QUATERNIONIC HEISENBERG GROUPS AS NATURALLY REDUCTIVE HOMOGENEOUS SPACES

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ABSTRACT. In this note, we describe the geometry of the quaternionic Heisenberg groups from a Riemannian viewpoint. We show, in all dimensions, that they carry an almost 3-contact metric structure which allows us to define the metric connection that equips these groups with the structure of a naturally reductive homogeneous space. It turns out that this connection, which we shall call the canonical connection because of its analogy to the 3-Sasaki case, preserves the horizontal and vertical distributions and even the quaternionic contact structure of the quaternionic Heisenberg groups. We focus on the 7-dimensional case and prove that the canonical connection can also be obtained by means of a cocalibrated G_2 structure. We then study the spinorial properties of this group and present the noteworthy fact that it is the only known example of a manifold which carries generalized Killing spinors with three different eigenvalues.

1. INTRODUCTION

Among all homogenous Riemannian manifolds, naturally reductive spaces are a class of particular interest. Traditionally, they are defined as Riemannian manifolds $(M = G/K, g)$ with a reductive complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ such that

$$(1) \qquad \langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle \;=\; 0 \;\; \text{for all } X, Y, Z \in \mathfrak{m},$$

where $\langle -, - \rangle$ denotes the inner product on $\mathfrak{m}$ induced from g. For any reductive homogeneous space, the submersion $G \to G/K$ induces a connection that is called the *canonical connection*. It is a metric connection ∇ with torsion $T(X, Y) = -[X, Y]_{\mathfrak{m}}$ wich satisfies $\nabla T = \nabla \mathcal{R} = 0$, and condition (1) thus states that a naturally reductive homogeneous space is a reductive space for which the torsion $T(X, Y, Z) := g(T(X, Y), Z)$ (viewed as a $(3, 0)$-tensor) is a 3-form on G/K (see [14, Ch. X] as a general reference). Classical examples of naturally reductive homogeneous spaces include irreducible symmetric spaces, isotropy irreducible homogeneous manifolds, Lie groups with a biinvariant metric, and Riemannian 3-symmetric spaces.

In the recent article [2], the first two authors together with Thomas Friedrich (Berlin) initiated a systematic investigation and, up to dimension six, achieved the classification of naturally reductive homogeneous spaces. This is done by applying recent results and techniques from the holonomy theory of metric connections with skew torsion.

Definition 1. We call a Riemannian manifold (M, g) *naturally reductive* if it is a homogeneous space $M = G/K$ endowed with a metric connection ∇ with skew torsion T such that its torsion and curvature $\mathcal{R}$ are ∇-parallel, i. e. $\nabla T = \nabla \mathcal{R} = 0$.

If M is connected, complete, and simply connected, a result of Tricerri asserts that the space is indeed naturally reductive in the traditional sense [16].

2. Geometry of quaternionic Heisenberg groups

2.1. **Lie groups of type H.** Lie groups of type H or generalized Heisenberg groups are, as the name indicates, a generalization of the classical Heisenberg groups. We will give a brief overview of such groups and treat the naturally reductive ones from the point of view of contact geometry. We start by describing the Lie algebra of such a group. Let Z and V be two (real) vector spaces of any positive dimension. Equip such vector spaces with some inner product, which shall be denoted for both spaces as $\langle -, - \rangle$. Suppose there is a linear map $k : Z \longrightarrow \mathrm{End}(V)$ such that

$$\|k(a)x\| = \|x\|\|a\| \quad \text{and} \quad k(a)^2 = -\|a\|^2 \mathrm{Id}$$

for $a \in Z$, $x, y \in V$. We can use the map k to define the Lie algebra $\mathfrak{n}$ as the direct sum $\mathfrak{n} = Z \oplus V$ together with the bracket defined by

$$[a + x, b + y] = [x, y] \quad \text{and} \quad \langle [x, y], a \rangle = \langle k(a)x, y \rangle,$$

where $a, b \in Z$ and $x, y \in V$. Then $\mathfrak{n}$ is said to be a Lie algebra of type H. Remark that $\mathfrak{n}$ is a 2-step nilpotent Lie algebra with center Z. There are infinitely many Lie algebras of type H with center of any given dimension. The connected, simply-connected Lie group N with Lie algebra $\mathfrak{n}$ is said to be a Lie group of type H. Observe also that the Lie algebra $\mathfrak{n}$ can be equipped with an inner product such that the decomposition $Z \oplus V$ is orthogonal and N is endowed with a left invariant metric g induced by the inner product on $\mathfrak{n}$. Of particular interest are the Lie algebras which are obtained from the composition algebras W – the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$ as follows: Z is the space formed by purely imaginary numbers, V is a power of W, i.e., $V = W^n$ and $k : Z \longrightarrow \mathrm{End}(V)$ is simply the linear map given by ordinary scalar multiplication. The corresponding groups are the Heisenberg groups or their quaternionic or octonionic analogs. As far as naturally reductive spaces go, we have the following theorem of Tricerri and Vanhecke.

Theorem 1 ([17], Theorem 9.1, page 96)**.** *The Lie group N with its left invariant metric g is naturally reductive if and only if N is a Heisenberg group or a quaternionic Heisenberg group.*

Heisenberg groups of dimension $2n + 1$ were described as naturally reductive spaces in [2]. The Heisenberg group of dimension 5 was the first known example of a manifold with parallel skew torsion carrying a Killing spinor with torsion that does not admit a Riemannian Killing spinor [3], [6].

2.2. **The quaternionic Heisenberg group N_p of dimension** $4p + 3$**.** Let $p \in \mathbb{N}$, V be the space of quaternions and Z be the space of imaginary quaternions. Consider the Lie algebra $\mathfrak{n}_p = Z \oplus V^p$, of dimension $4p + 3$, and denote by N_p its corresponding connected, simply connected Lie group. For ease of notation we will denote by z_1, z_2, z_3 the standard elements i, j, k in Z and by $\tau_r, \tau_{p+r}, \tau_{2p+r}, \tau_{3p+r}$ the elements $1, i, j, k$ in each copy of V^p, respectively, $r = 1, \ldots, p$. More concisely, we make the following identifications for $r = 1, \ldots, p$

$$\tau_r \longmapsto 1, \quad z_1, \ \tau_{p+r} \longmapsto i, \quad z_2, \ \tau_{2p+r} \longmapsto j, \quad z_3, \ \tau_{3p+r} \longmapsto k.$$

We introduce a parameter λ in our metric by declaring that the set $\xi_i := \frac{z_i}{\lambda}$ ($1 \leq i \leq 3$), τ_l ($1 \leq l \leq 4p$) is an orthonormal frame for the metric g_λ, $\lambda > 0$. The commutator relations are now written as

$$
\begin{array}{lll}
[\tau_r, \tau_{p+r}] = \lambda\, \xi_1 & [\tau_r, \tau_{2p+r}] = \lambda\, \xi_2 & [\tau_r, \tau_{3p+r}] = \lambda\, \xi_3 \\
[\tau_{2p+r}, \tau_{3p+r}] = \lambda\, \xi_1 & [\tau_{3p+r}, \tau_{p+r}] = \lambda\, \xi_2 & [\tau_{p+r}, \tau_{2p+r}] = \lambda\, \xi_3
\end{array}
$$

with all the remaining commutators begin zero. The Levi-Civita connection can be computed, but it is not very insightful, so we will not reproduce this calculation here. Let us just point out that we have three Riemannian Killing fields, namely ξ_1, ξ_2, ξ_3. We do not have a distinguished direction, but a distinguished 3-dimensional distribution in the tangent bundle, so the best approach to study the geometry of these groups is to consider 3-contact structures. Let η_i be the dual form of ξ_i, $i = 1, 2, 3$, and θ_l be the dual form of τ_l, respectively, $l = 1, \ldots, 4p$. Define the $(1,1)$-tensors

$$\varphi_1 = \eta_2 \otimes \xi_3 - \eta_3 \otimes \xi_2 + \sum_{r=1}^{p}[\theta_r \otimes \tau_{p+r} - \theta_{p+r} \otimes \tau_r + \theta_{2p+r} \otimes \tau_{3p+r} - \theta_{3p+r} \otimes \tau_{2p+r}],$$
$$\varphi_2 = \eta_3 \otimes \xi_1 - \eta_1 \otimes \xi_3 + \sum_{r=1}^{p}[\theta_r \otimes \tau_{2p+r} - \theta_{2p+r} \otimes \tau_r + \theta_{3p+r} \otimes \tau_{p+r} - \theta_r \otimes \tau_{3p+r}],$$
$$\varphi_3 = \eta_1 \otimes \xi_2 - \eta_2 \otimes \xi_1 + \sum_{r=1}^{p}[\theta_r \otimes \tau_{3p+r} - \theta_{3p+r} \otimes \tau_r + \theta_{p+r} \otimes \tau_{2p+r} - \theta_{2p+r} \otimes \tau_{p+r}].$$

It is easy to check that the triple $(\varphi_1, \varphi_2, \varphi_3)$ satisfies the compatibility equations

$$\varphi_i = \varphi_j\varphi_k - \eta_k \otimes \xi_j = -\varphi_k\varphi_j + \eta_j \otimes \xi_k,$$

(for $(i, j, k) = (1, 2, 3)$ and cyclic permutations) and also that all three almost contact structures are compatible with the metric. All in all, $(N_p, \varphi_i, \xi_i, g_\lambda)$ is an almost 3-contact metric manifold (see the classical monography [8] for more information on this topic). Note that none of the structures φ_i is quasi-Sasaki, so N_p is not a 3-(quasi)-Sasaki manifold, but all three are normal (vanishing Nijenhuis tensor). The vertical subbundle T^v is spanned by ξ_1, ξ_2, ξ_3, the horizontal subbundle T^h is its orthogonal complement. For later use, let us write down the formulas for the differentials $d\eta_i$:

$$(2) \qquad d\eta_i = -\lambda\sum_{r=1}^{p}[\theta_{r,ip+r} + \theta_{(i+1)p+r,(i+2)p+r}], \qquad i = 1, 2, 3.$$

Henceforth, we write θ_{ij} for $\theta_i \wedge \theta_j$ (and similarly for η), and the index i is understood modulo 3, i. e. for $i = 2$, $i + 2 = 1 \mod 3$, thus $(i+2)p + r$ is to be read as $p + r$. In particular, for $p = 1$, we have the simple formulas

$$(3) \qquad d\eta_1 = -\lambda(\theta_{12} + \theta_{34}), \quad d\eta_2 = -\lambda(\theta_{13} - \theta_{24}), \quad d\eta_3 = -\lambda(\theta_{14} + \theta_{23}).$$

Each of the three almost contact structures (φ_i, ξ_i) $(i = 1, 2, 3)$ of N_p has a characteristic connection [11], but they do not coincide, and each of them is not well adapted to the underlying 3-contact structure. In the article [4], a notion of canonical connection was proposed for 7-dimensional 3-Sasaki manifolds. This was the metric connection ∇ with skew torsion $T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i$; it was shown to preserve the vertical and horizontal subbundles, and to admit a ∇-parallel spinor field ψ with the property that the fields $\xi_i \cdot \psi$ were the Riemannian Killing spinors of the manifold. The construction was done by using an intermediate cocalibrated G_2-structure.

Even though we are not in the 3-Sasaki case, we will now show that a similar connection can be constructed on N_p, and that this connection gives N_p a naturally reductive homogeneous structure. The special case $p = 1$ will be considered separately in Section 2.4.

Theorem 2. *On the almost 3-contact metric manifold $(N_p, \varphi_i, \xi_i, g_\lambda)$, the metric connection ∇ with skew torsion*

$$(4) \qquad T = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3 - 4\lambda\eta_{123}$$

has the following properties:

(1) *Its torsion and curvature are ∇-parallel, $\nabla T = \nabla \mathcal{R} = 0$;*

(2) *Its holonomy algebra is isomorphic to $\mathfrak{su}(2)$, acting irreducibly on T^v and on T^h by p copies of its 4-dimensional representation; in particular, ∇ preserves the vertical and horizontal subbundles T^v, T^h.*

Proof. One computes that the metric connection with skew torsion T is described by the map $\Omega : \mathfrak{n}_p \longrightarrow \Lambda^2 \mathfrak{n}_p = \mathfrak{so}(\mathfrak{n}_p)$

$$\Omega(\xi_i) = -\lambda H_i, \qquad \Omega(\tau_l) = 0, \qquad i = 1, 2, 3, \ l = 1, \dots, 4p$$

where H_1, H_2, H_3 are given by

$$H_1 = -\frac{1}{\lambda} d\eta_1 + 2\eta_2 \wedge \eta_3, \quad H_2 = -\frac{1}{\lambda} d\eta_2 - 2\eta_1 \wedge \eta_3, \quad H_3 = -\frac{1}{\lambda} d\eta_1 + 2\eta_1 \wedge \eta_2.$$

The elements H_1, H_2, H_3 satisfy the commutator relation of $\mathfrak{su}(2)$, that is, $[H_1, H_2] = 2H_3$, $[H_3, H_1] = 2H_2$, $[H_2, H_3] = 2H_1$. This fact yields that the curvature tensor can be readily computed to be

$$\mathcal{R} = \lambda^2 [H_1 \otimes H_1 + H_2 \otimes H_2 + H_3 \otimes H_3]$$

and the holonomy algebra $\mathfrak{h}$ of our connection is $\mathfrak{su}(2)$. Clearly, both T and $\mathcal{R}$ are $\mathfrak{h}$-invariant, so we can conclude that both T and $\mathcal{R}$ are parallel objects. It is then established that every quaternionic Heisenberg group has the structure of a naturally reductive homogeneous space.

The elements H_j of the holonomy algebra $\mathfrak{h}$ act on a vector field ξ_i or τ_l by inner product, i.e. $H_j \cdot \xi_i = \eta_i \lrcorner H_j$ and $H_j \cdot \tau_l = \theta_l \lrcorner H_j$. Thus, the explicit formulas (2) for $d\eta_i$ and therefore H_i imply that $\mathfrak{h}$ acts irreducibly on $T^v = \mathrm{Span}(\xi_1, \xi_2, \xi_3)$ and leaves invariant the space $\mathrm{Span}(\theta_r, \theta_{p+r}, \theta_{2p+r}, \theta_{3p+r})$ for each $r = 1 \dots, p$. In particular, this means that not only T is ∇-parallel, but also η_{123} and each of the $\theta_r \wedge \theta_{r,p+r} \wedge \theta_{2p+r} \wedge \theta_{3p+r}$ as well. $\qquad \square$

Definition 2. The connection ∇ described in the previous theorem will be called the *canonical connection* of the almost 3-contact metric manifold $(N_p, \varphi_i, \xi_i, g_\lambda)$.

Remark 1. It is interesting to observe that ∇-Ricci curvature is a diagonal matrix

$$\mathrm{Ric}^\nabla = \mathrm{diag}(-8\lambda^2, -8\lambda^2, -8\lambda^2, -3\lambda^2, \dots, -3\lambda^2),$$

even though it is never a multiple of the identity. We can also deduce the ∇-scalar curvature and the Riemannian scalar curvature to be negative, more precisely, $s^\nabla = -12\lambda^2(p+2)$ and $s^g = s^\nabla + (3/2)\|T\|^2 = -3p\lambda^2$.

2.3. Compatibility of ∇ with the qc structure and comparison to the Biquard connection.
The quaternionic Heisenberg group is the standard example of a non compact quaternionic contact (qc for short) manifold. Standard references are [7] or [10], we shall mainly follow the notations and definitions (which vary slightly) from [12]. Instead of giving the abstract definition of a qc structure, we quickly define all relevant quantities for the quaternionic Heisenberg group and check the required properties:

(1) The endomorphism fields $I_i := \varphi_i|_{T^h}$ are complex structures on T^h satisfying the commutation relations of the quaternions, $I_1 I_2 I_3 = -\mathrm{Id}$;
(2) the triple of 1-forms $\tilde{\eta}_i := -\frac{2}{\lambda} \eta_i$ satisfies $T^h = \cap_i \ker \tilde{\eta}_i$;
(3) the differentials of the 1-forms $\tilde{\eta}_i$ and the vector fields $\tilde{\xi}_j := -\frac{1}{2} z_j$ satisfy the identities $d\tilde{\eta}_j(X, Y) = 2\, g(I_j X, Y)$ for $X, Y \in T^h$ and $d\tilde{\eta}_j(\tilde{\xi}_k, X) = -d\tilde{\eta}_k(\tilde{\xi}_j, X)$ for $X \in T^h$, $j, k = 1, 2, 3$.

In general, the condition for a metric connection ∇ to preserve the qc structure reduces to the requirement that ∇ preserves the splitting $T^h \oplus T^v$ and has the additional two properties

(1) $\nabla(I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3) = 0$;
(2) $\nabla(\tilde{\xi}_1 \otimes I_1 + \tilde{\xi}_2 \otimes I_2 + \tilde{\xi}_3 \otimes I_3) = 0$.

From the formulas given in Theorem 2 and its proof, it is clear that the canonical connection ∇ satisfies these condition, hence we conclude:

Corollary 2.1. *The canonical connection ∇ preserves the above defined qc structure of the quaternionic Heisenberg group N_p.*

In fact, one can show that ∇ is the unique metric connection preserving the qc structure whose torsion is skew-symmetric. The Biquard connection is the most commonly used qc connection; on N_p, it is given by the trivial map $\Omega^B : \mathfrak{n}_p \longrightarrow \Lambda^2 \mathfrak{n}_p = \mathfrak{so}(\mathfrak{n}_p)$, $\Omega^B = 0$. Obviously, it's flat and has vanishing holonomy. The canonical connection of Theorem 2 can be defined in the same way on any qc-Einstein manifold with zero qc-scalar curvature. Other examples can be obtained by considering $\mathbb{R}^3$-bundles over hyper-Kähler manifolds [13, Remark 5.2].

2.4. **The 7-dimensional quaternionic Heisenberg group, G_2-geometry, and its spinorial properties.** Let us explain how the connection ∇ appears naturally as the characteristic connection of a G_2 structure in dimension 7, i.e. when $p = 1$. Consider the three-form

$$\omega = -\eta_1 \wedge (\theta_{12} + \theta_{34}) - \eta_2 \wedge (\theta_{13} + \theta_{42}) - \eta_3 \wedge (\theta_{14} + \theta_{23}) + \eta_{123}.$$

This is a globally defined three-form of generic type and therefore equips our 7-dimensional Lie group N_1 with a G_2 structure. Notice that

$$T = \lambda(\omega - 5\eta_{123}).$$

It is easy to check that this structure is cocalibrated, that is, $d^*\omega = 0$. Therefore, it is known that our manifold has characteristic connection ∇ with torsion [11],

$$T^c = \frac{1}{6}(d\omega, *\omega)\omega - *d\omega.$$

It is a simple calculation to check that $T = T^c$, but observe also that this torsion is extremely similar in spirit to the one given for the 3-Sasaki structure in [4]. Being globally diffeomorphic to $\mathbb{R}^7$, the quaternionic Heisenberg group N_1 carries a unique left-invariant spin structure. As a matter of fact, the 7-dimensional spin representation is real, so denote by Σ the 8-dimensional real spinor bundle over N_1. As a G_2-manifold, it has a ∇-parallel spinor field ψ_0 that may be used to split Σ into three summands,

$$\Sigma \;=\; \Sigma_0 \oplus \Sigma_v \oplus \Sigma_h, \quad \Sigma_0 \;:=\; \mathbb{R} \cdot \psi_0, \quad \Sigma_{v,h} \;:=\; \{X \cdot \psi_0 \;:\; X \in T^{v,h}\}.$$

The following identities are tensorial, they can therefore be checked in any realisation of the real spin representation. In one such representation, one views T as an endomorphism of the spin bundle (replacing all wedge products by Clifford products) and computes, in this basis, an explicit formula for ψ_0 (it is the unique spinor preserved by T). An explicit purely algebraic computer calculation yields then the following result:

Lemma 1. *The spinor field ψ_0 is a T-eigenspinor, $T \cdot \psi_0 = -2\lambda \psi_0$, and the Clifford product $T \cdot X$ acts on ψ_0 as follows:*

$$T \cdot X \cdot \psi_0 = 6\lambda \, X \cdot \psi_0 \;\; for \; X \in T^h, \quad T \cdot X \cdot \psi_0 = -4\lambda \, X \cdot \psi_0 \; for \; X \in T^v.$$

Thus, T acts as multiplication on each of the subbundles of Σ. The spinor field ψ_0 has to be a generalized Killing spinor because the G_2 structure is cocalibrated [9]. The preceding lemma together with $\nabla\psi_0 = 0$ allows us to compute the explicit differential equation of ψ_0:

Corollary 2.2. *The spinor field ψ_0 is a generalized Killing spinor satisfying the differential equation*

$$\nabla^g_X \psi_0 = \frac{\lambda}{2} \, X \cdot \psi_0 \;\; for \; X \in T^v, \quad \nabla^g_X \psi_0 = -\frac{3\lambda}{4} \, X \cdot \psi_0 \;\; for \; X \in T^h.$$

In the 3-Sasaki case, the spinor fields $\xi_i \cdot \psi_0$ are exactly the Riemannian Killing spinors, and they define a nearly parallel G_2 structure. This cannot hold on N_1 (being a nilpotent Lie group, it cannot carry an Einstein metric), so it becomes an interesting question to compute instead the field equation that these three spinors satisfy. We prove:

Corollary 2.3. *The spinor fields $\psi_i := \xi_i \cdot \psi_0$, $i = 1, 2, 3$, are generalized Killing spinors satisfying the differential equation*

$$\nabla^g_{\xi_i} \psi_i = \frac{\lambda}{2} \, \xi_i \cdot \psi_i, \quad \nabla^g_{\xi_j} \psi_i = -\frac{\lambda}{2} \, \xi_j \cdot \psi_i \ (i \neq j), \quad \nabla^g_X \psi_i = \frac{5\lambda}{4} \, X \cdot \psi_i \ \ for \ X \in T^h.$$

Proof. Denote by $s(X)$ the eigenvalue such that $\nabla^g_X \psi_0 = s(X) X \cdot \psi_0$ as in Corollary 2.2, i.e. $s(X) = \frac{\lambda}{2}$ resp. $s(X) = -\frac{3\lambda}{4}$ for X in T^v resp. T^h. As duals of Killing vector fields, the one-forms η_i satisfy the equation $\nabla^g_X \eta_i = \frac{1}{2} X \lrcorner \, d\eta_i$, hence

$$(5) \qquad \nabla^g_X (\xi_i \cdot \psi_0) = \nabla^g_X (\xi_i) \cdot \psi_0 + \xi_i \cdot \nabla^g_X \psi_0 = \frac{1}{2} (X \lrcorner \, d\eta_i) \cdot \psi_0 + s(X) \, \xi_i \cdot X \cdot \psi_0.$$

Consider first the case that $X \in T^v$. In this situation, the first term vanishes because of the expressions for $d\eta_i$, see Eq. (3). Furthermore, $s(X) = \lambda/2$, so we obtain

$$\nabla^g_X (\xi_i \cdot \psi_0) = \frac{\lambda}{2} \xi_i \cdot X \cdot \psi_0.$$

If $X = \xi_i$, the first statement of the corollary follows, and for $X = \xi_j$ with $i \neq j$, the second statement follows after use of the identity $\xi_i \cdot \xi_j = -\xi_j \cdot \xi_i$.

Now assume $X \in T^h$. We can set $s(X) = -3\lambda/4$, and a computer computation in the real spin representation proves $(X \lrcorner \, d\eta_i) \cdot \psi_0 = X \cdot \xi_i \cdot \psi_0$. Therefore, Eq. (5) becomes

$$\nabla^g_X (\xi_i \cdot \psi_0) = \frac{\lambda}{2} X \xi_i \cdot \psi_0 - s(X) \, X \cdot \xi_i \cdot \psi_0 = \frac{5\lambda}{4} X \cdot \xi_i \cdot \psi_0$$

(observe that $\xi_i \notin T^h$, so $X \cdot \xi_i = -\xi_i \cdot X$ for all possible X). $\square$

Remark 2. Generalized Killing spinors with three distinct eigenvalues seem to be very rare—the authors do not know of any other examples. Moroianu and Semmelmann investigated the existence of generalized Killing spinors with two distinct eigenvalues on spheres S^n and proved that they only exist if $n = 3$ or $n = 7$ [15]; in the latter case, they are induced from the canonical spinor of the underlying 3-Sasaki structure introduced in [4].

The first author and J. Höll discussed cones of G manifolds and their spinorial properties in the article [5]. In Section 3.5, they constructed three almost Hermitian structures on the cone of a metric almost contact 3-structure. Using this construction, we prove:

Theorem 3. *The cone $(\bar{N}_1, \bar{g}) = (N_1 \times \mathbb{R}^+, \lambda^2 r^2 g + dr^2)$ is an 8-dimensional hyper-Kähler manifold with torsion ('HKT manifold').*

Proof. Let T_i be the torsion of the characteristic connection of the normal almost contact structures (φ_i, ξ_i) $(i = 1, 2, 3)$ of N_1. One checks that

$$T_i = \eta_i \wedge d\eta_i - \sum_{j=1, j \neq i}^{3} \eta_j \wedge d\eta_j.$$

We apply Theorem 3.23 of [5]. The crucial point is that we have to show the existence of a positive constant a (the cone constant) such that the three tensors $S_i := T_i - 2a \, \eta_i \wedge F_i$ coincide. This happens exactly when a equals the metric parameter λ, and the claim follows. $\square$

This generalizes of course the well-known fact that the cone of a 3-Sasaki manifold is a hyper-Kähler manifold.

Remark 3. Corollary 4.14 of the same article [5] yields immediatly that the connection ∇ on N_1, viewed as the characteristic connection of a cocalibrated G_2-structure, induces a Spin(7)-structure on the same cone, and that the spinor ψ_0 lifts to a spinor that is parallel for the characteristic Spin(7)-connection. Thus, the HKT structure is compatible in a very subtle sense with a Spin(7)-structure.

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