MANIFOLDS WITH VECTORIAL TORSION

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Abstract. The present note deals with the properties of metric connections ∇ with vectorial torsion V on semi-Riemannian manifolds (M^n, g). We show that the ∇-curvature is symmetric if and only if $V^\flat$ is closed, and that $V^\perp$ then defines an $(n-1)$-dimensional integrable distribution on M^n. If the vector field V is exact, we show that the V-curvature coincides up to global rescaling with the Riemannian curvature of a conformally equivalent metric. We prove that it is possible to construct connections with vectorial torsion on warped products of arbitrary dimension matching a given Riemannian or Lorentzian curvature—for example, a V-Ricci-flat connection with vectorial torsion in dimension 4, explaining some constructions occurring in general relativity. Finally, we investigate the Dirac operator D of a connection with vectorial torsion. We prove that for exact vector fields, the V-Dirac spectrum coincides with the spectrum of the Riemannian Dirac operator. We investigate in detail the existence of V-parallel spinor fields; several examples are constructed. It is known that the existence of a V-parallel spinor field implies $dV^\flat = 0$ for $n = 3$ or $n \geq 5$; for $n = 4$, this is only true on compact manifolds. We prove an identity relating the V-Ricci curvature to the curvature in the spinor bundle. This result allows us to prove that if there exists a nontrivial V-parallel spinor, then $\mathrm{Ric}^V = 0$ for $n \neq 4$ and $\mathrm{Ric}^V(X) = X \lrcorner\, dV^\flat$ for $n = 4$. We conclude that the manifold is conformally equivalent either to a manifold with Riemannian parallel spinor or to a manifold whose universal cover is the product of $\mathbb{R}$ and an Einstein space of positive scalar curvature. We also prove that if $dV^\flat = 0$, there are no non-trivial ∇-Killing spinor fields.

1. Introduction

The present note deals with metric connections on semi-Riemannian manifold (M^n, g) of the form

$$\nabla_X Y = \nabla^g_X Y + g(X, Y)V - g(V, Y)X,$$

where V denotes a fixed vector field on M and ∇^g is the usual Levi-Civita connection. This is one of the three basic types of metric connections introduced by Élie Cartan (see Section 2), and is called a metric connection with *vectorial torsion*. These connections are particularly interesting on surfaces, in as much that *every* metric connection on a surface is of this type. By a seminal theorem of Ambrose and Singer [AS58], a simply connected Riemannian manifold is homogeneous if and only if its canonical connection ∇^c has ∇^c-parallel torsion and curvature. Cartan's result implies therefore that there are three possible basic types of non-symmetric homogeneous spaces (see also [TV83]). If the canonical connection has vectorial torsion, the condition ∇^c-parallel torsion is equivalent to $\nabla^c V = 0$. Such homogeneous spaces were intensively studied in the past, while only few results on general semi-Riemannian manifolds equipped with a metric connection ∇ with vectorial torsion are available. In [TV83, Thm 5.2], F. Tricerri and L. Vanhecke showed that if M is connected, complete, simply-connected and $\nabla V = \nabla\mathcal{R} = 0$, then (M, g) has to be

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isometric to hyperbolic space (see also [CGS13] for an alternative modern proof). Hence, it is too strong a condition to require that V is ∇-parallel.

Denote by $V^\flat$ the dual 1-form of V; we shall loosely call the vector field V *closed* if $dV^\flat = 0$. In Section 3, we prove that ∇ has symmetric curvature if and only if V is closed, and that this is equivalent to the condition that the $(n-1)$-dimensional distribution $D := V^\perp$ is involutive. We thus believe that $dV^\flat = 0$ is a richer, geometrically interesting replacement of the condition $\nabla V = 0$. If the vector field is exact, we show that the V-curvature coincides up to global rescaling with the Riemannian curvature of a conformally equivalent metric. We then investigate more in detail the case of warped products, both with Riemannian and Lorentzian signature. In particular, we prove that it is possible to construct connections with vectorial torsion on warped products of arbitrary dimension matching a given Riemannian or Lorentzian curvature—for example, a V-Ricci-flat connection with vectorial torsion in dimension 4. This explains the occurrence of some examples of V-Ricci-flat manifolds known in physics [OM97].

In the last part of the paper, we investigate the Dirac operator D of a connection with vectorial torsion. As already observed by Friedrich in [Fr79], the Dirac operator is not formally self-adjoint anymore, thus making its analysis much harder. Nevertheless, we prove that for exact vector fields, the V-Dirac spectrum coincides with the spectrum of the Riemannian Dirac operator. Based on results from [PS11], we derive a formula of Schrödinger-Lichnerowicz type formula for D^*D. We investigate in detail the existence of V-parallel spinor fields (i. e. parallel for the metric connection with vectorial torsion V); several examples are constructed on warped products. By a result of [Mo96], it is known that the existence of a V-parallel spinor field implies $dV^\flat = 0$ for $n = 3$ or $n \geq 5$; for $n = 4$, this is only true on compact manifolds. To get a more detailed picture, we prove an identity relating the V-Ricci curvature to the curvature in the spinor bundle. This result allows us to prove that if there exists a nontrivial V-parallel spinor, then $\mathrm{Ric}^V = 0$ for $n \neq 4$ and $\mathrm{Ric}^V(X) = X \lrcorner\, dV^\flat$ for $n = 4$—in particular, the V-Ricci curvature is totally skew-symmetric in the latter case, a rather unfamiliar situation. We conclude that the manifold is conformally equivalent either to a manifold with parallel spinor or to a manifold whose universal cover is the product of $\mathbb{R}$ and an Einstein space of positive scalar curvature. We also prove that if $dV^\flat = 0$, there are no non-trivial ∇-Killing spinor fields.

2. Metric connections with torsion

Consider a semi-Riemannian manifold (M^n, g) of index k. The difference between its Levi-Civita connection ∇^g and any linear connection ∇ is a $(2,1)$-tensor field A,

$$\nabla_X Y \;=\; \nabla^g_X Y + A(X,Y), \quad X, Y \in TM^n.$$

Following Cartan, we study the algebraic types of the torsion tensor for a metric connection. Denote by the same symbol the $(3,0)$-tensor derived from a $(2,1)$-tensor by contraction with the metric. We identify TM^n with $(TM^n)^*$ using g from now on. Let $\mathcal{T}$ be the $n^2(n-1)/2$-dimensional space of all possible torsion tensors,

$$\mathcal{T} \;=\; \{T \in \otimes^3 TM^n \mid T(X,Y,Z) = -T(Y,X,Z)\} \;\cong\; \Lambda^2 TM^n \otimes TM^n\,.$$

A connection ∇ is metric if and only if A belongs to the space

$$\mathcal{A}^g \;:=\; TM^n \otimes (\Lambda^2 TM^n) \;=\; \{A \in \otimes^3 TM^n \mid A(X,V,W) + A(X,W,V) \;=\; 0\}\,.$$

In particular, $\dim \mathcal{A}^g = \dim \mathcal{T}$, reflecting the fact that metric connections can be uniquely characterized by their torsion. The following proposition has been proven in [Ca25, p.51], [TV83] in the Riemannian case, but one easily checks that it holds also for semi-Riemannian manifolds.

Proposition 2.1. *The spaces $\mathcal{T}$ and $\mathcal{A}^g$ are isomorphic as $\mathrm{O}(n,k)$ representations, an equivariant bijection being*

$$T(X,Y,Z) = A(X,Y,Z) - A(Y,X,Z), \quad 2\,A(X,Y,Z) = T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y).$$

For $n = 2$, $\mathcal{T} \cong \mathcal{A}^g \cong \mathbb{R}^2$ is $\mathrm{O}(2,k)$-irreducible, while for $n \geq 3$, it splits under the action of $\mathrm{O}(n,k)$ into the sum of three irreducible representations,

$$\mathcal{T} \cong TM^n \oplus \Lambda^3(M^n) \oplus \mathcal{T}'.$$

The connection ∇ is said to have *vectorial torsion* if its torsion tensor lies in the first space of the decomposition in Proposition 2.1, i. e. if it is essentially defined by some vector field V on M. The tensors A and T can then be directly expressed through V as

$$(1) \qquad A_V(X)Y \;=\; g(X,Y)V - g(V,Y)X, \quad T_V(X,Y,Z) \;=\; g\big(g(V,X)Y - g(V,Y)X, Z\big).$$

The connection ∇ is said to have *skew-symmetric torsion* or just *skew torsion* if its torsion tensor lies in the second component of the decomposition in Proposition 2.1, i. e. it is given by a 3-form. The third torsion component has no geometric interpretation. Recall that homogeneous spaces whose canonical connection has skew torsion are usually known as *naturally reductive* homogeneous spaces; their holonomy properties and their classification are currently topics of great interest. While metric connections with ∇-parallel skew torsion have a rich geometry (for example, the characteristic connections of Sasaki manifolds, of nearly Kähler manifolds, and of nearly parallel G_2 manifolds have this property), metric connections with ∇-parallel vectorial torsion are rare–the underlying manifold has to be covered by hyperbolic space. Alas, this means that the general holonomy principle will not be applicable for the investigation of metric connections with vectorial torsion.

3. Curvature

Let (M,g) be an n-dimensional semi-Riemannian manifold and ∇ a metric connection on M, possibly with torsion T^∇. We denote the Levi-Civita connection on M by ∇^g. For a vector field V on M, we define a 1-form $A_V \in \Omega^1(M, \mathrm{End}(TM))$ by

$$(2) \qquad A_V(X)Y \;:=\; g(X,Y)V - g(V,Y)X,$$

hence $\nabla^g + A_V$ becomes a connection with vectorial torsion on M. Then the following formulas relating the curvatures of ∇ and $\nabla + A_V$ hold:

Lemma 3.1. *The curvature quantities of the connections ∇ and $\nabla + A_V$ satisfy the following relations:*

(1) *Curvature transformation:*

$$\begin{aligned}
\mathcal{R}^{\nabla + A_V}(X,Y)Z \;=\;& \mathcal{R}^\nabla(X,Y)Z + g(Y,Z)\nabla_X V - g(X,Z)\nabla_Y V \\
&+ \big[g(\nabla_Y V, Z) - g(Y,Z)\|V\|^2 + g(V,Z)g(V,Y)\big]X \\
&- \big[g(\nabla_X V, Z) - g(X,Z)\|V\|^2 + g(V,Z)g(V,X)\big]Y \\
&+ \big[g(Y,Z)g(V,X) - g(X,Z)g(V,Y)\big]V \\
&+ g(T^\nabla(X,Y),Z)V - g(Z,V)T^\nabla(X,Y).
\end{aligned}$$

(2) *Ricci curvature:*

$$\begin{aligned}
\mathrm{Ric}^{\nabla + A_V}(X,Y) \;=\;& \mathrm{Ric}^\nabla(X,Y) + [\mathrm{div}^\nabla V + (2-n)\|V\|^2]\, g(X,Y) \\
&+ (n-2)\,[g(V,X)g(V,Y) + g(\nabla_X V, Y)] \\
&+ g(V,Y)\mathrm{tr}(X \lrcorner T^\nabla) + g(T^\nabla(V,X),Y).
\end{aligned}$$

(3) *Scalar curvature:*

$$s^{\nabla + A_V} \;=\; s^{\nabla} + 2(n-1)\mathrm{div}^{\nabla} V + (n-1)(2-n)\|V\|^2 + 2\mathrm{tr}(V \lrcorner T^{\nabla}).$$

Proof. The formulas follow by a routine computation (for $\nabla = \nabla^g$, the terms involving T^{∇} vanish, see also the Appendix of [Ag06]). We normalize the Ricci curvature as follows

$$\mathrm{Ric}^{\nabla + A_V}(X,Y) \;=\; \sum_{i=1}^{n} \varepsilon_i g(\mathcal{R}^{\nabla + A_V}(X,e_i)e_i, Y)$$

for an orthonormal basis $e_1,\ldots,e_n$ $(g(e_i,e_j) = \varepsilon_i\delta_{ij},\ \varepsilon_i = \pm 1)$. The trace and divergence are defined by

$$\mathrm{tr}(V \lrcorner T^{\nabla}) \;:=\; \sum_{i=1}^{n} \varepsilon_i g(T^{\nabla}(V,e_i),e_i)), \quad \mathrm{div}^{\nabla} V \;:=\; \sum_{i=1}^{n} \varepsilon_i g(\nabla_{e_i} V, e_i). \qquad \square$$

In case that ∇ has skew symmetric torsion, $\mathrm{div}^{\nabla} V = \mathrm{div}^{\nabla^g} =: \mathrm{div} V$ and $\mathrm{tr}(X \lrcorner T^{\nabla}) = 0$.
Let us now consider the case where $\nabla = \nabla^g$ the Levi-Civita connection. We denote the curvature of ∇^g by $\mathcal{R}^g$ and the curvature of $\nabla := \nabla^g + A_V$ by $\mathcal{R}^V$, and analogously for the Ricci and scalar curvatures. We call a manifold V-flat if $\mathcal{R}^V = 0$ and use the words V-Einstein and so on in the same way.

Corollary 3.1. *Let (M,g) be an n-dimensional semi-Riemannian manifold, ∇ a connection with vectorial torsion V. Then*

(1) *Ricci curvature:*

$$\mathrm{Ric}^V(X,Y) = \mathrm{Ric}^g(X,Y) + [\mathrm{div}\, V + (2-n)\|V\|^2]\, g(X,Y) + (n-2)\, [g(V,X)g(V,Y) + g(\nabla^g_X V, Y)].$$

(2) *Scalar curvature:*

$$s^V \;=\; s^g + 2(n-1)\mathrm{div}\, V + (n-1)(2-n)\|V\|^2.$$

For surfaces, the formulas simplify

Corollary 3.2. *Let (M,g) be an 2-dimensional semi-Riemannian manifold, ∇ a connection with vectorial torsion V. Then*

(1) *Ricci curvature:* $\mathrm{Ric}^V(X,Y) = \mathrm{Ric}^g(X,Y) + \mathrm{div}\, V g(X,Y)$
(2) *Scalar curvature:* $s^V \;=\; s^g + 2\,\mathrm{div} V$

In particular, a surface M is V-Einstein if and only if M is Einstein.

Remark 3.1. If M is a closed surface, the total V-scalar curvature $S^V(M) := \int_M s^V d\mu$ is equal to $2\pi\chi(M)$, and therefore independent of V. Especially if M is V-flat then M is a torus.

In dimension $n > 2$ we consider the the Schouten tensor

$$C^g = \frac{1}{n-2}\Big(\frac{1}{2(n-1)} s^g g - \mathrm{Ric}^g\Big) \owedge g$$

and the Weyl tensor W^g in Dimension $n > 3$. Here $\owedge$ denotes the Kulkarni-Nomizu Product, which is usually defined for symmetric tensors, but we will also use it for two arbitrary tensors:

$$(\alpha \owedge \beta)(X,Y,Z,W) := \alpha(X,Z)\beta(Y,W) + \alpha(Y,W)\beta(X,Z) - \alpha(X,W)\beta(Y,Z) - \alpha(Y,Z)\beta(X,W).$$

Note that this is a tensor which is still antisymmetric in the first as well as in the last two variables. We define the Schouten Tensor for the V-curvature in the same way as the usual Schouten Tensor:

$$C^V = \frac{1}{n-2}\Big(\frac{1}{2(n-1)} s^V g - \mathrm{Ric}^V\Big) \owedge g$$

Then the equation for the V-curvature simplifies as follows:

Proposition 3.1. *Let (M,g) be a semi-Riemannian manifold of dimension $n>2$, ∇ a connection with vectorial torsion V. Then for the V-curvature tensor holds*

(1) $\mathcal{R}^V = W^g + C^V$ *if* $n>3$
(2) $\mathcal{R}^V = C^V$ *if* $n=3$

Proof. Let $(e_1,\dots,e_n)$ an orthonormal basis. We denote the components of the curvature tensore by $g(\mathcal{R}(e_i,e_j)e_k,e_l) = \mathcal{R}_{ijkl}$ and analogously for the other tensors. We always suppose that $i\neq j\neq k\neq l$. Note that both sides of the equations are pairwise antisymmetric in the first two and in the last two arguments. Therefore it suffices to calculate the following cases:

$$\begin{aligned}
\mathcal{R}^V_{ijji} &= \mathcal{R}^g_{ijji} + g(\nabla_{e_i}V,e_i) + g(\nabla_{e_j}V,e_j) - \|V\|^2 + V_i^2 + V_j^2 \\
&= \mathcal{R}^g_{ijji} + \frac{1}{n-2}(\mathrm{Ric}^V_{ii} - \mathrm{Ric}^g_{ii} + \mathrm{Ric}^V_{jj} - \mathrm{Ric}^g_{jj} + \frac{s^g - s^V}{n-1}) \\
&= \mathcal{R}^g_{ijji} - C^g_{ijji} + C^V_{ijji} \;=\; W^g_{ijji} + C^V_{ijji} \\
\mathcal{R}^V_{ijjk} &= \mathcal{R}^g_{ijjk} + g(\nabla_{e_i}V,e_k) + V_iV_k \;=\; \mathcal{R}^g_{ijjk} + \frac{1}{n-2}(\mathrm{Ric}^V_{ik} - \mathrm{Ric}^g_{ik}) \\
&= W^g_{ijjk} + C^V_{ijjk} \\
\mathcal{R}^V_{ijkl} &= \mathcal{R}^g_{ijkl} = W^g_{ijkl} + C^V_{ijkl}
\end{aligned}$$

Here we inserted the formulas for the Ricci curvature of Corollary 3.1 in the formula for the curvature tensor. Obviously, for $n=3$ the last equation as well as the Weyl tensor vanish. $\square$

Corollary 3.3. *Let (M,g) a Riemannian manifold of Dimension $n>3$, V a vector field. If M is V-flat, then M is also conformally flat. Conversely, if M is conformally flat and V-Ricci flat, then M is V-flat.*

Remark 3.2. The triple (M,g,V) defines a Weyl structure, i. e. a conformal manifold with a torsion free connection which is compatible with the conformal class. The corresponding Weyl curvature satisfies $\mathcal{R}^{Weyl} = \mathcal{R}^V + dV^\flat$, therefore it coincides with the V-curvature if $dV^\flat = 0$ [Gau95]. This was the argument used in [AF10] to prove the first part of the corollary if $dV^\flat = 0$.

Our aim is to show now that indeed, assuming the vector field V to be closed is a very natural condition. The V-Ricci curvature for surfaces is symmetric, but in higher dimensions this is not necessarily the case. For reference, let us recall that the first Bianchi identity for a metric connection with vectorial torsion V is (see for example [Ag06])

$$\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}^V(X,Y)Z \;=\; \overset{X,Y,Z}{\mathfrak{S}}\ dV^\flat(X,Y)Z. \tag{3}$$

Proposition 3.2. *Let (M,g) be a semi-Riemannian manifold of dimension $n>2$, ∇ a connection with vectorial torsion V. Then the following conditions are equivalent:*

(1) *The V-curvature $\mathcal{R}^V(X,Y,Z,W)$ is symmetric with respect to the pairwise exchange of (X,Y) and (Z,W),*
(2) *the V-Ricci tensor Ric^V is symmetric,*
(3) *the 1-form $V^\flat$ is closed.*

Proof. We proceed as in the Riemannian case: we take the inner product of the first Bianchi identity with some fourth vector field W and sum cyclically over X,Y,Z, and W. This yields on the left hand side

$$\overset{X,Y,Z,W}{\mathfrak{S}}\ \big(\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}^V(X,Y,Z,W)\big) \;=\; 2\mathcal{R}^V(Z,X,Y,W) - 2\mathcal{R}^V(Y,W,Z,X),$$

hence the V-curvature is 'pairwise' symmetric if this quantity vanishes identically. For the right hand side, we compute

$$\overset{X,Y,Z,W}{\mathfrak{S}}\Big(\overset{X,Y,Z}{\mathfrak{S}}\, dV^\flat(X,Y)g(Z,W)\Big) \;=\; 2g(\overset{X,Y,Z}{\mathfrak{S}}\, dV^\flat(X,Y)Z,W).$$

Since the inner product is non-degenerate and X,Y,Z can be chosen to be linearly independent, we conclude that the right hand side vanishes if and only if $dV^\flat = 0$.
Of course, Ric^V will be symmetric if $\mathcal{R}^V$ is symmetric; but let us prove that this is indeed the only situation where this happens. By Corollary 3.1, Ric^V is symmetric if, for all vector fields X,Y on M, $g(\nabla^g_X V,Y) = g(\nabla^g_Y V,X)$ holds. For the 1-form $V^\flat$ corresponding to V, the Cartan differential is calculated as $dV^\flat(X,Y) = g(\nabla^g_X V,Y) - g(\nabla^g_Y V,X)$. This gives the result. □

Remark 3.3. A metric connection with skew torsion $T \in \Lambda^3(M)$ has symmetric curvature if T is parallel, $\nabla T = 0$ [Ag06, p.32]. This is again a hint that $dV^\flat = 0$ is a more natural geometric condition than $\nabla V = 0$.

Definition 3.1. Let ∇ be a metric connection with vectorial torsion V. We agree to call ∇ a connection with *closed* vectorial torsion if, in addition, $dV^\flat = 0$ holds. More generally, we will call a vector field *closed* if its dual 1-form is closed.

A second geometric interpretation of closed vectorial torsion is given in the following proposition.

Proposition 3.3. *Let (M,g) be a semi-Riemannian manifold. Suppose that V is a vector field on M with $g(V,V) = \|V\|^2 \neq 0$ everywhere, and ∇ the metric connection with vectorial torsion V. If V is closed, the distribution $D = V^\perp$ is involutive and for every vector field X in D holds:*

$$X(\|V\|^2) \;=\; g(V,[X,V]). \tag{4}$$

Conversely, if D is an involutive $(n-1)$-dimensional distribution on (M,g) and V a vector field of nowhere vanishing length orthogonal to D such that $V \lrcorner\, dV^\flat = 0$ holds, then the V-curvature is symmetric. In this case, the V-Ricci tensor satisfies for $\xi,\eta \in D$:

$$\begin{aligned}
\mathrm{Ric}^V(\xi,\eta) - \mathrm{Ric}^g(\xi,\eta) &= \left[H\|V\| + \frac{1}{|V|}(V(\|V\|)) + (2-n)\|V\|^2\right] g(\xi,\eta) + (n-2)\|V\|\mathrm{II}\,(\xi,\eta),\\
\mathrm{Ric}^V(\xi,V) - \mathrm{Ric}^g(\xi,V) &= \frac{n-2}{2}\xi(\|V\|^2),\\
\mathrm{Ric}^V(V,V) - \mathrm{Ric}^g(V,V) &= \|V\|^2(H\|V\| + (n-1)\frac{1}{\|V\|}V(\|V\|)),
\end{aligned}$$

where II *is the second fundamental form,* $H = \mathrm{tr}\,\mathrm{II}$ *the mean curvature on the leaves of D, and* Ric^g *the Ricci curvature of the Levi-Civita connection on M.*

Proof. For $X,Y \in V^\perp$ from $dV^\flat = 0$ follows

$$0 = g(V, \nabla^g_X Y - \nabla^g_Y X) = g(V,[X,Y]),$$

which means that $[X,Y] \in V^\perp$ and so $V^\perp$ is involutive and (4) is equivalent to $V \lrcorner\, dV^\flat = 0$. Conversely, if $V^\perp$ is involutive then for $X,Y \in V^\perp$

$$dV^\flat(X,Y) = g(V, \nabla^g_X Y - \nabla^g_Y X) = g(V,[X,Y]) = 0.$$

Together with $V \lrcorner\, dV^\flat = 0$ follows the symmetry of the V-Ricci curvature. □

Remark 3.4. By now, manifolds admitting a homogeneous structure of strict type $TM \oplus \mathcal{T}'$ (see Proposition 2.1) are called *cyclic homogeneous manifolds.* Pastore and Verroca proved in [PV91] that cyclic homogeneous manifolds whose vectorial torsion part satisfies $dV^\flat = 0$ are foliated by isoparametric $(n-1)$-dimensional submanifolds—the assumption that $\mathcal{R}^V$ is ∇-parallel is crucial for this stronger result. It is known that cyclic homogeneous manifolds are never compact [TV88], which fits into the general picture (see the examples) below that all interesting examples are non compact. For dimension $n \leq 4$, they are classified in [GGO14]. Examples on Lie groups can be found in [GGO15].

Remark 3.5. For any metric connection ∇ with torsion T, the differential of a 1-form ω satisfies

$$d\omega(X,Y) \;=\; \nabla_X\omega(Y) - \nabla_Y\omega(X) + \omega(T(X,Y)).$$

Formula (1) implies that the last term vanishes for a connection with vectorial torsion and $\omega = V^\flat$, hence we obtain the remarkable identity

$$dV^\flat(X,Y) \;=\; \nabla_X V^\flat(Y) - \nabla_Y V^\flat(X) \;=\; g(\nabla_X V, Y) - g(\nabla_Y V, X), \tag{5}$$

in complete analogy to the classical formula expressing $dV^\flat$ through $\nabla^g V^\flat$. In particular, V is closed if and only if the tensor $S(X,Y) := g(\nabla_X V, Y)$ is symmetric in X and Y.

We now consider the V-curvature for some special vector fields.

Example 3.1. If V is a Killing vector field, the condition that V be closed implies

$$g(\nabla^g_X V, Y) = 0$$

for all vector fields X, Y on M and therefore V is a Riemannian parallel vector field. In this case the second fundamental form II of the orthogonal distribution is zero. Then $\mathrm{Ric}^V = 0$ iff $\mathrm{Ric}^g(X,Y) = (n-2)\|V\|^2 g(X,Y)$ for X and Y in the orthogonal distribution and $\mathrm{Ric}^g(V,X) = 0$ for any vector X. Therefore if M is simply connected, it is the product of a Einstein space of positiv scalar curvature and $\mathbb{R}$. Moreover if M is V-flat, it is the product of a sphere and $\mathbb{R}$.

Example 3.2. If the vector field V is ∇-parallel, it is closed by equation (5) and $\mathrm{div}(V) = (n-1)g(V,V)$ is constant. If one integrates this identity over a closed Riemannian manifold, $V = 0$ follows, proving that this condition is rather restrictive. The curvature formulas are particularly simple,

1. Ricci curvature: $\mathrm{Ric}^V(X,Y) = \mathrm{Ric}^g(X,Y) + (n-1)g(V,V)g(X,Y)$
2. Scalar curvature: $s^V \;=\; s^g + n(n-1)g(V,V)$

Therefore, M is V-Ricci flat exactly if M is Einstein with Ricci curvature $-(n-1)g(V,V)g$ which is, up to a constant, the Ricci curvature of hyperbolic space. Indeed, if M is the hyperbolic space noted as warped product $\mathbb{R} \times_{e^t} \mathbb{R}^n$, then $V = \partial_t$ is V-parallel and $\mathrm{Ric}^V = 0$. Actually hyperbolic space satisfies even $\mathcal{R}^V = 0$ and according to Tricerri and Vanhecke it is locally the only V-flat space with $\nabla V = 0$

Example 3.3. If V is a closed conformal vector field, $\nabla^g_X V = \lambda X$ for all vector fields X on M and a function λ on M. Thus the leaves of the distribution $V^\perp$ outside the set of zeroes of V are umbilic with mean curvature $\frac{n-1}{\|V\|}\lambda$ and

$$(\mathrm{Ric}^V - \mathrm{Ric}^g)(X,Y) = (2(n-1)\lambda + \|V\|^2(2-n))g(X,Y) + (n-2)g(V,X)g(V,Y)$$

and

$$s^V - s^g = 2n(n-1)\lambda - (n-1)(n-2)\|V\|^2.$$

Now we consider the case that the orthogonal distribution is still umbilic, but the vector field is not necessarily conformal. Umbilic distributions are precisely $\mathrm{SO}(n-1)$-structures of vectorial type in the sense of [AF06].

Proposition 3.4. *Let M be an n-dimensional Riemannian manifold with an involutive $(n-1)$-dimensional D, whose leaves are totally umbilic with mean curvature $H = (n-1)\lambda$, where $\lambda \in C^\infty(M)$. Let N a normal unit field on the leaves of D.*

For $n > 2$, a vector field V is orthogonal to the distribution D and closed if and only if $V = hN$ with $h \in C^\infty(M)$ and h satisfies the equation

$$dh(\xi) = hg(N, [\xi, N]) \tag{6}$$

for all vector fields ξ in the distribution D. Then for all vector fields ξ, η in D

$$\begin{aligned}
\mathrm{Ric}^V(\xi,\eta) - \mathrm{Ric}^g(\xi,\eta) &= ((2n-3)h\cdot\lambda + (2-n)h^2 + N(h))g(\xi,\eta)\\
\mathrm{Ric}^V(N,\xi) - \mathrm{Ric}^g(N,\xi) &= (n-2)\xi(h)\\
\mathrm{Ric}^V(N,N) - \mathrm{Ric}^g(N,N) &= (n-1)(h\lambda + N(h))
\end{aligned}$$

Proof. Since D is umbilic, $\nabla^g_\xi N = \lambda\xi$ for $\xi \in D$. For $V = hN$, we obtain

$$\nabla^g_\xi V = dh(\xi)N + h\lambda\xi.$$

Therefore, $V \lrcorner\, dV^\flat = 0$ if and only if (6) holds. Then $\nabla^g_N V = \mathrm{grad}\, h$ because

$$g(\nabla^g_N V, \xi) = -g(V, \nabla^g_N \xi) = -g(V, \nabla^g_\xi N) - g(V, [N, \xi]) = dh(\xi)$$

according to (6) and $g(\nabla^g_N V, N) = dh(N)$. □

From Hodge theory it is well known that the space of closed vector fields on a closed Riemannian manifold is the orthogonal sum of the space of harmonic vector fields and the space of exact vector fields. Let us first consider the case that V is a gradient vector field. In [AT04], it has been shown that in this case the V-geodesics coincide with the geodesics of a conformally equivalent metric up to reparametrisation. Here we show that the V-curvature coincides up to rescaling with the Riemannian curvature of a conformally equivalent manifold. From the point of view of Weyl geometry this follows from the fact that the Weyl structures $(W, e^{2f}g, V - df)$ are all equivalent and have therefore the same Weyl curvature. Alternatively, one compares the definition of A_V with the formulas for the connection of the conformally changed metric [Be08] and concludes that $\nabla^{\tilde g} = \nabla^V + B$ with $B(X) = -g(V,X)\mathrm{Id}$; the following formulas are then obtained by a straightforward calculation.

Proposition 3.5. *Let M be a manifold with metric g, $V = -\mathrm{grad}\, f$ and $\tilde g = e^{2f}g$. Then*

$$\mathcal{R}^V = e^{-2f}\mathcal{R}^{\tilde g}, \quad \mathrm{Ric}^V = \mathrm{Ric}^{\tilde g}, \quad s^V = e^{2f}s^{\tilde g}.$$

In particular, M is $(-\mathrm{grad}\, f)$-flat (resp. $(-\mathrm{grad}\, f)$-Einstein resp. $(-\mathrm{grad}\, f)$-Ricci flat) if and only if $(M, \tilde g)$ is flat (resp. Einstein resp. Ricci flat).

Example 3.4. Let (F, g_F) an $(n-1)$-dimensional Riemannian manifold and $f \in C^\infty(\mathbb{R}, \mathbb{R}_+)$. For $\varepsilon = \pm 1$ we consider on the differentiable manifold $M = \mathbb{R} \times F$ the product metric $g_\varepsilon = \varepsilon dt^2 + g_F$ and the warped product metric $\tilde g_\varepsilon = \varepsilon dt^2 + f^2 g_F$; hence, the sign ε is introduced to cover Riemannian and Lorentzian signature in one expression. In these examples the distribution $\mathbb{R} \times TF$ has totally umbilic leaves $\{t\} \times F$, with mean curvature $(n-1)\lambda = (n-1)\frac{\dot f}{f}$ for the warped product.
For the Ricci curvature of the Levi-Civita connection on $(M, \tilde g_{\pm 1})$, it is well known that

$$\begin{aligned}
\mathrm{Ric}^{\tilde g_\varepsilon}(\partial_t, \partial_t) &= -\frac{(n-1)}{f}\ddot f\\
\mathrm{Ric}^{\tilde g_\varepsilon}(\xi, \partial_t) &= 0\\
\mathrm{Ric}^{\tilde g_\varepsilon} &= \mathrm{Ric}^{g_F}(\xi,\eta) - \epsilon(\frac{\ddot f}{f} + (n-2)\frac{\dot f^2}{f^2})g(\xi,\eta)
\end{aligned} \tag{7}$$

where ξ, η are vector fields on F.

We are interested in connections $\widetilde{\nabla}^V := \nabla^{\tilde{g}_\varepsilon} + A_V$ with symmetric Ricci tensor $\widetilde{\mathrm{Ric}}_\varepsilon^V$ and orthogonal distribution given by $\mathbb{R} \times TF$. By equation (6) this amounts to choosing $V = h \cdot \partial_t$ for some $h \in C^\infty(\mathbb{R})$, because for the normal unit vector field $N = \partial_t$, we have $g(\partial_t, [\partial_t, \xi]) = 0$ for vector fields ξ in the leaves. If we ask moreover for a factor h such that

$$\widetilde{\mathrm{Ric}}_\varepsilon^V = \mathrm{Ric}^{g_\varepsilon},$$

the only possible choice is $h = \varepsilon \frac{\dot{f}}{f}$, because $(M, \tilde{g}_\varepsilon)$ and (M, g_ε) are conformally equivalent with the conformal factor f^{-2}. By Proposition 3.5, the vector field we were looking for is thus $V = \mathrm{grad}\,(\ln f) = \varepsilon \frac{\dot{f}}{f} \partial_t$. This is exactly the choice of vector field that appeared in [AT04].

Although we are mainly interested in connections with pure vectorial torsion, it is straightforward to generalize our results to connections with an additional part in the torsion. In [OM97], the authors constructed on the warped product $M \times_f S^3$ a Lorentzian spacetime with nontrivial Nieh-Yan 4-forms with nonvanishing torsion, but zero Ricci curvature. We generalize this construction in the following proposition. To this purpose we replace in Example 3.4 the Levi-Civita connection by a connection with skew symmetric torsion induced by a connection on F. More exactly, let ∇^F be the Levi-Civita connection on F and $\nabla^F + \omega$ a connection on F with skew symmetric torsion, and $\pi : M \to F$ the projection. Then $\widetilde{\nabla}^\omega = \nabla^{\tilde{g}_\varepsilon} + \pi^*\omega$ is a metric connection on $(M, \tilde{g}_\varepsilon)$ with skew symmetric torsion. In [OM97], ω is chosen such that the Ricci curvature of $\nabla^{g_{-1}} + \pi^*\omega$ vanishes.

Proposition 3.6. *Denote by* $\widetilde{\mathrm{Ric}}_\varepsilon^{\omega,V}$ *the Ricci curvature of* $\widetilde{\nabla}^\omega + A_V$ *on* $(M, \tilde{g}_\varepsilon)$ *and by* $\mathrm{Ric}_\varepsilon^\omega$ *the Riemannian Ricci curvature of* $\nabla^{g_\varepsilon} + \pi^*\omega$ *on the product* (M, g_ε)*. Then*

$$\widetilde{\mathrm{Ric}}_\varepsilon^{\omega,V} = \mathrm{Ric}_\varepsilon^\omega \text{ if and only if } V = \varepsilon \frac{\dot{f}}{f} \partial_t.$$

Proof. We generalize Example 3.4. If $\mathrm{Ric}^{(F,\omega)}$ denotes the Ricci tensor of the connection $\nabla^F + \omega$ on F, then according to the formulas for the curvature of connection with skew symmetric torsion [Ag06] the Ricci tensor of $\widetilde{\nabla}^\omega$ on $M \times_f F$ is the same as (7) with Ric^{g_F} replaced by $\mathrm{Ric}^{(F,\omega)}$. Because $V \lrcorner T = 0$ and T is skew symmetric, the last two terms in (3) vanish in this situation, $\mathrm{div}^{\nabla^\omega} V = \mathrm{div} V$ and $\widetilde{\nabla}^\omega V = \widetilde{\nabla}^g V$ for $V = h\partial_t$. Therefore the same argument as in Example 3.4 shows the result. □

Remark 3.6. The importance of this result stems from the fact that it implies, in particular, that it is possible to construct connections with vectorial torsion on warped products of arbitrary dimension matching a given Riemannian or Lorentzian curvature—for example, a Ricci-flat connection with vectorial torsion in dimension 4. This is totally different from the case of skew torsion, where it was proved that a Riemannian Einstein manifold can never be Einstein with skew torsion if $n = 4, 5$ [AF14, Exa 2.14].

Now we consider the case of a harmonic vector field V, which means a closed vector field with vanishing divergence. For surfaces, it follows from Corollary 3.2 that the V-Ricci curvature coincides with the Riemannian Ricci curvature.
In the case of a non trivial spacelike vector field with vanishing divergence for $n > 2$

$$s^V = s^g - (n-1)(n-2)\|V\|^2 \leq s^g$$

holds. But if M is closed manifold we can always choose a conformal equivalent metric such that the corresponding vectorfield is divergence free. More precisely for $n > 2$ in [Gau95] has been shown that on a closed Weyl manifold (M, g, V) there exists a *standard metric* $\tilde{g} = e^{2f} g$ such that $\tilde{V} = V - df$ is divergence free related to the metric $\tilde{g}$.

Corollary 3.4. *For any vector field V on a closed Riemannian manifold M of dimension $n > 2$, the inequality $e^{2f} s^{\tilde g} \geq s^V$ holds, where $\tilde g = e^{2f} g$ is the standard metric for the corresponding Weyl structure.*

Proof. From Corollary 3.1 and proposition 3.5, we conclude that $s^V = e^{2f} s^{\tilde V} \leq e^{2f} s^{\tilde g}$ where $\tilde V = V - df$. □

Example 3.5. In this example, we do not assume a priori that the V-Ricci curvature is symmetric. Consider a S^1-fiber bundle $M \to B$ with S^1-invariant metric on M and a vertical vector field V on M and the connection on M induced by the metric. If V is a fundamental vector field on M given by the S^1-action, ξ is a vector field on B and $\bar\xi$ its horizontal lift to M, then $[V, \bar\xi] = 0$. Therefore

$$\operatorname{div} V \;=\; 0, \quad dV^\flat(\bar\xi, \bar\eta) \;=\; 2g(\nabla_{\bar\xi} V, \bar\eta) \;\text{ and } dV^\flat(V, \bar\xi) \;=\; -\bar\xi \|V\|^2.$$

For the curvature quantities this implies:

$$\begin{aligned}
\mathrm{Ric}^V(\xi,\eta) - \mathrm{Ric}^g(\xi,\eta) &= (n-2)(\frac{1}{2} dV^\flat(\bar\xi,\bar\eta) - \|V\|^2 g(\xi,\eta)) \\
\mathrm{Ric}^V(V,\xi) - \mathrm{Ric}^g(V,\xi) &= -\frac{(n-2)}{2} dV^\flat(V,\bar\xi) \\
\mathrm{Ric}^V(V,V) - \mathrm{Ric}^g(V,V) &= 0 \\
s^V - s^g &= -(n-1)(n-2)\|V\|^2
\end{aligned}$$

Now let us study what happens when the V-Ricci tensor is symmetric, i. e. $dV^\flat = 0$. The equivalent geometric criterion (4) from Proposition 3.3 is fulfilled if and only if $\bar\xi(\|V\|) = 0$ for all horizontal vector fields ξ and a fundamental vector field V; because of the S^1-invariance of the metric this is equivalent to $\|V\|$ =const. Thus, the V-Ricci tensor is symmetric if and only if V has constant length. Since Proposition 3.3 also shows that then situation the horizontal distribution is involutive, we can furthermore conclude that the connection of the bundle is flat. By a well-known result of algebraic topology, this is equivalent to the vanishing of the real Euler class of the S^1-bundle.

Proposition 3.7. *Let V be a vector field with vanishing divergence on a closed Riemannian manifold. Then*

$$\int_M \mathrm{Ric}^V(V,V) d\mu = \int_M \mathrm{Ric}^g(V,V) d\mu.$$

Moreover if V is harmonic and $\mathrm{Ric}^V = 0$ then V is ∇^g-parallel and therefore the universal cover of M is the product of an Einstein space of positive scalar curvature and $\mathbb{R}$.

Proof. The formula of Corollary 3.1 yields for a vector field with vanishing divergence $\mathrm{Ric}^V(V,V) = \mathrm{Ric}^g(V,V) + \frac{n-2}{2} V(\|V\|)^2$ and the first result follows from Stokes' theorem by integration. If V is harmonic then the Bochner formula yields for a V-Ricci flat manifold

$$0 = \int_M (\mathrm{Ric}^g(V,V) + \|\nabla^g V\|^2) d\mu = \int_M \|\nabla^g V\|^2 d\mu. \qquad \square$$

4. Dirac operators of connections with vectorial torsion

Already in 1979, Thomas Friedrich observed that the Dirac operator D associated with a metric connection ∇ with torsion T has different properties depending on the torsion type. We summarize the result of [Fr79] in the following table:

	$T \in TM$	$T \in \Lambda^3(M)$	$T \in \mathcal{T}'$
D formally self-adjoint	no	yes	yes
$D = D^g$	no	no	yes

Assume from now on that (M, g) is spin, and denote by ΣM its spinor bundle. For the connection $\nabla^g + A_V(X)Y$ defined before, the lift to ΣM is given by [Ag06, p.18]

$$\nabla_X\psi \;=\; \nabla^g_X\psi + \frac{1}{2}(X \wedge V)\cdot\psi. \tag{8}$$

One then easily computes that

$$D\psi \;=\; D^g\psi - \frac{n-1}{2}V\cdot\psi, \quad D^*\psi \;=\; D^g\psi + \frac{n-1}{2}V\cdot\psi, \tag{9}$$

because the Clifford multiplication by a vector field is skew-adjoint with respect to the hermitian product on the spin bundle. We start by constructing a few examples of manifolds with a V-parallel spinor field.

Example 4.1. Let F be an $(n-1)$-dimensional manifold with Killing spinor with Killing number $\frac{1}{2}$. Then F is an Einstein space of scalar curvature $(n-1)(n-2)$. The induced spinor on $M = \mathbb{R}\times F$ is $-\partial_t$-parallel, because for n odd and X tangent to F

$$\begin{aligned}\nabla^g_{\partial_t}\psi &= 0\\ \nabla^g_X\psi &= \nabla^F_X(\psi|F) = \frac{1}{2}X\cdot\partial_t\cdot\psi = \frac{1}{2}X\wedge\partial_t\cdot\psi\end{aligned}$$

For n even, one performs a similar calculation to obtain the result. Note that the manifold is ∂_t-Ricci flat; in Theorem 4.2, we will show that this is always true (with the additional assumption $dV^\flat = 0$ in dimension 4, which is satisfied in this example).

Example 4.2. If F is a manifold with a parallel spinor, the induced spinor on the warped product $\mathbb{R}\times_f F$ is $\frac{\dot f}{f}\partial_t$-parallel. Remember that in Example 3.4 we have shown that these manifolds are $-\frac{\dot f}{f}\partial_t$-Ricci flat, as they should be.

Remark 4.1. Since the connection is metric, V-parallel spinors have automatically constant length, so we may assume the length to be one. If $V = \frac{1}{2}\operatorname{grad} u$ is a gradient vector field, V-parallel spinor fields coincide with weakly T-parallel spinors as defined in [Ki06] for the choice of parameter $\beta = \mathrm{Id}$. These spinors are used to construct solutions of the Einstein Dirac equation.

Proposition 4.1. *If V is an exact vector field, the V-Dirac spectrum is the same as the spectrum of the Riemannian Dirac operator.*

Proof. By assumption, there exists a function f such that $V = \operatorname{grad} f$. Choose the conformal factor $h = e^{\frac{(n-1)}{2}f}$; then, according to [BFGK91, p. 19],

$$Dh\psi \;=\; hD^g\psi + \operatorname{grad} h\psi - \frac{n-1}{2}\operatorname{grad} f h\psi \;=\; hD^g\psi. \tag{10}$$

Therefore, if ψ is an eigenspinor for D^g, then $h\psi$ is an eigenspinor for D and vice versa, because h has no zeros. □

Remark 4.2. We recall the behaviour of spinor bundles under a conformal change of metric. If $\tilde g = e^{2f}g$, then there is an identification of the two spinor bundles which we denote by

$\sim$. Comparing the formulas [BFGK91, p. 16] with (8) shows that for the connection (8) with $V = -\mathrm{grad}\, f$

$$\nabla^{\tilde{g}}_{\tilde{X}}\tilde{\psi} = e^{-f}\widetilde{\nabla_X\psi} \quad \text{and} \quad D^{\tilde{g}}\tilde{\psi} = e^{-f}\widetilde{D\psi}.$$

Therefore, V-parallel spinors and V-harmonic spinors can be identified with the parallel and harmonic spinors for the Levi-Civita on the corresponding conformally equivalent Riemannian manifold.

In [PS11], the authors derived the following formula for $D_t^* D_t$,

$$D_t^* D_t\psi \;=\; \Delta^{\nabla}_{(n-1)t}\psi + \frac{1}{4}s^g\psi + \frac{(n-1)t}{2}\mathrm{div}^g V\psi + t^2\left(\frac{n-1}{2}\right)^2(2-n)\|V\|^2\psi \tag{11}$$

where Δ^{∇}_t is the Laplacian associated with the connection

$$\nabla^t_X\psi \;=\; \nabla^g_X\psi + \frac{t}{2}(X\wedge V)\cdot\psi \tag{12}$$

and D_t the corresponding Dirac operator. As for connections with skew torsion, one observes that a rescaling of the vector field is necessary; but while this rescaling was by a constant in the skew torsion case, it turns out to be dimension dependent here. In particular, for $n = 2$ the scaling factor is equal to one! According to [PS11], the Laplacian of the rescaled connection satisfies

$$\Delta^{\nabla}_{n-1} = (D^g)^2 - \frac{1}{4}s^g + \frac{n-1}{2}[2V\cdot D^g + 2\nabla^g_V - d(V^b)] + (\frac{n-1}{2})^2(n-1)\|V\|^2 \tag{13}$$

and therefore the Laplacian Δ^{∇}_t associated with $\nabla^g + A_{tV}$ is given by

$$\Delta^{\nabla}_t = (D^g)^2 - \frac{1}{4}s^g + tVD^g + t\nabla^g_V - t\frac{1}{2}d(V^b) + t^2\frac{1}{4}(n-1)\|V\|^2. \tag{14}$$

Proposition 4.2. *Let V a vector field on a spin manifold. Then for the Dirac operator D_t corresponding to the connection (12) we get:*
(15)

$$D_t^* D_t\psi \;=\; \Delta^{\nabla}_t\psi + \frac{1}{4}s^g\psi + t\frac{n-1}{2}\mathrm{div}^g V\psi + t^2\frac{(n-1)(n-2)}{4}\|V\|^2\psi + t(n-2)(VD^g + \nabla^g_V - \frac{1}{2}dV^\flat)$$

Proof. We deduce from (14)

$$\Delta^{\nabla}_{(n-1)t}\psi \;=\; \Delta^{\nabla}_t\psi + t^2 n\frac{(n-1)(n-2)}{4}\|V\|^2\psi + t(n-2)(VD^g + \nabla^g_V - \frac{1}{2}dV^\flat).$$

By replacing this expression in (11) and using (9), one obtains the result. □

We shall now investigate what can be said about V-parallel spinor fields. In Weyl geometry, there is a lift of the Weyl connectection to weighted spinor bundles. V-parallel spinor fields can be identified with parallel spinors of weight zero for the Weyl connection. These have been studied by [Mo96]. The author showed the following proposition:

Proposition 4.3. *Let (M, g, V) be a Weyl manifold of dimension $n \geq 3$, and in addition compact if $n = 4$. If it admits a non trivial parallel spinor of weight zero, then V is closed.*

The following proposition may already be found in [AF06, Thm 2.1.], we reproduce a proof for completeness.

Proposition 4.4. *If ψ is a V-parallel spinor field, the following identities hold:*

(1) *The Riemannian scalar curvature is given by*

$$s^g = (n-1)(n-2)\|V\|^2 - 2(n-1)\mathrm{div}V, \quad \textit{i.e.} \quad s^V \;=\; 0.$$

(2) *The square of the Riemannian Dirac operator acts on ψ by*

$$(D^g)^2\psi = (\frac{n-1}{2}||V||)^2\psi - \frac{n-1}{2}\mathrm{div}V\psi = \frac{1}{4}(s^g + (n-1)\|V\|^2)\psi.$$

Proof. For a V-parallel spinor, equation (8) implies for $X = V$:

$$\nabla^g_V\psi = 0$$

and equation (9) is reduced to $D^g\psi = \frac{n-1}{2}V \cdot \psi$. By applying the Riemannian identity for $D^g(V \cdot \psi)$ ([BFGK91, p. 19]), we obtain

$$(D^g)^2\psi = (\frac{n-1}{2}||V||)^2\psi - \frac{n-1}{2}\mathrm{div}V\psi + \frac{n-1}{2}d(V^b)\psi, \tag{16}$$

and from (14) we get

$$s^g\psi = \left[(n-1)(n-2)||V||^2 - 2(n-1)\mathrm{div}V + 2(n-2)d(V^b)\right]\psi. \tag{17}$$

But for any 2-form α, the quantity $(\alpha\psi, \psi)$ is purely imaginary, therefore the last term vanishes. This gives $d(V^b)\psi = 0$—in fact, we already know that $d(V^b) = 0$ from Moroianu's result except when $n = 2, 4$. Either way, inserting $d(V^b)\psi = 0$ in (16) und (17) yields the proposition after a short calculation. □

For a surface, we conclude that V has to be closed, $s^g = -2\,\mathrm{div}V$ and Stokes' formula implies:

Corollary 4.1. *The only closed surface admitting non-trivial V-parallel spinor fields is the torus.*

The following example shows that on the flat 2-dimensional torus, there exists a vector field V admitting V-parallel spinor fields for every spin structure.

Example 4.3. On the 2-dimensional flat torus $T^2 = [0,1]/\sim$ there are four spin structures, which we label with $(\varepsilon_1, \varepsilon_2)$ with $\varepsilon_i \in \{0,1\}$. The space of continuous spinors can be identified with

$$\{\psi \in C^0([0,1]^2, \mathbb{C}^2) \,|\, \psi(x,0) = (-1)^{\varepsilon_2}\psi(x,1) \text{ and } \psi(0,y) = (-1)^{\varepsilon_1}\psi(1,y)\}.$$

Therefore, Example 4.4 implies that on T^2 there are (v_1, v_2)-parallel spinors if and only if $v_1 = 2(k+\varepsilon_2)\pi$ and $v_2 = 2(m+\varepsilon_2)\pi$ where $k, m \in \mathbb{Z}$.

If there exists a V-parallel spinor field on $\mathbb{R}^2$ then $V = \mathrm{grad}\, f$ for a harmonic function f anf therefore V-parallel spinor fields correspond to parallel spinor fields on the corresponding conformal equivalent manifold. We give a more concrete example.

Example 4.4. On $\mathbb{R}^2$, every constant vector field $V = (v_1, v_2)$ admits a 2-dimensional space of V-parallel spinors, since the spinor derivative is given by

$$\nabla^V_{\partial_x}\psi \;=\; \partial_x\psi + \frac{1}{2}v_2\omega\cdot\psi, \quad \nabla^V_{\partial_y}\psi \;=\; \partial_y\psi - \frac{1}{2}v_1\omega\cdot\psi$$

with $\omega = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$. Therefore, the space of V-parallel spinors on $\mathbb{R}^2$ is spanned by $e^{\frac{i}{2}(v_2x - v_1y)}e_1$ and $e^{-\frac{i}{2}(v_2x - v_1y)}e_2$.

Remark 4.3. In [Mo96] parallel spinors of Weyl structures (M, g, V) have been investigated, in particular in dimension 4. In this case the author shows that a compact Weyl 4-manifold with a V-parallel spinor is a hyperhermitian manifold and thus, by a result of Boyer, conformally equivalent to a torus, a K3 surface, or a Hopf surface.

Remark 4.4. Suppose that ψ is a V-parallel spinor field, $n > 2$. If $\mathrm{div} V = 0$ then M has constant positive scalar curvature

$$s^g = (n-1)(n-2)\|V\|^2$$

and therefore the expression (2) of the previous Proposition is reduced to

$$(D^g)^2\psi \;=\; \Big(\frac{n-1}{2}||V||\Big)^2\psi \;=\; \frac{n-1}{4(n-2)}s^g\psi.$$

A D^g-eigenspinor is given by $\tilde{\psi} = \sqrt{\frac{(n-1)s^g}{4(n-2)}}\,\psi + D^g\psi$.

A deeper analysis will now show that as in the case of the Levi-Civita connection, the V-Ricci curvature of M will vanish if it is symmetric and if M admits a nontrivial V- parallel spinor. Recall that the curvature operator of any spin connection can be understood as a endomorphism-valued 2-form,

$$\mathcal{R}(X,Y)\psi \;=\; \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi.$$

One checks that it is related to the curvature operator on 2-forms defined through

$$\mathcal{R}(e_i\wedge e_j) \;:=\; \sum_{k<l} R_{ijkl}\, e_k\wedge e_l$$

by the relation

$$\mathcal{R}(X,Y)\psi \;=\; \frac{1}{2}\mathcal{R}(X\wedge Y)\cdot\psi.$$

The following identity is crucial for deriving integrability conditions for spinor fields satisfying first order differential equations. It generalizes a well-known result of Friedrich [Fr80, Satz 5.2] for the Levi-Civita connection ($V = 0$).

Theorem 4.1. *Let ∇ be a metric spin connection with vectorial torsion. Then, the following identity holds for any spinor field ψ and any vector field X:*

$$\mathrm{Ric}^V(X)\cdot\psi = -2\sum_{k=1}^{n} e_k\mathcal{R}^V(X,e_k)\psi - dV^\flat\wedge X\cdot\psi.$$

Proof. Rewrite the first term on the right hand side (without the numerical factor) as

$$\sum_{k=1}^{n} e_k\mathcal{R}^V(e_l,e_k) \;=\; \frac{1}{2}\sum_{k=1}^{n} e_k\cdot\mathcal{R}^V(e_l\wedge e_k) \;=\; \frac{1}{2}\sum_{k=1}^{n}\sum_{i<j} R^V_{lkij}e_ke_ie_j \;=:\; R_1+R_2,$$

where R_1 denotes all terms with three different indices k,i,j, and R_2 all terms with at least one repeated index. We first discuss R_1:

$$R_1 \;=\; \frac{1}{2}\sum_{i<j}\left[\sum_{k<i} R^V_{lkij}e_ke_ie_j + \sum_{i<k<j} R^V_{lkij}e_ke_ie_j + \sum_{j<k} R^V_{lkij}e_ke_ie_j\right] \;=\; \frac{1}{2}\sum_{k<i<j}\overset{k,i,j}{\mathfrak{S}}\; R^V_{lkij}e_ke_ie_j,$$

where the symbol $\mathfrak{S}$ denotes the cyclic sum. The first Bianchi identity for a metric connection with vectorial torsion (see equation (3)) implies then $R_1 = -\frac{1}{2}dV^\flat\wedge e_l^\flat$. The second term does not depend on the detailed type of the connection, so a similar argument as in [ABBK13] shows

$$R_2 \;=\; -\frac{1}{2}\sum_{r=1}^{n}\left[\sum_{p=1}^{r-1} R^c_{lppr}e_e + \sum_{q=r+1}^{n} R^c_{lqqr}e_r\right].$$

But since the Ricci tensor is exactly the contraction of the curvature, $R_2 = -\mathrm{Ric}^V(e_l)/2$. This ends the proof. □

For closed vector fields V, Theorem 4.1 amounts therefore to an identity that looks formally as for $\nabla = \nabla^g$.

Theorem 4.2. *Let (M, g) be a semi-Riemannian spin manifold, ∇ a connection with vectorial torsion V, $n \geq 2$.*

1. *If M admits a nontrivial V-parallel spinor field, then $\mathrm{Ric}^V = 0$ and $dV^\flat = 0$ hold, and in particular, M is locally conformally Ricci flat. If $n = 4$, Ric^V is totally skew symmetric and given by $\mathrm{Ric}^V(X) = X \lrcorner\, dV^\flat$.*
2. *If $dV^\flat = 0$, there are no nontrivial ∇-Killing spinor fields, i. e. spinor fields satisfying the equation $\nabla_X\psi = \beta\, X \cdot \psi$ for some $\beta \neq 0$.*

Proof. For both claims, let's suppose ψ is a spinor satisfying the equation $\nabla_X\psi = \beta\, X \cdot \psi$ for all vector fields X (∇-parallel spinor fields will be obtained by choosing $\beta = 0$). Such a spinor field has automatically constant length, because it is parallel for the metric spinorial connection $\tilde{\nabla}_X := \nabla_X - \beta\, X\cdot$. Then

$$\begin{aligned}
\mathcal{R}(X,Y)\psi &= \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi \\
&= \nabla_X(\beta\, Y\cdot\psi) - \nabla_Y(\beta\, X\cdot\psi) - \beta[X,Y]\cdot\psi \\
&= \beta(\nabla_X Y - \nabla_Y X - [X,Y])\cdot\psi + \beta(Y\cdot\nabla_X\psi - X\cdot\nabla_Y\psi) \\
&= \beta(g(V,X)Y - g(V,Y)X)\cdot\psi + \beta^2(Y\cdot X - X\cdot Y)\cdot\psi \\
&= \beta(g(V,X)Y - g(V,Y)X)\cdot\psi + 2\,\beta^2(Y\cdot X + g(X,Y))\cdot\psi.
\end{aligned}$$

Therefore, the curvature contraction may be computed,

$$\sum_{k=1}^{n} e_k \mathcal{R}^V(X, e_k) \;=\; -\beta(ng(V,X) + V\cdot X)\cdot\psi + 2\beta^2(1-n)X\cdot\psi.$$

By Theorem 4.1, we can conclude that the following equation is an integrability condition for the existence of such a spinor field,

$$\mathrm{Ric}^V(X)\cdot\psi \;=\; 2\beta n\, g(V,X)\,\psi + 2\beta\, V\cdot X\cdot\psi + 4\beta^2(n-1)X\cdot\psi - (dV^\flat \wedge X)\cdot\psi. \tag{18}$$

We now discuss the two situations occuring in the statement. Let's treat the easier case first, i. e. $\beta \neq 0$ and $dV^\flat = 0$. The last term of the integrability condition hence vanishes. Let $(-,-)$ be the positive definite scalar product on spinor fields induced by the canonical hermitian product of the spinor bundle, and recall that it satisfies $(X\cdot\varphi, \varphi) = 0$ for any spinor field φ. We take the scalar product of the remaining identity (18) with ψ and, by the previous remark, we are finally left with

$$0 \;=\; \beta n\, g(V,X)\|\psi\|^2 + \beta\,(V\cdot X\cdot\psi, \psi) + 0.$$

But $(V\cdot X\cdot\psi,\psi) = -(X\cdot\psi, V\cdot\psi) = -g(X,V)\|\psi\|^2$, so this identity cannot hold for $\beta \neq 0$ if $n \neq 1$.

Now let's consider the case of a V-parallel spinor field, i. e. $\beta = 0$. The integrability condition (18) is thus reduced to

$$\mathrm{Ric}^V(X)\cdot\psi \;=\; -(dV^\flat \wedge X)\cdot\psi.$$

Inner and exterior product are related by $X\cdot\omega = X\wedge\omega - X\lrcorner\,\omega$. From Prosition 4.4, we know that $dV^\flat\cdot\psi = 0$, hence $X\cdot dV^\flat\cdot\psi = 0$ and thus

$$-(dV^\flat\wedge X)\cdot\psi \;=\; +(X\wedge dV^\flat)\cdot\psi \;=\; (X\lrcorner\, dV^\flat)\cdot\psi.$$

Hence, the integrability condition is reduced to $\mathrm{Ric}^V(X)\cdot\psi = (X\lrcorner dV^\flat)\cdot\psi$. Since ψ has constant length, this implies

$$\mathrm{Ric}^V(X) \;=\; X\lrcorner\, dV^\flat.$$

Viewed as an endomorphism, $X \lrcorner dV^\flat$ is antisymmetric, whereas Ric^V may be split into its symmetric and antisymmetric part according to Corollary 3.1,

$$\mathrm{Ric}^V(X) \;=\; \mathrm{Ric}^V_{\mathrm{sym}}(X) + \frac{n-2}{2} X \lrcorner dV^\flat.$$

We conclude $\mathrm{Ric}^V_{\mathrm{sym}}(X) = \frac{4-n}{2} X \lrcorner dV^\flat$. One is symmetric, one is antisymmetric, so both have to vanish. If $n \neq 4$, this implies $dV^\flat = 0$ and then also $\mathrm{Ric}^V(X) = 0$. For $n = 4$, we can only conclude $\mathrm{Ric}^V_{\mathrm{sym}}(X) = 0$, hence Ric^V has only antisymmetric part given by $\mathrm{Ric}^V(X) = X \lrcorner dV^\flat$. □

Remark 4.5. The dimension distinction in statement (1) of the Corollary cannot be removed. Indeed, in Section 7 of [Mo96], an example of a non closed vector field V admitting non-trivial V-parallel spinors is given on an open subset of $\mathbb{C}^2$. Thus, $\mathrm{Ric}^V(X) = X \lrcorner dV^\flat \neq 0$ and statement (1) cannot be improved to $\mathrm{Ric}^V(X) = 0$ for $n = 4$ without the assumption of compactness (compare Proposition 4.3). However, our result proves that $dV^\flat = 0$ if and only if $\mathrm{Ric}^V = 0$ in the non-compact case, and gives a formula expressing one quantity through the other. Oddly enough, we recover that $s^V = 0$ even in dimension 4 (as it should be by Proposition 4.4), since Ric^V, although possibly non-zero, is always skew-symmetric and therefore trace-free.

Recall that a closed manifold of dimension $n > 2$ with vector field V carries a conformally equivalent metric for which the corresponding vector field has vanishing divergence. Applying Proposition 3.7 thus yields

Corollary 4.2. *Let M be a closed Riemannian manifold of dimension $n > 2$ with a V-parallel spinor. Then M is conformally equivalent either to a manifold with parallel spinor or to a manifold whose universal cover is the product of $\mathbb{R}$ and an Einstein space of positive scalar curvature.*

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