = \sqrt{3/\gamma}(e_{12} + +e_{34}) \wedge e_5,$$
$$R = \left[\frac{3}{\gamma} - 3\right](e_{12} + e_{34})^2 - [(e_{13} + e_{24})^2 + (e_{14} - e_{23})^2 + (e_{12} - e_{34})^2].$$

Comparing with Theorem 7.8 and formula (7.2), we see that $a = -1, b = 3/\gamma - 3$ and the coefficient $\varrho = \lambda$ of the torsion is given by $\varrho^2 = 3/\gamma$. One verifies that T and R are indeed ∇-parallel. For $\gamma = 1/2$, the ∇-Ricci curvature vanishes, while for $\gamma = 3/4$, the metric is Riemannian Einstein. This is a particular case of the deformation of Einstein-Sasaki metrics that yield ∇-Ricci-flat connections described in [AF14].

9.3. **The Heisenberg group of dimension** $2n+1$. The Heisenberg group of dimension $2n+1$ is the subgroup of $GL(n+2, \mathbb{R})$ given by upper triangular matrices of the form

$$H^{2n+1} = \left\{ \begin{bmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\}.$$

It appears in the classifications in dimension 3 (Theorem 5.2) and dimension 5 (Theorem 7.6, case B.1). Clearly H^{2n+1} is diffeomorphic to $\mathbb{R}^{2n+1}$. It can be readily checked that the following sets are, respectively, a basis of left-invariant vector fields and its dual basis of left-invariant 1-forms,

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial z}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} + x_2 \frac{\partial}{\partial z} \cdots, \frac{\partial}{\partial z} \right\}, \quad \left\{ dx_1, dy_1, dx_2, dy_2, \cdots, dz - \sum_{i=1}^{n} x_i dy_i \right\}.$$

We have an odd-dimensional manifold with a clearly preferred direction and we can define a contact structure as follows. Choose $\xi = \frac{\partial}{\partial z}$ to be the Reeb field and $\eta = dz - \sum_{i=1}^{n} x_i dy_i$ to be the contact form. The (1,1)-tensor

$$\varphi = \sum_{i=1}^{n} \left(dx_i \otimes \left(\frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z} \right) - dy_i \otimes \frac{\partial}{\partial x_i} \right)$$

is then a contact structure on H^{2n+1}, since $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$. For each $i = 1, \cdots, n$ let λ_i be a positive scalar and consider the n-tuple $\lambda = (\lambda_1, \cdots, \lambda_n)$. Then

$$g_\lambda = \sum_{i=1}^{n} \frac{1}{\lambda_i} (dx_i^2 + dy_i^2) + \left[dz - \sum_{j=1}^{n} x_j dy_j \right]^2$$

defines an n-parameter family of metrics which are compatible with φ. Hence $(H^{2n+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. An orthonormal frame $\{u_i, v_i, \xi\}$ of TH^{2n+1} and its dual frame $\{\alpha_i, \beta_i, \eta\}$ are given by

$$u_i = \sqrt{\lambda_i} \frac{\partial}{\partial x_i}, \quad v_i = \sqrt{\lambda_i} \left(\frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z} \right) \text{ and } \alpha_i = \frac{1}{\sqrt{\lambda_i}} dx_i, \quad \beta_i = \frac{1}{\sqrt{\lambda_i}} dy_i \text{ for } i = 1, \cdots, n.$$

The only non-vanishing commutators are $[u_i, v_i] = \lambda_i \xi$, for all $i = 1, \cdots, n$. An easy but tedious computation shows that the Nijenhuis tensor N of φ vanishes. Moreover, $d\eta$ and the fundamental form $F(X, Y) = g(X, \varphi(Y))$ are expressed by

$$d\eta = -\sum_{i=1}^{n} \lambda_i \alpha_i \wedge \beta_i \quad \text{and} \quad F = -\sum_{i=1}^{n} \alpha_i \wedge \beta_i.$$

Therefore F is always closed and H^{2n+1} is quasi-Sasaki for all parameters. Furthermore H^{2n+1} is α-Sasaki if and only if all parameters coincide and, in particular, Sasaki for $\lambda_1, \ldots \lambda_n = 2$. Using the first structure equation of Cartan, we can see that the non-vanishing Levi-Civita connection forms are given by

$$\omega_{u_i v_i}^g = -\frac{\lambda_i}{2} \eta, \quad \omega_{u_i \xi}^g = -\frac{\lambda_i}{2} \beta_i, \quad \omega_{v_i \xi}^g = \frac{\lambda_i}{2} \alpha_i.$$

Thus our Reeb field ξ is a Killing vector field and since $N = 0$, a theorem from [FI02] guarantees the existence of the characteristic connection whith torsion 3-form $T = \eta \wedge d\eta$. The non-vanishing connection forms for ∇ are then simply $\omega_{u_i v_i} = -\lambda_i \eta$ $(i = 1, \ldots, n)$. Using now the second structure equation of Cartan, we can calculate the curvature forms for the characteristic connection,

$\Omega_{u_i v_i} = -\lambda_i \, d\eta$. Equivalently, the curvature tensor can be written as

$$\mathcal{R} = \sum_{i \leq j}^{n} \lambda_i \lambda_j (\alpha_i \wedge \beta_i) \odot (\alpha_j \wedge \beta_j).$$

Observe that the holonomy algebra of ∇ is one-dimensional, and therefore in particular abelian. We can readily check that ∇ satisfies $\nabla T = 0 = \nabla \mathcal{R}$ and therefore we have determined an n-parameter family of naturally reductive homogeneous structures on the Heisenberg group of dimension $2n + 1$.

Remark 9.1. Heisenberg groups of dimension $2n+1 \geq 5$ with their naturally reductive structure defined above are the first known examples of manifolds with parallel skew torsion carrying a Killing spinor with torsion that do *not* admit a Riemannian Killing spinor (in fact, they do not even carry a Riemannian Einstein metric, which would be a necessary requirement for such a spinor) – see [ABK12] and [AH14]. This is a typical example of how naturally reductive spaces are used in differential geometry as a vast reservoir of examples.

9.4. **The Lie group $S^3 \times S^3$.** In this section, we explain the naturally reductive structures on $S^3 \times S^3$ with $\mathfrak{hol}^\nabla = \mathfrak{so}(3)$ (case D.2, Theorem 8.12). We realise $S^3 \times S^3$ as the homogeneous space G/H where $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times SU(2)$ and $H = \mathrm{SU}(2)$ is embedded into G diagonally, that is,

$$H = \{(h, h, h) : h \in \mathrm{SU}(2)\}.$$

Let $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ be the Lie algebra of G and $\mathfrak{h} = \mathfrak{su}(2) = \{(C, C, C) : C \in \mathfrak{su}(2)\}$ be the Lie algebra of H. Consider the following spaces

$$\mathfrak{m}_1 = \{(A, aA, bA) : a, b \in \mathbb{R}, A \in \mathfrak{su}(2)\}, \quad \mathfrak{m}_2 = \{(B, cB, dB) : c, d \in \mathbb{R}, B \in \mathfrak{su}(2)\}$$

and let $\mathfrak{m}$ be the direct sum of $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then $\mathfrak{m}$ is a reductive complement of $\mathfrak{h}$ inside $\mathfrak{g}$ if and only if

$$\Delta := \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{bmatrix} = (a-1)(d-1) - (b-1)(c-1) \neq 0.$$

Let $K(X, Y) = -\frac{1}{2}\mathrm{tr}(XY)$ denote the (rescaled) Killing form on $\mathfrak{su}(2)$ and define an inner product on $\mathfrak{m}$, for each parameter $\lambda > 0$, as

$$\langle (A_1 + B_1, aA_1 + cB_1, bA_1 + dB_1), (A_2 + B_2, aA_2 + cB_2, bA_2 + dB_2) \rangle = K(A_1, A_2) + \frac{1}{\lambda^2} K(B_1, B_2).$$

We define an almost complex structure J on $\mathfrak{m}$ by

$$J((A, aA, bA) + (B, cB, dB)) = -\frac{1}{\lambda}(B, aB, bB) + \lambda(A, cA, dA).$$

Let

$$Y_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Y_5 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

be the standard basis of $\mathfrak{su}(2)$. Recall that we have the following commutator relations

$$[Y_1, Y_3] = -2Y_5, \quad [Y_1, Y_5] = 2Y_3, \quad [Y_3, Y_5] = -2Y_1.$$

Take $h_i = (Y_i, Y_i, Y_i)$, $i = 1, 3, 5$. Then $\{h_1, h_3, h_5\}$ is a basis of $\mathfrak{h}$. Consider also the following orthonormal basis of $\mathfrak{m}$

$$e_i = (Y_i, aY_i, bY_i) \quad \text{and} \quad e_{i+1} = \lambda(Y_i, cY_i, dY_i) \quad \text{for } i = 1, 3, 5.$$

Remark that J is given, in this basis, by the 2-form $\Omega = -(e_{12} + e_{34} + e_{56})$. The isotropy representation is given by

$$\mathrm{ad}\, h_1 = -2(E_{35}+E_{46}) =: -2A_1, \ \mathrm{ad}\, h_3 = 2(E_{15}+E_{26}) =: 2A_3, \ \mathrm{ad}\, h_5 = -2(E_{13}+E_{24}) =: -2A_5.$$

The commutator structure is somewhat complicated. For ease of notation we introduce the following coefficients

$$\mu = -\tfrac{2}{\Delta}((a^2 - 1)(d-1) - (b^2 - 1)(c-1)) \qquad \nu = -\tfrac{2}{\Delta}(b-1)(a-1)(b-a)$$

$$\gamma = -\tfrac{2}{\Delta}(a(d - b^2) + a^2(b-d) + (b^2 - b)c) \qquad \delta = -\tfrac{2}{\Delta}(c(a(d-1) - bd + 1) + (b-1)d)$$

$$\sigma = \tfrac{2}{\Delta}((a-1)(1 - bd) + (ac - 1)(b-1)) \qquad \tau = \tfrac{2}{\Delta}(ac(d-b) + cb(1-d) + ad(b-1))$$

$$\xi = -\tfrac{2}{\Delta}(c-1)(d-1)(c-d) \qquad \eta = -\tfrac{2}{\Delta}((d^2 - 1)(a-1) - (c^2 - 1)(b-1))$$

$$\theta = -\tfrac{2}{\Delta}(d^2(c-a) + c^2(b-d) + (da - cb)).$$

Then we can write the nonvanishing brackets as

$$[e_1, e_3] = \mu e_5 + \tfrac{\nu}{\lambda}e_6 + \gamma h_5, \qquad\qquad [e_1, e_4] = [e_2, e_3] = \lambda\delta e_5 + \sigma e_6 + \lambda\tau h_5,$$

$$[e_1, e_5] = -\mu e_3 - \tfrac{\nu}{\lambda}e_4 + \gamma h_3, \qquad\qquad [e_1, e_6] = [e_2, e_5] = -\lambda\delta e_3 - \sigma e_4 - \lambda\tau h_3,$$

$$[e_2, e_4] = \lambda^2\xi e_5 + \lambda\eta e_6 + \lambda^2\theta h_5, \qquad [e_2, e_6] = -\lambda^2\xi e_3 - \lambda\eta e_4 - \lambda^2\theta h_3,$$

$$[e_3, e_5] = \mu e_1 + \tfrac{\nu}{\lambda}e_2 + \gamma h_1, \qquad\qquad [e_3, e_6] = [e_4, e_5]\lambda\delta e_1 + \sigma e_2 + \lambda\tau h_1,$$

$$[e_4, e_6] = \lambda^2\xi e_1 + \lambda\eta e_2 + \lambda^2\theta h_1.$$

The Nijenhuis tensor N is totally skew-symmetric and given by

$$N = [\lambda^2\xi + 2\sigma - \mu](e_{135} - e_{146} - e_{236} - e_{245}) + \left[\tfrac{\nu}{\lambda} + \lambda(2\delta - \eta)\right](e_{246} - e_{136} - e_{145} - e_{235}).$$

We can also compute that

$$d^c\Omega = -3\lambda^2\xi e_{135} - 3\tfrac{\nu}{\lambda}e_{246} + (2\sigma - \mu)(e_{146} + e_{245} + e_{236}) + \lambda(2\delta - \eta)(e_{145} + e_{136} + e_{235}).$$

Therefore the torsion tensor $T = N + d\Omega \circ J$ of the almost complex structure is [FI02, Thm 10.1]

$$T = [-2\lambda^2\xi + 2\sigma - \mu]e_{135} + \left[-2\tfrac{\nu}{\lambda} + \lambda(2\delta - \eta)\right]e_{246} - \lambda^2\xi(e_{146} + e_{236} + e_{245}) - \tfrac{\nu}{\lambda}(e_{136} + e_{145} + e_{235}).$$

For all parameters, the Hermitian structure is of type $\mathcal{W}_1 \oplus \mathcal{W}_3$. Its characteristic connection ∇ is given by the map $\Lambda : \mathfrak{m} \longrightarrow \mathfrak{so}(\mathfrak{m})$

$$\Lambda(e_i) = (-\lambda^2\xi + \sigma)A_i \ \text{ and } \ \Lambda(e_{i+1}) = \left(-\tfrac{\nu}{\lambda} + \lambda\delta\right)A_i \text{ for } i = 1, 3, 5.$$

It is then a straightforward computation to check that $\nabla T = 0$. As for the curvature tensor, we obtain

$$\mathcal{R} \ = \ \Sigma\,[A_1^2 + A_3^2 + A_3^2], \quad \text{where } \ \Sigma := \tfrac{\nu^2}{\lambda^2} + \lambda^4\xi^2 - \lambda^2\xi(2\sigma - \mu) - \nu(2\delta - \eta).$$

Thus, $\mathcal{R}$ is a projection on $\mathrm{span}(A_1, A_3, A_5) = \mathfrak{so}(3) = \mathfrak{hol}(\nabla)$, compare with equation (8.4). This shows (and one easily checks) that $\nabla\mathcal{R} = 0$. The constant Σ also appears in σ_T, for we have $\sigma_T = -\Sigma(*\Omega)$. Therefore $\mathcal{R} = 0$ if and only if $\sigma_T = 0$.

In this description, some parameters play no significant geometrical role. The torsion T is a linear combination of the four SO(3)-invariant 3-forms in $\mathcal{W}_1 \oplus \mathcal{W}_3$, as given in the proof of Theorem 8.9. A suitable base change would make the last term disappear, but we can obtain the same result by choosing our parameters a, b, c, and d so that the coefficient ν vanishes. The choice $a = b = 1$ is not allowed, since it yields $\Delta = 0$, but either $a = 1$ or $b = 1$ is an admissible choice. So let us set $b = 1$ ($\Delta \neq 0$ is then equivalent to $a \neq 1$ and $d \neq 1$). Most, but-alas-not all, coefficients simplify considerable.

Let us identify a few particularly interesting choices of parameters. First, there are several solutions for $\Sigma = 0$, the simplest being $c = 1$ or $c = d$ (because both imply $\xi = 0$). Recall that if α, α', β denote the coefficients of the torsion as in equation (8.4), we described in the proof of Theorem 8.9 and Remark 8.11 three noteworthy situations. They all require $\alpha' = 0$, which is equivalent to $2c = d + 1$. The discussion of the additional conditions is summarized in the following table:

Geometric description	condition	equivalent to
bi-inv. Riemannian Einstein metric	$\alpha = +\beta$	$\lambda = \frac{2|a-1|}{|d-1|}$
pure type $\mathcal{W}_1$ (nearly Kähler)	$\alpha = -\beta$	$\lambda = \frac{2|a-1|}{\sqrt{3}|d-1|}$
pure type $\mathcal{W}_3$	$\alpha = 3\beta$	impossible

The next example has exactly the opposite behaviour: the naturally reductive metrics on $\mathrm{SL}(2, \mathbb{C})$ are either of type $\mathcal{W}_3$ or $\mathcal{W}_1 \oplus \mathcal{W}_3$, but never of pure type $\mathcal{W}_1$.

9.5. The Lie group $\mathrm{SL}(2, \mathbb{C})$.

The complex Lie group $\mathrm{SL}(2, \mathbb{C})$ can be understood as a real 6-dimensional non compact manifold. Its standard complex structure and Killing form can be deformed to yield an almost Hermitian structure with parallel torsion of Gray-Hervella class $\mathcal{W}_1 \oplus \mathcal{W}_3$. Its $\mathcal{W}_3$ structure was first discovered in [AFS05], and enlarged to $\mathcal{W}_1 \oplus \mathcal{W}_3$ (albeit rather laconically) in [Sch07]. Since this is a rather unusual (and quite tricky) case of a naturally reductive space, we will give a self-contained account of its geometry here. Recall that

$$\mathfrak{sl}(2, \mathbb{C}) \; = \; \Big\{ A \in \mathcal{M}_2(\mathbb{C}) \, : \, \mathrm{tr} A = 0 \Big\} \; = \; \mathfrak{su}(2) \oplus i \, \mathfrak{su}(2).$$

We realize $\mathrm{SL}(2, \mathbb{C})$ as the quotient $G/H = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SU}(2)/\mathrm{SU}(2)$ with $H = \mathrm{SU}(2)$ embedded diagonally and a reductive complement $\mathfrak{m}_\alpha$ of $\mathfrak{h}$ inside $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{su}(2)$ depending on a real parameter $\alpha \neq 1$, i.e.

$$\mathfrak{h} \; = \; \{(B, B) \, : \, B \in \mathfrak{su}(2)\}, \quad \mathfrak{m}_\alpha \; := \; \{(A + \alpha B, B) \, : \, A \in i \, \mathfrak{su}(2), \; B \in \mathfrak{su}(2)\}.$$

Observe that elements of the form $(A + \alpha_1 B, \alpha_2 B)$ would still define a reductive complement of $\mathfrak{h}$ (as long as $\alpha_1 \neq \alpha_2$), but $\alpha_2 = 0$ is uninteresting, since $\mathfrak{m}$ would then be a subalgebra of $\mathfrak{g}$. Hence, our Ansatz for $\mathfrak{m}_\alpha$ sets this second constant equal to one. We choose as basis of $\mathfrak{sl}(2, \mathbb{C})$ over the reals $Y_1, Y_3, Y_5, Y_2 = i\, Y_1, Y_4 = i\, Y_3, Y_6 = i\, Y_5$, where the first three elements are defined as in the previous Example 9.4. Thus, the elements $h_i = (Y_i, Y_i)$, $i = 1, 3, 5$ form a basis of $\mathfrak{h}$. The Killing form $2\, \beta(X, Y) = \mathrm{Re}\, \mathrm{tr}(XY)$ of $\mathfrak{sl}(2, \mathbb{C})$ is negative definite on $\mathfrak{su}(2)$, positive definite on $i\, \mathfrak{su}(2)$ and these two spaces are orthogonal. Therefore, the formula

$$g_\lambda\big((A_1 + \alpha B_1, B_1), (A_2 + \alpha B_2, B_2)\big) \; := \; \beta(A_1, A_2) - \frac{1}{\lambda^2} \beta(B_1, B_2), \quad \lambda > 0$$

defines a one-parameter family of Riemannian metric on $\mathrm{SL}(2, \mathbb{C}) \cong G/H$. We shall prove later that G/H with such a metric is in fact a naturally reductive space for all λ. The elements

$$x_i \; = \; \lambda(\alpha\, Y_i, Y_i), \; i = 1, 3, 5 \; \text{ and } \; x_j \; = \; (Y_j, 0), \; j = 2, 4, 6$$

form an orthonormal frame of $\mathfrak{m}_\alpha$ with respect to g_λ. With respect to this frame, the differential $\mathrm{ad} \, : \, \mathfrak{h} \to \mathfrak{so}(\mathfrak{m}_\alpha)$ of the isotropy representation is (as in Example 9.4) given by

$$\mathrm{ad}\, h_1 \; = \; -2(E_{35} + E_{46}) \; =: \; H_1, \; \; \mathrm{ad}\, h_3 \; = \; 2(E_{15} + E_{26}) \; =: \; H_3, \; \; \mathrm{ad}\, h_5 \; = \; -2(E_{13} + E_{24}) \; =: \; H_5.$$

The non-vanishing commutators of elements in $\mathfrak{m}_\alpha$ are

$$[x_1, x_3] = +2\alpha\lambda^2 h_5 - 2\lambda(1+\alpha)x_5, \qquad [x_2, x_4] = +\frac{2}{1-\alpha}h_5 - \frac{2}{\lambda(1-\alpha)}x_5$$

$$[x_1, x_5] = -2\alpha\lambda^2 h_3 + 2\lambda(1+\alpha)x_3, \qquad [x_2, x_6] = -\frac{2}{1-\alpha}h_3 + \frac{2}{\lambda(1-\alpha)}x_3$$

$$[x_3, x_5] = +2\alpha\lambda^2 h_1 - 2\lambda(1+\alpha)x_1, \qquad [x_4, x_6] = +\frac{2}{1-\alpha}h_1 - \frac{2}{\lambda(1-\alpha)}x_1$$

as well as

$$[x_1, x_4] = [x_2, x_3] = -2\lambda\alpha x_6, \quad [x_1, x_6] = [x_2, x_5] = 2\lambda\alpha x_4, \quad [x_3, x_6] = [x_4, x_5] = -2\lambda\alpha x_2.$$

An almost Hermitian structure may be defined by the Kähler form $\Omega := x_{12} + x_{34} + x_{56}$; its Nijenhuis tensor turns out to be a 3-form,

$$(9.1) \qquad N \;=\; 2\left[\lambda(1-\alpha) - \frac{1}{\lambda(1-\alpha)}\right][x_{135} - x_{146} - x_{236} - x_{245}].$$

By [FI02, Thm 10.1], this almost complex structure admits therefore a unique characteristic connection ∇ with torsion

$$T \;=\; N + d\Omega \circ J \;=\; \left[2\lambda(1-\alpha) + \frac{4}{\lambda(1-\alpha)}\right]x_{135} + \frac{2}{\lambda(1-\alpha)}[x_{146} + x_{236} + x_{245}].$$

This torsion is $\mathrm{ad}(\mathfrak{h})$-invariant for all parameter values, hence $\nabla T = 0$. One checks that this almost complex structure has no contribution of Gray-Hervella type $\mathcal{W}_4$, and from formula (9.1), one can conclude that it is of type $\mathcal{W}_3$ if and only if $\lambda(1-\alpha) = \pm 1$. The map $\Lambda: \mathfrak{m}_\alpha \to \mathfrak{so}(\mathfrak{m}_\alpha)$ characterizing ∇ (see [KN69, Ch.X]) is given by

$$\Lambda(x_i) \;=\; \left[\lambda\alpha - \frac{1}{\lambda(1-\alpha)}\right]H_i \;\;\text{for } i = 1, 3, 5 \;\;\text{and}\;\; \Lambda(x_j) \;=\; 0 \text{ for } j = 2, 4, 6.$$

Let us give the expression for the curvature of ∇:

$$\mathcal{R} \;=\; 4\left(1 + \frac{1}{\lambda^2(1-\alpha)^2}\right)[(x_{13} + x_{24})^2 + (x_{15} + x_{26})^2 + (x_{35} + x_{46})^2],$$

which, as should be expected, is nothing else than the projection on the subalgebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ generated by H_1, H_3, H_5, i.e. the holonomy algebra of ∇. Thus, the metric g_λ is naturally reductive for all parameters.

APPENDIX A. THE NOMIZU CONSTRUCTION

We give here an algebraic construction of an infinitesimal model of a naturally reductive structure out of a given algebraic curvature and a skew torsion. This construction is known by the name *Nomizu construction* and is used in different places of the literature (see for example [Tr93] and [CS04]), but not in the form that we need for our purpose.

Let $\mathfrak{h}$ be a real Lie algebra, V a real finite-dimensional $\mathfrak{h}$-module with an $\mathfrak{h}$-invariant positive definite scalar product $\langle,\rangle$, i.e. we assume that $\mathfrak{h} \subset \mathfrak{so}(V) \cong \Lambda^2 V$. Let $\mathcal{R}: \Lambda^2 V \to \mathfrak{h}$ be an $\mathfrak{h}$-equivariant map and $T \in (\Lambda^3 V)^{\mathfrak{h}}$ an $\mathfrak{h}$-invariant 3-form. The 4-form σ_T is defined as usual. We would like to define a Lie algebra structure on $\mathfrak{g} := \mathfrak{h} \oplus V$ by setting

$$(A.1) \quad [A+X, B+Y] \;:=\; ([A,B]_{\mathfrak{h}} - \mathcal{R}(X,Y)) + (AY - BX - T(X,Y)), \qquad A, B \in \mathfrak{h},\; X, Y \in V.$$

This amounts to finding necessary and sufficient conditions for the Jacobi identity to hold in $\mathfrak{g}$. The first result is classical, hence we omit the easy proof:

Lemma A.1. *The bracket defined by (A.1) satisfies the Jacobi identity if and only if the following two conditions hold:*

$$(1) \quad \mathop{\mathfrak{S}}^{X,Y,Z} \mathcal{R}(X,Y,Z,V) \;=\; \sigma_T(X,Y,Z,V).$$

(2) $\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(T(X,Y),Z) = 0.$

We recognize that these two conditions are precisely the first and second Bianchi identity for a metric connection with parallel torsion and curvature (eqs. (2.3) and (2.5) in Section 2). Hence, we shall call conditions (1) and (2) the *first and second Bianchi conditions.*

We now give an interpretation of the first Bianchi condition in terms of the Clifford algebra. We embed $T \in \Lambda^3 V$ and $\mathcal{R} \in \Lambda^2 V \otimes \Lambda^2 V$ in the Clifford algebra $\mathcal{C}(V) := \mathcal{C}(V, -\langle,\rangle)$ by replacing the tensor product and the exterior product by the Clifford product. Similarly, we define T^2 as the composition of the endomorphism T of $\mathcal{C}(V)$ with itself. Finally, we define the (algebraic) Ricci tensor associated with $\mathcal{R}$ as $\mathrm{Ric}(X,Y) := \sum_{i=1}^{n} \mathcal{R}(X, e_i, e_i, Y)$, where e_i denotes an orthonormal frame of V. Recall that the curvature operator $\mathcal{R}$ is a symmetric endomorphism for a connection with parallel torsion, so these are the algebraic curvature operators we're interested in.

Theorem A.2. *If $\mathcal{R} : \Lambda^2 V \to \mathfrak{h} \subset \Lambda^2 V$ is symmetric, the first Bianchi condition is equivalent to $T^2 + \mathcal{R} \in \mathbb{R} \subset \mathcal{C}(V)$, and the second Bianchi condition holds automatically.*

Proof. We express the curvature with respect to an orthonormal frame $e_1, \ldots, e_n$ of V as

$$\mathcal{R} = \frac{1}{4} \sum_{i,j,k,l} R_{ijkl}\, e_{ij} \otimes e_{kl}$$

A priori, $\mathcal{R}$ has components of degree 0, 2, and 4 in $\mathcal{C}(V)$. The 0-degree part vanishes, because it corresponds to summands with four equal indices, and $R_{iiii} = 0 \ \forall\ 1 \leq i \leq n$. The degree-2 part comes from summands with exactly two equal indices that add up to the antisymmetric part of the algebraic Ricci tensor, which is zero for a symmetric curvature operator (just as in the Riemannian case). Thus, the crucial part is to rewrite the degree 4-part $\mathcal{R}_4$ of $\mathcal{R}$ in an appropriate way. It corresponds to summands with i, j, k, l all different. To clarify ideas, let us consider the term e_{1234}. Rewrite

$$\mathcal{R}_4 = \frac{1}{4} \sum_{i,j,k,l,\text{ all diff.}} R_{ijkl}\, e_{ij} \otimes e_{kl} = \mu\, e_{1234} + R,$$

where R are terms that are not proportional to e_{1234}. There is a contribution coming from all terms with $l = 4$, namely,

$$\frac{1}{4} \left[R_{1234} e_{1234} + R_{1324} e_{1324} + R_{2134} e_{2134} + R_{2314} e_{2314} + R_{3124} e_{3124} + R_{3214} e_{3214} \right]$$

$$= \frac{1}{4} \left[R_{1234} - R_{1324} - R_{2134} + R_{2314} + R_{3124} - R_{3214} \right] e_{1234}$$

$$= \frac{1}{4} \left[2R_{1234} + R_{2314} + 2R_{3124} - R_{3214} \right] e_{1234}.$$

Similarly, one computes the contributions coming from terms with $i, j, k = 4$ and obtains

$$\mu = 2(R_{1234} + R_{3124} + R_{2314}).$$

We emphasize that we did use the symmetry of the curvature tensor in this computation. The torsion can be written

$$T = \sum_{i<j<k} T_{ijk} e_{ijk},$$

and the standard identity $T^2 = -2\sigma_T + \|T\|^2$ ([Agr03, Prop.3.1], [Ag06, Prop.A.1]) means that the contribution in T^2 proportional to e_{1234} has coefficient

$$-2 \sum_{\gamma} (T_{12\gamma} T_{\gamma 34} + T_{23\gamma} T_{\gamma 14} + T_{31\gamma} T_{\gamma 24}).$$

Finally, the first Bianchi condition states for $X = e_1,\ Y = e_2,\ Z = e_3, V = e_4$:

$$R_{1234} + R_{3124} + R_{2314} = \sum_{\gamma} (T_{12\gamma} T_{\gamma 34} + T_{23\gamma} T_{\gamma 14} + T_{31\gamma} T_{\gamma 24}).$$

Thus, we see that the term proportional to e_{1234} in $T^2 + \mathcal{R}_4$ vanishes if ond only if the first Bianchi condition holds. We now prove that the second Bianchi condition is automatically satisfied. We express $\mathcal{R}$ through its action on 2-forms, i. e. $\mathcal{R}(T(X,Y),Z) = \mathcal{R}(T(X,Y) \wedge Z)$. Since $\mathcal{R} : \Lambda^2 V \to \mathfrak{h} \subset \Lambda^2 V$ is symmetric, it can be written as the composition $\mathcal{R} = \psi \circ \pi_{\mathfrak{h}}$, where $\pi_{\mathfrak{h}}$ denotes the projection $\Lambda^2 V \to \mathfrak{h}$ (with the orthogonal complement of $\mathfrak{h}$ inside $\mathfrak{so}(V)$ as complementary space) and $\psi : \mathfrak{h} \to \mathfrak{h}$ is some $\mathfrak{h}$-equivariant endomorphism. Hence it suffices to show that

$$\pi_{\mathfrak{h}}\big(\overset{X,Y,Z}{\mathfrak{S}} \ T(X,Y) \wedge Z \big) \ = \ 0,$$

or, equivalently, that $\overset{X,Y,Z}{\mathfrak{S}} \ T(X,Y) \wedge Z$ is orthogonal to $\mathfrak{h}$. Let α be an element of $\mathfrak{h}$, viewed as a skew-symmetric endomorphism in $\mathfrak{so}(V)$, and $\tilde{\alpha}$ the corresponding 2-form. By definition, these satisfy

$$\langle \alpha(X), Y \rangle \ = \ \tilde{\alpha}(X,Y) \ = \ \langle \tilde{\alpha}, X \wedge Y \rangle.$$

Thus, the scalar product of this cyclic sum with α may be rewritten

$$\overset{X,Y,Z}{\mathfrak{S}} \ \langle \tilde{\alpha}, T(X,Y) \wedge Z \rangle \ = \ - \ \overset{X,Y,Z}{\mathfrak{S}} \ \langle \alpha(Z), T(X,Y) \rangle \ = \ \overset{X,Y,Z}{\mathfrak{S}} \ T(X,Y,\alpha(Z)).$$

But the vanishing of this sum is precisely the $\mathfrak{h}$-invariance of T, which we had assumed from the very beginning. $\qquad\square$

Remark A.3. This result exists in the literature in various formulations. It is based on an algebraic identity in the Clifford algebra that was first observed by B. Kostant in [Ko99] and that is the crucial step in a formula of Parthasarathy type for the square of the Dirac operator. The link to naturally reductive homogeneous spaces and connections with skew torsion was established in the first author's work [Agr03] and is explained in detail in the survey [Ag06]. As formulated here, the result was previously used by the last author and N. Schoemann in [Fr07] and [Sch07], but without a clear statement nor a proof.

Appendix B. Skew holonomy systems

Our characterisation of irreducible manifolds with vanishing σ_T as Lie groups (Theorem 4.1) relies on the concept of skew holonomy system, that is very much inspired by Simons' geometric approach to the proof of Berger's holonomy theorem [Si62]. In our exposition, we follow (mostly) the approach and notation from [OR12a]; similar results were proved independently in [Na13]. Partial results may already be found in [AF04a].

Definition B.1 ([OR12a]). Let $G \subset \mathrm{SO}(n)$ be a connected Lie subgroup, $V = \mathbb{R}^n$ the corresponding G-module, and $T \in \Lambda^3(V)$ a 3-form such that $X \lrcorner T \in \mathfrak{g} \subset \mathfrak{so}(V)$ for all $X \in V$. Such a triple (G, V, T) is called a *skew-torsion holonomy system*. A skew torsion holonomy system is said to be *irreducible* if G acts irreducibly on V, *transitive* if G acts transitively on the unit sphere of V, and *symmetric* if T is G-invariant.

Theorem B.2 (Skew Holonomy Theorem [OR12a, Thm 1.4, Thm 4.1], [Na13]). *Let (G, V, T), $T \neq 0$ be an irreducible skew-torsion holonomy system. If it is transitive, $G = \mathrm{SO}(n)$. If it is not transitive, it is symmetric, and*

(1) *V is an orthogonal simple Lie algebra of rank at least two with respect to the bracket $[X,Y] = T(X,Y)$,*
(2) *$G = \mathrm{Ad}\,(H)$, where H is the connectec Lie group whose Lie algebra is $(V, [\cdot, \cdot])$,*
(3) *T is unique, up to a scalar multiple.*

We give here one further important application of this result. It is well-known that some manifolds carry several connections making it naturally reductive or, equivalently, they can be presented differently as naturally reductive quotients of groups. The easiest examples are probably the odd-dimensional spheres $S^{2n+1} = \mathrm{SO}(2n+2)/\mathrm{SO}(2n+1) = \mathrm{SU}(n+1)/\mathrm{SU}(n)$, whose first presentation is that of a symmetric space (the canonical connection coincides then with the

Levi-Civita-connection), the second presentation is that as a 'Berger sphere' (see Section 9.2). Furthermore, there are exceptional examples like

$$S^6 \ = \ G_2/\mathrm{SU}(3), \quad S^7 \ = \ \mathrm{Spin}(7)/G_2, \quad S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7).$$

These presentations are far from accidental, they all induce interesting G-structures and play a crucial role in the investigation of manifolds with characteristic connections; S^6 is a nearly Kähler manifold and appears in our Classification Theorem 8.9, see also Remark 8.10. A consequence of the Skew Holonomy Theorem is that spheres and projective spaces are basically the only manifolds for which this effect can happen.

Theorem B.3 ([OR12a, Thm 1.2], [OR12b, Thm 2.1])**.** *Let (M^n, g) be a simply connected and irreducible Riemannian manifold that is not isometric to a sphere, nor to a Lie group with a bi-invariant metric or its symmetric dual. Then (M^n, g) admits at most <u>one</u> naturally reductive homogeneous structure.*

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Ilka Agricola, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Strasse, D-35032 Marburg, Germany, `agricola@mathematik.uni-marburg.de`

Ana Cristina Ferreira, Centro de Matemática, Universidade de Minho, Campus de Gualtar, 4710-057 Braga, Portugal, `anaferreira@math.uminho.pt`

Thomas Friedrich, Institut für Mathematik, Humboldt-Universität zu Berlin, Sitz: WBC Adlershof, D-10099 Berlin, Germany, `friedric@mathematik.hu-berlin.de`