TANGENT LIE GROUPS ARE RIEMANNIAN NATURALLY REDUCTIVE SPACES

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Abstract. Given a compact Lie group G with Lie algebra $\mathfrak{g}$, we consider its tangent Lie group $TG \cong G \ltimes_{\mathrm{Ad}} \mathfrak{g}$. In this short note, we prove that TG admits a left-invariant naturally reductive Riemannian metric g and a metric connection with skew torsion ∇ such that (TG, g, ∇) is naturally reductive. An alternative spinorial description of the same connection on the direct product $G \times \mathfrak{g}$ generalizes in a rather subtle way to TS^7, which is in many senses almost a tangent Lie group.

1. Introduction

Among all homogenous Riemannian manifolds, naturally reductive spaces are a class of particular interest. Traditionally, they are defined as Riemannian manifolds $(M = G/K, g)$ with a reductive complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ such that

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle \ = \ 0 \ \text{ for all } X, Y, Z \in \mathfrak{m}, \tag{1}$$

where $\langle -, -\rangle$ denotes the inner product on $\mathfrak{m}$ induced from g. For any reductive homogeneous space, the submersion $G \to G/K$ induces a connection that is called the *canonical connection*. It is a metric connection ∇ with torsion $T(X,Y) = -[X,Y]_{\mathfrak{m}}$ which satisfies $\nabla T = \nabla \mathcal{R} = 0$, and condition (1) thus states that a naturally reductive homogeneous space is a reductive space for which the torsion $T(X,Y,Z) := g(T(X,Y),Z)$ is a 3-form on G/K (see [KN69, Ch. X] as a general reference). Classical examples of naturally reductive homogeneous spaces include irreducible symmetric spaces, isotropy irreducible homogeneous manifolds, Lie groups with a bi-invariant metric, and Riemannian 3-symmetric spaces.

In the recent article [AFF15], the authors together with Thomas Friedrich (Berlin) initiated a systematic investigation and, in dimension six, achieved the classification of naturally reductive homogeneous spaces. This is done by applying recent results and techniques from the holonomy theory of metric connections with skew torsion. Lie Groups (and spheres) appearing in the classification always play a special role because they may admit several naturally reductive structures (see [OR12], [OR13] for details on these rather subtle points). The motivation for this paper was to describe explicitly the naturally reductive structure on $S^3 \ltimes \mathbb{R}^3$ discovered in [AFF15], and to investigate whether it can be generalized. This turned out not to be as straight forward as expected. In fact, while a lot is known about left-invariant naturally reductive metrics on *compact* Lie groups (see [AZ79], [Ch16]), much less information is available on *non-compact* Lie groups (C. Gordon gives a general description of naturally reductive nilmanifolds in [Go85], the more recent article [AFS15] investigates quaternionic Heisenberg groups as naturally reductive spaces). Instead of the traditional approach, we shall often work with the following definition.

Definition 1.1. A Riemannian manifold (M, g) is said to be *naturally reductive* if it is a homogeneous space $M = G/K$ endowed with a metric connection ∇ with skew torsion T such that its torsion and curvature $\mathcal{R}$ are ∇-parallel, i. e. $\nabla T = \nabla \mathcal{R} = 0$. The connection ∇ will then be called the *Ambrose-Singer connection* or, loosely, the *naturally reductive structure* of (M, g).

If M is connected, complete, and simply connected, a theorem of Ambrose and Singer asserts that the space is indeed naturally reductive in the traditional sense [AS58].

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In this note, we prove that the tangent bundle $TG \cong G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ of a compact Lie group G with Lie algebra $\mathfrak{g}$ admits a left-invariant naturally reductive Riemannian metric g and a metric connection with skew torsion ∇ such that (TG, g, ∇) is naturally reductive. Furthermore, we will define a suitable almost Hermitian structure on TG such that ∇ is its characteristic connection [FI02, Ag06]. An alternative spinorial description of the same connection on the direct product $G \times \mathfrak{g}$ generalizes in a rather subtle way to TS^7, which is in some sense almost a tangent Lie group. Our construction will make use of the following well-known fact.

Remark 1.2. A compact Lie group G of dimension n endowed with a bi-invariant metric admits an orthonormal basis $e_1, \dots, e_n$ of $\mathfrak{g}$ such that its structure constants C_{ij}^k, defined by $[e_i, e_j] = \sum_{k=1}^n C_{ij}^k e_k$, are totally skew-symmetric in all indices. All structure constants on compact Lie groups appearing in this paper will be chosen in this way.

It is well known that G itself carries a family of naturally reductive structures whose torsion a multiple of $T(X,Y,Z) := \langle [X,Y], Z\rangle$—which is indeed a 3-form by the structure constants' property that we just described [CSch26a, KN69]. Naively, it is this antisymmetry property that allows us to define canonical 3-forms on TG as well, which are then candidates for torsion tensors. However, the picture is more sophisticated than this. We emphasize that our metric is neither a product metric, a g-natural metric (see Remark 2.6) nor a warped product metric or of any 'general' known type. We find it rather surprising that this rather 'asymmetric' metric has any special properties at all, and we believe that it is worth investigating its occurrence further.

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2. Tangent Lie groups as naturally reductive spaces

2.1. **The Lie group** $\mathrm{SU}(2) \ltimes \mathbb{R}^3$**.** In [AFF15, Theorems 8.9, 8.12], we proved the following classification result for 6-dimensional naturally reductive spaces and, more generally, spaces with parallel skew torsion:

Theorem 2.1. *Let (M^6, g, T) be a complete, simply connected Riemannian 6-manifold with a metric connection ∇ with parallel skew torsion T, $\mathrm{rk}\,(*\sigma_T) = 6$ and $\ker T = 0$. Then one of the following cases occurs:*

Case D.1: (M^6, g) is isometric to a nearly Kähler 6-manifold.
Case D.2: $\mathfrak{hol}^\nabla = \mathfrak{so}(3) \subset \mathfrak{su}(3)$, the manifold (M^6, g) is naturally reductive, almost Hermitian of Gray-Hervella type $\mathcal{W}_1 \oplus \mathcal{W}_3$, and isometric to one of the following Lie groups with a suitable family of left-invariant metrics:

- *(a) The nilpotent Lie group with Lie algebra $\mathbb{R}^3 \times \mathbb{R}^3$ with commutator $[(v_1, w_1), (v_2, w_2)] = (0, v_1 \times v_2)$ (see* [Sch07]*),*
- *(b) the direct or the semidirect product of S^3 with $\mathbb{R}^3$,*
- *(c) the product $S^3 \times S^3$ (described in Section* [AFF15, 9.4]*),*
- *(d) the Lie group $\mathrm{SL}(2, \mathbb{C})$ viewed as a 6-dimensional real manifold (described in Section* [AFF15, 9.5]*).*

Case (b) turned out to be rather non-intuitive and was therefore not described any further. In the proof, $S^3 = \mathrm{SU}(2)$ appeared as the isometry group of the 3-dimensional Euclidean space. We start by giving an explicit description of this space, and afterwards a generalisation of it.
Recall that our Ansatz was as follows: (M^6, g, T) is a complete, simply connected Riemannian 6-manifold with parallel skew torsion T such that $\mathrm{rk}(*\sigma_T) = 6$ and $\ker T = 0$. In [AFF15, Theorem 8.9, case D.2], it is proved that there is a local orthonormal frame $\{e_1, \cdots, e_6\}$ such that the torsion form can be written as

$$T = \alpha\, e_{135} + \alpha'\, e_{246} + \beta\,(e_{245} + e_{236} + e_{146}), \text{ hence } \sigma_T = \beta(\beta - \alpha)\,(e_{1256} + e_{1234} + e_{3456}) \tag{2}$$

and the curvature is given by

$$\mathcal{R} = \beta(\alpha-\beta)\,[(e_{35}+e_{46})^2 + (e_{15}+e_{26})^2 + (e_{13}+e_{24})^2] \tag{3}$$

with $\beta \neq 0$ and $\alpha \neq \beta$ to ensure that $*\sigma_T$ defines an almost complex structure. Our group G corresponds to the parameters α, α' and β such that $\alpha \neq 2\beta$ and $4\beta(\alpha-2\beta) - {\alpha'}^2 = 0$.
Here is the explicit realization of $\mathrm{SU}(2) \ltimes \mathbb{R}^3$. Let $\{f_1, f_2, f_3\}$ be the canonical basis of $\mathbb{R}^3$ and $\{e_{12}, e_{13}, e_{23}\}$ be the standard basis of the Lie algebra $\mathfrak{su}(2) = \Lambda^2\mathbb{R}^3$. Consider real parameters $a, b \neq 0$ and the basis of $\mathfrak{su}(2) \ltimes \mathbb{R}^3$ given by

$$x_1 = a(e_{12}, 0),\ x_3 = a(e_{13}, 0),\ x_5 = a(e_{23}, 0),\ x_2 = (be_{12}, f_3),\ x_4 = (be_{13}, -f_2),\ x_6 = (be_{23}, f_1).$$

with the following commutator relations given by the structure of semi-direct product

$$\begin{aligned}
&[x_1,x_3] = ax_5, \quad [x_2,x_4] = 2bx_6 - \tfrac{b^2}{a}x_5, \quad [x_1,x_4] = [x_2,x_3] = ax_6,\\
&[x_5,x_1] = ax_3, \quad [x_6,x_2] = 2bx_4 - \tfrac{b^2}{a}x_3, \quad [x_6,x_1] = [x_5,x_2] = ax_4,\\
&[x_3,x_5] = ax_1, \quad [x_4,x_6] = 2bx_2 - \tfrac{b^2}{a}x_1, \quad [x_3,x_6] = [x_4,x_5] = ax_2.
\end{aligned}$$

We define an almost Hermitian structure (G, g, J) (indexed to the parameters a and b) such that $x_1, \ldots, x_6$ are an orthonormal basis of the Riemannian metric g and the almost complex structure J is such that its fundamental form is given by $\Omega = -(x_{12} + x_{34} + x_{56})$; here and in the sequel, we will write x_{ijk} for $x_i \wedge x_j \wedge x_k$, etc. We now compute the torsion form of the characteristic connection of (g, J). The Nijenhuis tensor of J is easily seen to be

$$N_J = \left(a - \frac{b^2}{a}\right)x_{135} + 2b(x_{136} + x_{145} + x_{235}) - 2bx_{246} + \left(\frac{b^2}{a} - a\right)(x_{146} + x_{236} + x_{245})$$

and the twisted derivative of Ω is

$$d^J\Omega = 3\frac{b^2}{a}x_{135} + a(x_{245} + x_{236} + x_{146}) - 2b(x_{235} + x_{145} + x_{136})$$

and thus the characteristic torsion is [FI02, Ag06]

$$T = N_J + d^J\Omega = \left(a + 2\frac{b^2}{a}\right)x_{135} - 2bx_{246} + \frac{b^2}{a}(x_{146} + x_{236} + x_{245}). \tag{4}$$

The connections forms are easily computed to be

$$\Lambda(x_i) \;=\; \left(a + \frac{b^2}{a}\right)H_i \;\text{ for } i = 1,3,5 \;\text{ and }\; \Lambda(x_j) \;=\; 0 \;\text{ for } j = 2,4,6,$$

where $H_1 = x_{35} + x_{46}$, $-H_3 = x_{15} + x_{26}$ and $H_5 = x_{13} + x_{24}$. These elements satisfy the bracket relations $[H_1, H_3] = H_5$, $[H_5, H_1] = H_3$ and $[H_3, H_5] = H_1$, which means that the characteristic connection has holonomy algebra $\mathfrak{su}(2)$, as it should. The curvature is then a constant multiple of the projection onto the holonomy algebra, namely

$$\mathcal{R} = \frac{b^2}{a^2}(a^2 + b^2)[H_1 \otimes H_1 + H_3 \otimes H_3 + H_5 \otimes H_5]. \tag{5}$$

It is a simple calculation to verify that $\nabla T = \nabla R = 0$ so, indeed, $G = \mathrm{SU}(2) \ltimes \mathbb{R}^3$ is equipped with a two-parameter family of naturally reductive metrics. Comparing our explicit formulas of Eq. 4 and Eq. 5 with the Ansatz of Eq. 2 and Eq. 3, if we take $\alpha = a + 2\frac{b^2}{a}$, $\beta = \frac{b^2}{a}$ and $\alpha' = 2b$ then indeed $(\alpha')^2 - 4\beta(\alpha - 2\beta) = 0$ and $\beta(\alpha - \beta) = \frac{b^2}{a^2}(a^2 + b^2)$ so our computations check out.

Remark 2.2. In fact, more can be said about the almost complex structure of the examples occurring in Case D.2 of Theorem 2.1. The U(3) structure turns out to be an SU(3) structure, hence it may be defined by a real spinor φ of constant length [ACFH15]. A computer-aided computation reveals that φ satisfies the equation

$$\nabla^g_X\varphi \;=\; \eta(X)\varphi + S(X)\varphi$$

with $\eta = 0$ and

$$S = -\frac{\alpha'}{8}J + \frac{3\beta-\alpha}{8}\mathrm{Id} + \begin{bmatrix} -\frac{\beta+\alpha}{8} & \frac{\alpha'}{8} & 0 & 0 & 0 & 0 \\ \frac{\alpha'}{8} & \frac{\beta+\alpha}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta+\alpha}{8} & \frac{\alpha'}{8} & 0 & 0 \\ 0 & 0 & \frac{\alpha'}{8} & \frac{\beta+\alpha}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\beta+\alpha}{8} & \frac{\alpha'}{8} \\ 0 & 0 & 0 & 0 & \frac{\alpha'}{8} & \frac{\beta+\alpha}{8} \end{bmatrix},$$

where the last summand anticommutes with J. Therefore, these three summands are the χ_1, $\chi_{\bar{1}}$ and χ_3 components, respectively, of our SU(3)-structure. An SU(3)-structure is said to be half-flat if the endomorphism S is symmetric – in our example this happens if and only if $\alpha' = 0$. Let D^g be the Dirac operator associated to the Levi-Civita connection. We can readily check that

$$D^g(\varphi) = 3\frac{\alpha - 3\beta}{4}\varphi - 3\frac{\alpha'}{4}\tilde{\varphi},$$

where $\tilde{\varphi}$ is a second spinor linearly independent of φ. We see that if $\alpha' = 0$, then φ is an eigenspinor, and if in addition $\alpha - 3\beta = 0$, then $\varphi \in \ker D^g$.

2.2. **The semidirect product $TG = G \ltimes_{\mathrm{Ad}} \mathfrak{g}$.** Our aim is now to show how the previous example can be generalized to a construction that starts with any compact Lie group endowed with a bi-invariant metric.

Let N and H be two connected Lie groups, $\varphi : H \to \mathrm{Aut}(N)$ a non-trivial group homomorphism, and assume that N is abelian. The semidirect product of H and N with respect to φ, denoted $H \ltimes_\varphi N$, is the manifold $H \times N$ endowed with the multiplication

$$(h_1, n_1)(h_2, n_2) \;=\; (h_1h_2, n_1 + \varphi(h_1)n_2) \;\text{ for all } h_1, h_2 \in H \text{ and } n_1, n_2 \in N.$$

At the Lie algebra level, we have an induced map $\varphi_* : \mathfrak{h} \to \mathrm{End}(\mathfrak{n})$ and the bracket of two elements $(A, u), (B, v) \in \mathfrak{h} \oplus \mathfrak{n}$ is

$$[(A, u), (B, v)] \;=\; ([A, B], \varphi_*(A)(v) - \varphi_*(B)(u)).$$

Since N is abelian, $\mathfrak{n}$ is just a vector space with trivial Lie bracket. We shall shortly construct explicitly a left-invariant metric on $H \ltimes_\varphi N$ for some particular choice of H and N. Instead of checking manually that it is not bi-invariant, let us quickly prove that, more generally, $H \ltimes_\varphi N$ does not admit any bi-invariant metrics at all.

Lemma 2.3. *The semidirect product $H \ltimes_\varphi N$ does not admit bi-invariant Riemannian metrics.*

Proof. Left-invariant Riemannian metrics always exist, as they can be identified with positive definite scalar products on $\mathfrak{h} \oplus \mathfrak{n}$; let $\langle , \rangle$ be one such product. Since $H \ltimes_\varphi N$ is connected, $\langle , \rangle$ will be bi-invariant if and only if $\mathrm{ad}\, X$ is a skew-symmetric endomorphism for any $X \in \mathfrak{h} \oplus \mathfrak{n}$. Let $A \in \mathfrak{h},\ u \in \mathfrak{n}$ be such that $\varphi_*(A)u =: v \neq 0$; this is possible, since we assumed φ non-trivial. Then

$$\begin{aligned} 0 \;&<\; \langle (0, v), (0, v)\rangle \;=\; \langle (0, \varphi_*(A)u), (0, v)\rangle \;=\; \langle [(A, 0), (0, u)], (0, v)\rangle \\ &=\; -\langle [(0, u), (A, 0)], (0, v)\rangle = +\langle (A, 0), [(0, u), (0, v)]\rangle = 0, \end{aligned}$$

since N is abelian. This yields a contradiction and finishes the proof. ☐

By a classical result of Milnor [Mi76], bi-invariant metrics exist only on groups isomorphic to a direct product of a compact Lie group and a vector space (viewed as an abelian group), so actually our proof implies that $H \ltimes_\varphi N$ is not isomorphic to such a product.[1]

Definition 2.4. The Lie group $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ obtained when choosing $H = G$ any connected Lie group, $N = \mathfrak{g}$ its Lie algebra (viewed as an abelian Lie group with respect to addition) and $\varphi = \mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ the adjoint representation will be called the *tangent Lie group of* G and will be denoted by TG.

Of course, TG is indeed the tangent bundle of G. The geometry of tangent Lie groups is described in [YK66a, YK66b] and [Se86]. The Lie algebra of TG is spanned by the complete lifts and the vertical lifts to TG of the left invariant vector fields on G, denoted by an upper index c and v respectively,

$$\mathrm{Lie}(TG) \ = \ \{A^c + B^v \ : \ A, B \in \mathfrak{g}\},$$

and their commutator structure coincides with that of $\mathrm{Lie}(G \ltimes_{\mathrm{Ad}} \mathfrak{g})$ described above (see [YK66a] for a proof),

$$[A^c, B^c] \ = \ [A, B]^c, \quad [A^v, B^v] \ = \ 0, \quad [A^c, B^v] \ = \ [A, B]^v.$$

Hence, given a basis $d_1, \ldots, d_n$ of $\mathfrak{g}$, the $2n$ elements $e_i := d_i^c$ and $f_i := d_i^v$ $(1 \leq i \leq n)$ form a basis of $\mathrm{Lie}(TG) \cong \mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$.
It is known that the complete lift of a semi-Riemannian metric g on M^n to TM^n is a semi-Riemannian metric g^c of split signature (n, n), and that a connection ∇ making (M^n, g) a naturally reductive space lifts to a connection ∇^c that turns (TM^n, g^c) into a naturally reductive space [YK66a, Propositions 6.3, 7.8], [Se86, Theorem 3.6]. Similar lifting results for constructing general *Riemannian* metrics on TM^n with a naturally reductive structure are unknown. We shall construct such a metric in the case where $M^n = G$ is a Lie group.

Theorem 2.5. *Let G be a compact connected Lie group equipped with a bi-invariant metric, $TG = G \ltimes_{\mathrm{ad}} \mathfrak{g}$ its tangent group. Then there is a two parameter family $g_{a,b}$ $(a, b \in \mathbb{R}^*)$ of left-invariant Riemannian metrics on TG such that TG is naturally reductive. More precisely, there is a two parameter family of almost Hermitian structures of Gray-Hervella class $\mathcal{W}_1 \oplus \mathcal{W}_3$ such that their characteristic connection ∇, its torsion T, and curvature $\mathcal{R}$, satisfy $\nabla T = \nabla \mathcal{R} = 0$ and $\mathfrak{hol}(\nabla) = [\mathfrak{g}, \mathfrak{g}]$.*

Proof. Let $n = \dim G$ and $\{d_1, \cdots, d_n\}$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$ with respect to the chosen bi-invariant metric, $e_i := d_i^c$ and $f_i := d_i^v$ their complete and vertical lifts, respectively, as explained above $(1 \leq i \leq n)$. We may assume that the structure constants C_{ij}^k are totally antisymmetric in all indices, see Remark 1.2.
We define a two-parameter family of left-invariant Riemannian metrics $g_{a,b}$ $(a, b \neq 0)$ on TG by setting the following elements of $\mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$ to be orthonormal

$$x_i = a e_i \qquad \text{and} \qquad y_i = b e_i + f_i \quad (i = 1, \ldots, n).$$

Also, we define an almost complex structure J on M by the two form

$$\Omega \ := \ -(x_1 \wedge y_1 + \cdots + x_n \wedge y_n).$$

We have the following commutator relations

$$[x_i, x_j] = a \sum_{k=1}^n C_{ij}^k x_k, \quad [x_i, y_j] = a \sum_{k=1}^n C_{ij}^k y_k, \quad [y_i, y_j] = b \sum_{k=1}^n C_{ij}^k (2 y_k - \frac{b}{a} x_k).$$

[1]As an aside, we observe that it is not difficult to construct examples of semidirect products with φ non-trivial and N non-abelian admitting bi-invariant metrics; as the proof goes, it is clear that these will have the property $\langle \mathfrak{h}, [\mathfrak{n}, \mathfrak{n}] \rangle \neq 0$, and by Milnor's result, they are actually isomorphic (in a non-trivial way) to direct products.

For ease of notation, we will omit the wedge product sign in general; for instance, we will write $x_{ij}y_k$ instead of $x_i \wedge x_j \wedge y_k$ etc. The Nijenhuis tensor of the almost complex structure is the skew-symmetric tensor

$$N \;=\; \sum_{i<j<k}^{n} C_{ij}^k \left[\left(a - \frac{b^2}{a}\right)(x_{ijk} - (x_i y_{jk} + y_i x_j y_k + y_{ij} x_k)) - 2b\,(y_{ijk} - (x_{ij} y_k + y_i x_{jk} + x_i y_j x_k))\right]$$

and the twisted derivative of Ω is

$$d^J\Omega \;=\; \sum_{i<j<k}^{n} C_{ij}^k \left[a\,(x_i y_{jk} + y_i x_j y_k + y_{ij} x_k) + 3\frac{b^2}{a}\, x_{ijk} - 2b\,(x_{ij} y_k + x_i y_j x_k + y_i x_{jk})\right].$$

Thus the torsion of the characteristic connection ∇ is given by

$$T \;=\; N + d^J\Omega \;=\; \sum_{i<j<k}^{n} C_{ij}^k \left[\left(a + 2\frac{b^2}{a}\right) x_{ijk} - 2b y_{ijk} + \frac{b^2}{a}(x_i y_{jk} + y_i x_j y_k + y_{ij} x_k)\right].$$

The connection forms of the characteristic connection are

$$\Lambda(x_i) = \left(a + \frac{b^2}{a}\right)\sum_{j<k}^{n} C_{ij}^k (x_{jk} + y_{jk}) \quad \text{ and } \quad \Lambda(y_i) = 0 \quad \text{for } \; i = 1, \ldots, n.$$

If we define $H_i = \sum_{j<k}^n C_{ij}^k (x_{jk} + y_{jk})$, these elements are linearly independent, and the Jacobi identity on G shows that $[H_i, H_j] = \sum_{k=1}^n C_{ij}^k H_k$. The holonomy Lie algebra $\mathfrak{hol}(\nabla)$ is spanned by [KN69, Ch.X, 4.1]

$$\mathfrak{m}_0 \;:=\; \{[\Lambda(X), \Lambda(Y)] - \Lambda([X,Y]) \;:\; X,\, Y \in \mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}\}$$

and its iterated commutators with $\Lambda(\mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g})$. The only contribution to $\mathfrak{m}_0$ comes from the elements

$$[\Lambda(x_i), \Lambda(x_j)] - \Lambda([x_i, x_j]) \;=\; \frac{b^2}{a^2}(a^2 + b^2)\sum_{k=1}^{n} C_{ij}^k H_k,$$

whose coefficient in front cannot be zero, since a and b are assumed to be non vanishing. Hence, the characteristic connection has holonomy algebra $[\mathfrak{g}, \mathfrak{g}]$. Furthermore, the curvature tensor is a constant multiple of the projection on the holonomy algebra, namely

$$\mathcal{R} = \frac{b^2}{a^2}(a^2 + b^2)\sum_{k=1}^{n} H_k \otimes H_k.$$

This immediately yields that both the torsion form and the curvature tensor are parallel w. r. t. ∇. It remains to prove that the almost complex structure is of type $\mathcal{W}_1 \oplus \mathcal{W}_3$. We already observed that N is a skew symmetric so we only need to show that the $\mathcal{W}_4$ component vanishes, i.e. $\delta^g\Omega = 0$. We use the formula (see [Ag06], for instance)

$$\delta^g\Omega = \delta^\nabla\Omega + \frac{1}{2}(e_i \lrcorner\, e_j \lrcorner\, T) \wedge (e_i \lrcorner\, e_j \lrcorner\, \Omega) \tag{6}$$

for any orthonormal basis $\{e_1, \cdots, e_{2n}\}$ of M. Using our adapted basis $\{x_1, y_1, \cdots, x_n, y_n\}$ and the fact that ∇ is the characteristic connection of (M, g, J), then we get that $\delta^g\Omega = 0$. □

Remark 2.6. This is not the canonical symplectic structure on T^*G (since $d\Omega \neq 0$). Observe that the Hermitian form can also be written

$$\Omega \;=\; -a\sum_{i=1}^{n} e_i \wedge f_i \;=\; -a\sum_{i=1}^{n} d_i^c \wedge d_i^v.$$

Thus, up to normalization the almost complex structure 'rotates' complete lifts of vector fields into vertical lifts. This is similar in spirit to the almost complex structure already constructed in [Do62], which acts in a similar way on horizontal (with respect to any affine connection) and vertical lifts.

However, the metric we constructed is not g-natural in any reasonable sense: Given an affine connection ∇ (usually the Levi-Civita connection of some Riemannian metric), recall that a g-natural metric is a metric on TM for which the vertical and ∇-horizontal distribution are orthogonal and the metric coincides with the original metric on horizontal lifts. But the vector fields y_i are not purely vertical, and there is no connection having the vector fields x_i as horizontal lifts.

Remark 2.7. We excluded $b = 0$ from the previous discussion; however, all formulas make sense, so we can investigate what happens in this limiting case. The value $b = 0$ corresponds to the direct product metric, but on the semidirect product Lie group, hence it is nevertheless only left-invariant, in accordance to Lemma 2.3. For $b = 0$, we see that $\mathfrak{hol}(\nabla) = 0$ and $\mathcal{R} = 0$, so the connection is flat. We will come back to this case in the next section.

Remark 2.8. Let us look more closely at the case where $\mathfrak{g}$ has non-trivial center $\mathfrak{z}$ of dimension $p > 0$. We can assume that we chose our orthonormal basis such that the first p elements $d_1, \ldots, d_p$ span $\mathfrak{z}$. Then all corresponding structure constants $C^k_{\alpha j}$ vanish, where $\alpha = 1, \ldots, p$, and therefore $H_\alpha = 0$ and

$$\operatorname{span}(e_\alpha, f_\beta \,|\, \alpha, \beta = 1, \ldots, p) \;=\; \ker T \;:=\; \{X \in \operatorname{Lie}(TG) \,|\, X \lrcorner T = 0\}.$$

Thus T has a $2p$-dimensional kernel and the splitting theorem [AFF15, Thm 3.4] implies that TG is locally a Riemannian product with vanishing torsion on one factor and torsion with trivial kernel on the other factor. The case $\mathfrak{z} \neq \{0\}$ may therefore be reduced to lower dimensional examples. Furthermore, if G is semisimple but not simple its Lie algebra $\mathfrak{g}$ splits as the sum of n $(n > 1)$ simple Lie algebras and clearly both the tangent group TG and the torsion form T will also split into n summands, reducing again the problem to lower dimensional spaces. All in all, we can conclude that the interesting cases are G simple: After the space $M^6 = T\mathrm{SU}(2)$ described before, the next examples are $M^{16} = T\mathrm{SU}(3)$ and $M^{28} = TG_2$.

2.3. **The direct product $G \times \mathfrak{g}$.** We shall now prove that the metric β constructed on TG in Theorem 2.5 is isometric to a left-invariant metric on the direct product $G \times \mathfrak{g}$, despite the fact that it is not isomorphic (as a Lie group) to $G \ltimes \mathfrak{g}$ by Lemma 2.3.

First, let us describe the relevant metric on the direct product $G \times \mathfrak{g}$. We assume, as in Theorem 2.5, that G is a connected Lie group of dimension n with bi-invariant metric, $\{e_1, \cdots, e_n\}$ an orthonormal basis of $\mathfrak{g}$ with respect to this metric, and C^k_{ij} the totally antisymmetric structure constants of G.

We define a two-parameter family of left-invariant Riemannian metrics $\tilde{\beta}$ (depending again on $a, b \in \mathbb{R},\ a \neq 0$) on $G \times \mathfrak{g}$ by setting the following elements of $\mathfrak{g} \oplus \mathfrak{g}$ to be orthonormal,

$$x_i = (ae_i, 0) \qquad \text{and} \qquad y_i = (be_i, e_i) \quad i = 1, \ldots, n.$$

As an inner product on $\mathfrak{g} \oplus \mathfrak{g}$, this coincides of course with the inner product defined on $\mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$ in Theorem 2.5. The case $b = 0$ corresponds now to the bi-invariant direct product metric on $G \times \mathfrak{g}$. For completeness, let us state the bracket relations given by the direct product structure,

$$[x_i, x_j] = a \sum_{k=1}^{n} C^k_{ij} x_k, \quad [x_i, y_j] = b \sum_{k=1}^{n} C^k_{ij} x_k, \quad [y_i, y_j] = \frac{b^2}{a} \sum_{k=1}^{n} C^k_{ij} x_k.$$

Vector fields on $G \ltimes \mathfrak{g}$ and $G \times \mathfrak{g}$ are defined by left translation from their respective Lie algebras (identified with the tangent space at the neutral element), so they are not the same on the set $G \times \mathfrak{g}$, because the group structures are different – as can be seen from the differing commutator relations. However, the left translation operators starting from the neutral element $(e, 0)$ on both groups coincide. To see this, start with any point $\mathcal{Y} := (h, Y)$ and consider the left translation in TG (denoted by L^s) and in $G \times \mathfrak{g}$ (denoted by L^d) by any group element $\mathcal{X} := (g, X)$. The letter L without index denotes the usual left translation in G respectively $\mathfrak{g}$ (the actual formula on $\mathfrak{g}$ does not matter for the purpose here). The definition of the group multiplication in $G \ltimes \mathfrak{g}$

and $G\times\mathfrak{g}$ is equivalent to

$$L^s_{\mathcal{X}}(\mathcal{Y}) \;=\; (L_g(h), L_X(\mathrm{Ad}_g Y)), \quad L^d_{\mathcal{X}}(\mathcal{Y}) \;=\; (L_g(h), L_X(Y)).$$

Therefore, for $\mathcal{Y}=(e,0)$, $\mathrm{Ad}_g 0 = 0$ and hence $L^s_{\mathcal{X}}(e,0)$ and $L^d_{\mathcal{X}}(e,0)$ coincide, and one easily checks that their differential at $(e,0)$ coincide as well. Therefore, left translation by $\mathcal{X}$ maps the origin to the *same* point in the set $G\times\mathfrak{g}$ regardless which group structure we consider. In particular, there is a natural identification of tangent spaces to TG and $G\times\mathfrak{g}$ at *all* points, and the metrics β, $\tilde\beta$ coincide in each of these tangent spaces. Recall now that a diffeomorphism $f: TG = G\ltimes\mathfrak{g}\to G\times\mathfrak{g}$ is an isometry if $\beta_{g,X}(U,V)=\tilde\beta_{f(g,X)}(df_{g,X}U, df_{g,X}V)$. We choose $f(g,X)=(g,X)$ the identity, hence $df=\mathrm{Id}$, and this becomes thus an isometry because of the identification of tangent spaces and metrics.

Remark 2.9. Within the set-up described above, define an almost complex structure on $(G\times\mathfrak{g},\tilde\beta)$ by the two form

$$\Omega \;=\; -(x_1\wedge y_1+\cdots+x_n\wedge y_n).$$

It is a straightforward computation to check that, up to sign, the characteristic torsion of this almost complex structure is the same as the one described in the proof of Theorem 2.5. The connection forms of the characteristic connection ∇ are expressed as

$$\Lambda(x_i) = -\frac{b^2}{a}\sum_{j<k}^{n} C^k_{ij}(x_{jk}+y_{jk}) \qquad\text{and}\qquad \Lambda(y_i) = b\sum_{j<k}^{n} C^k_{ij}(x_{jk}+y_{jk}),$$

One checks that the curvature tensor is given by the same expression as the in the proof of Theorem 2.5. A similar argument yields that both the torsion form and the curvature tensor are parallel with respect to ∇.

Now, we had just seen before that the metrics on TG and $G\times\mathfrak{g}$ corresponding to $b=0$ are isometric (compare Remark 2.7), and the connection ∇ is then flat. By a Theorem of Cartan and Schouten ([CSch26a], see also [AF10] for a modern proof), $G\ltimes\mathfrak{g}$ with a flat metric connection had to be isometric to a Lie group with a bi-invariant metric, so we could have anticipated the isometry at least in this special case.

Another argument to prove that $(G\ltimes\mathfrak{g},\beta)$ and $(G\times\mathfrak{g},\tilde\beta)$ are isometric is simply to use the Nomizu construction, see [AFF15, Appendix A] or [TV83], since the expressions for the torsion form and the curvature tensor are identical.

3. A peculiar connection on TS^7

One might ask which are the crucial properties that make the construction of the metric connection in Section 2 work. Certainly, the Lie group approach is straightforward and yields a complete description in the well-known formalism of Lie groups and Lie algebras.

A different look at $G\times\mathfrak{g}$ is to observe that both factors carry a remarkable *flat* metric connection — the former one of the two flat Cartan-Schouten connections (for the biinvariant metric), the latter the usual Levi-Civita connection (for the standard euclidean metric). By the theorem of Cartan and Schouten mentioned before [CSch26a, AF10], the only irreducible manifold carrying a flat metric connection ∇ with skew torsion T that is neither a Lie group nor a vector space is the sphere S^7. Leaving flat vector spaces aside, the crucial difference between a compact Lie group and S^7 lies in the behaviour of the torsion: for Lie groups, $\nabla T=0$, while on the 7-sphere, the torsion T fails to be parallel [AF10, p.484].

We shall sketch how this point of view yields a remarkable connection on TS^7 and where differences appear. First we summarize the construction of the flat metric connection with skew torsion on S^7. In dimension 7, the complex $\mathrm{Spin}(7)$-representation $\Delta_7^{\mathbb{C}}$ is the complexification of a real 8-dimensional representation $\kappa:\mathrm{Spin}(7)\to\mathrm{End}(\Delta_7)$, since the real Clifford algebra $\mathcal{C}(7)$ is isomorphic to $\mathcal{M}(8)\oplus\mathcal{M}(8)$. Thus, we may identify $\mathbb{R}^8$ with the vector space Δ_7 and embed therein the sphere S^7 as the set of all spinors of length one ($\langle\cdot,\cdot\rangle$ is the euclidean scalar product on $\Delta_7 = \mathbb{R}^8$). Fix your favourite explicit realization of the spin representation by skew matrices, $\kappa_i := \kappa(e_i) \in \mathfrak{so}(8) \subset \mathrm{End}(\mathbb{R}^8)$, $i = 1, \dots, 7$. Define vector fields $V_1, \dots, V_7$ on S^7 by

$$V_i(x) \;=\; \kappa_i \cdot x \text{ for } x \in S^7 \subset \Delta_7.$$

From the antisymmetry of $\kappa_1, \dots, \kappa_7$, one easily deduces that the vector fields $V_1(x), \dots, V_7(x)$ define a global orthonormal frame on S^7 consisting of Killing vector fields. The connection ∇ on S^7 is then defined by the requirement $\nabla V_i(x) = 0$. This connection is trivially flat and metric, and its torsion coefficients are given by $(i \neq j)$

$$T(V_i, V_j, V_k)(x) \;=\; -\langle [V_i(x), V_j(x)], V_k\rangle \;=\; -2\langle \kappa_i\kappa_j x, \kappa_k x\rangle \;=\; 2\langle \kappa_i\kappa_j\kappa_k x, x\rangle \;=:\; \tau_{ijk}(x).$$

Of course, the coefficients $\tau_{ijk}(x)$ are not constant, reflecting that T is not parallel. The Killing orthonormal frame $V_1(x), \dots, V_7(x)$ does *not* form a Lie algebra; rather,

$$[V_i(x), V_j(x)] \;=\; [\kappa_i, \kappa_j](x) \;=\; 2\,\kappa_i\kappa_j x \;=\; -\sum_k \tau_{ijk}(x) V_k(x) \quad \text{for } i \neq j. \tag{7}$$

The antisymmetric functions $\tau_{ijk}(x)$ replace (up to sign) the structure constants C_{ij}^k of the Lie group approach. These commutators $[V_i(x), V_j(x)]$ are, of course, again Killing vector fields, but not of constant length; however, they span a Lie algebra isomorphic to $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$.
Now choose the frame $f_i := \partial/\partial z_i$ with respect to standard euclidean coordinates $z_1, \dots, z_7$ on $\mathbb{R}^7$, which is of course orthonormal for the euclidean scalar product. Formally, the vector fields V_i and f_i obey commutator relations similar to the direct product situation described in Subsection 2.3. Introduce global vector fields on $S^7 \times \mathbb{R}^7 \ni (x, z)$ by

$$X_i(x,z) \;=\; (aV_i(x), 0), \quad Y_i(x,z) \;=\; (bV_i(x), f_i), \quad i = 1, \dots, 7 \quad (a, b \in \mathbb{R}, a \neq 0)$$

and define – just as before – a Riemannian metric g on $S^7 \times \mathbb{R}^7$ by requiring that these are orthonormal, and an almost complex structure via the Hermitian form $\Omega = -\sum X_i \wedge Y_i$. Similarly to the direct product case, the following bracket relations hold (we omit the base point (x, z)),

$$[X_i, X_j] = -a\sum_{k=1}^n \tau_{ijk} X_k, \quad [X_i, Y_j] = -b\sum_{k=1}^n \tau_{ijk}\, X_k, \quad [Y_i, Y_j] = -\frac{b^2}{a}\sum_{k=1}^n \tau_{ijk}\, X_k.$$

The Nijenhuis tensor of the almost hermitian structure is the skew-symmetric tensor

$$N \;=\; \left[a - \frac{b^2}{a}\right] \sum_{i<j<k} \tau_{ijk}(X_{ijk} - Y_{ij}X_k) + 2b \sum_{i<j<k} \tau_{ijk}(X_{ij}Y_k - Y_{ijk})$$

and the twisted derivative of Ω is

$$d^J\Omega = a \sum_{i<j<k} \tau_{ijk} X_k Y_{ij} + 3\frac{b^2}{a} \sum_{i<j<k} \tau_{ijk} X_{ijk} - 2b \sum_{i<j<k} \tau_{ijk} X_{ij} Y_k.$$

Thus the characteristic torsion of this almost hermitian structure is given by

$$\tilde{T} = N + d^J\Omega = \left[a + 2\frac{b^2}{a}\right] \sum_{i<j<k} \tau_{ijk} X_{ijk} - 2b \sum_{i<j<k} \tau_{ijk} Y_{ijk} + \frac{b^2}{a} \sum_{i<j<k} \tau_{ijk} X_k Y_{ij}.$$

Finally, the connection $\nabla = \nabla^g + \frac{1}{2}\tilde{T}$ on $S^7 \times \mathbb{R}^7$ is given by

$$\nabla_{Z_i} X_j = \frac{b^2}{a} \sum_{k=1}^7 \tau_{ijk} Z_k \qquad \text{and} \qquad \nabla_{Z_i} Y_j = -b \sum_{k=1}^7 \tau_{ijk} Z_k, \quad Z_i = X_i \text{ or } Y_i, \quad i = 1, \dots, 7.$$

If we define the 2-forms $H_i = -\frac{1}{2}\sum_{j,k} \tau_{ijk}(X_{jk} + Y_{jk})$, we can summarize these identities as

$$\nabla_{Z_i} X_j \;=\; \frac{b^2}{a}\,(Z_i \lrcorner H_j), \quad \nabla_{Z_i} Y_j \;=\; -b\,(Z_i \lrcorner H_j).$$

As on S^7 itself, the torsion $\tilde{T}$ is not parallel, but $\nabla_X\tilde{T}(Y,Z,V)$ is antisymmetric in all arguments, hence the curvature operator $\mathcal{R}$ is a symmetric operator $\Lambda^2(TS^7)\to\Lambda^2(TS^7)$ (see [Ag06, Remark 2.3] for details on this curvature argument). In fact, one computes

$$\mathcal{R}=\frac{4b^2}{a^2}(a^2+b^2)(H_1\otimes H_1+\cdots H_n\otimes H_n).$$

Together with the property (7), this implies that ∇ has holonomy $\mathfrak{spin}(7)$.

Appendix A. The two-fold product $G\times G$

We now discuss briefly the compact case. The direct product $G\times G$ also has families of naturally reductive structures – we sketch the construction to emphasize how this compares to the case described in Section 2. This is an explicit version of the constructions from [AZ79]; it generalizes the $S^3\times S^3$ example given in [AFF15, Section 9.4], see case (c) in Theorem 2.1.

Theorem A.1. *Let G be a connected compact Lie group with bi-invariant metric. The group $G\times G$ can be equipped with a five-parameter family of left invariant naturally reductive structures. More precisely, $G\times G$ can be endowed with a family of almost complex structures of type $\mathcal{W}_1\oplus\mathcal{W}_3$ such that its characteristic connection satisfies $\nabla T=\nabla\mathcal{R}=0$ and $\mathfrak{hol}(\nabla)$ is either $[\mathfrak{g},\mathfrak{g}]$ or trivial.*

Proof. We realise $G\times G$ as the homogeneous space K/L, where $K=G\times G\times G$ and $L=\Delta G$ is embedded into K diagonally. Let $\mathfrak{k}=\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$ be the Lie algebra of K and $\Delta\mathfrak{g}=\{(X,X,X): X\in\mathfrak{g}\}$ be the Lie algebra of ΔG. Consider the following $\Delta\mathfrak{g}$-modules

$$\mathfrak{m}_1=\{(X,aX,bX):a,b\in\mathbb{R},X\in\mathfrak{g}\},\quad \mathfrak{m}_2=\{(Y,cY,dY):c,d\in\mathbb{R},Y\in\mathfrak{g}\},\quad \mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2.$$

One checks that $\mathfrak{m}$ is a reductive complement of $\Delta\mathfrak{g}$ inside $\mathfrak{k}$ if and only if

$$\Delta:=(a-1)(d-1)-(b-1)(c-1)\neq 0.$$

Let B denote the negative Killing form on $\mathfrak{g}$ and define an inner product on $\mathfrak{m}$, for each parameter $\lambda>0$, as

$$\langle(X_1+Y_1,aX_1+cY_1,bX_1+dY_1),(X_2+Y_2,aX_2+cY_2,bX_2+dY_2)\rangle=B(X_1,X_2)+\frac{1}{\lambda^2}B(Y_1,Y_2).$$

We define also an almost complex structure J on $\mathfrak{m}$ by

$$J((X,aX,bX)+(Y,cY,dY))=-\frac{1}{\lambda}(Y,aY,bY)+\lambda(X,cX,dX).$$

Let $e_1,\dots,e_n$ be an orthonormal basis of $\mathfrak{g}$ with antisymmetric structure constants C_{ij}^k. Consider also the elements

$$x_i=(e_i,ae_i,be_i)\quad\text{and}\quad y_i=(e_i,ce_i,de_i),\qquad i=1,\dots,n.$$

The sets $\{x_i\}$ and $\{y_i\}$ are bases of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. Remark that J is given, in the basis $\{x_i,y_i\}$ of $\mathfrak{m}$, by the 2-form $\Omega=-(x_1\wedge y_1+\cdots+x_n\wedge y_n)$. Finally, let $h_i=(e_i,e_i,e_i)$. The isotropy representation $\lambda:\mathfrak{h}\longrightarrow\mathfrak{so}(\mathfrak{m})$ is given by

$$\lambda(h_i)=\sum_{j<k}C_{ij}^k(x_jx_k+y_jy_k):=H_i.$$

The structure of $\mathfrak{so}(\mathfrak{m})$ together with the Jacobi identity in $\mathfrak{g}$ imply that $[H_i,H_j]=\sum_{k=1}^n C_{ij}^k H_k$. The commutator structure is somewhat complicated. For ease of notation we introduce the following coefficients

$$\alpha = -\tfrac{2}{\Delta}((a^2-1)(d-1)-(b^2-1)(c-1)) \qquad \beta = -\tfrac{2}{\Delta}(b-1)(a-1)(b-a)$$
$$\gamma = -\tfrac{2}{\Delta}(a(d-b^2)+a^2(b-d)+(b^2-b)c) \qquad \delta = -\tfrac{2}{\Delta}(c(a(d-1)-bd+1)+(b-1)d)$$
$$\sigma = \tfrac{2}{\Delta}((a-1)(1-bd)+(ac-1)(b-1)) \qquad \tau = \tfrac{2}{\Delta}(ac(d-b)+cb(1-d)+ad(b-1))$$
$$\xi = -\tfrac{2}{\Delta}(c-1)(d-1)(c-d) \qquad \eta = -\tfrac{2}{\Delta}((d^2-1)(a-1)-(c^2-1)(b-1))$$
$$\theta = -\tfrac{2}{\Delta}(d^2(c-a)+c^2(b-d)+(da-cb))$$

Then we can write the nonvanishing brackets of elements of $\mathfrak{m}$ as

$$[x_i,x_j] \;=\; \textstyle\sum_{k=1}^n C_{ij}^k(\mu x_k+\frac{\nu}{\lambda}y_k+\gamma h_k), \qquad [x_i,y_j] \;=\; \sum_{k=1}^n C_{ij}^k(\lambda\delta x_k+\sigma y_k+\lambda\tau h_k),$$
$$[y_i,y_j] \;=\; \textstyle\sum_{k=1}^n C_{ij}^k(\lambda^2\xi x_k+\lambda\eta y_k+\lambda^2\theta h_k).$$

The Nijenhuis tensor N is totally skew-symmetric and given by

$$\begin{aligned} N \;&=\; \sum_{i<j<k} C_{ij}^k\Bigg[[\lambda^2\xi+2\sigma-\alpha](x_{ijk}-(x_iy_{jk}+y_ix_jy_k+y_{ij}x_k))+\\ &\quad+\left[\frac{\beta}{\lambda}+\lambda(2\delta-\eta)\right](y_{ijk}-(y_ix_{jk}+x_iy_jx_k+x_{ij}y_k))\Bigg].\end{aligned}$$

We can also compute that

$$\begin{aligned} d^c\Omega \;&=\; \sum_{i<j<k} C_{ij}^k\Bigg[-3\lambda^2\xi x_{ijk}-3\frac{\beta}{\lambda}y_{ijk}+(2\sigma-\alpha)(x_iy_{jk}+y_ix_jy_k+y_{ij}x_k)\\ &\quad+\lambda(2\delta-\eta)(y_ix_{jk}+x_iy_jx_k+x_{ij}y_k)\Bigg].\end{aligned}$$

Therefore the torsion tensor $T=N+d^c\Omega$ is given by

$$\begin{aligned} T \;&=\; \textstyle\sum_{i<j<k} C_{ij}^k\Big[[-2\lambda^2\xi+2\sigma-\alpha]x_{ijk}+\Big[-2\frac{\beta}{\lambda}+\lambda(2\delta-\eta)\Big]y_{ijk}-\lambda^2\xi(x_iy_{jk}+y_ix_jy_k+y_{ij}x_k)\\ &\quad-\tfrac{\beta}{\lambda}(y_ix_{jk}+x_iy_jx_k+x_{ij}y_k)\Big].\end{aligned}$$

The characteristic connection is given by the map $\Lambda:\mathfrak{m}\longrightarrow\mathfrak{so}(\mathfrak{m})$

$$\Lambda(x_i)=(-\lambda^2\xi+\sigma)H_i, \quad \Lambda(y_i)=\left(-\frac{\beta}{\lambda}+\lambda\delta\right)H_i, \quad i=1,\dots n.$$

It is then a straightforward computation to check that $\Lambda T=0$. As for the curvature tensor we have that

$$\mathcal{R}=\Sigma(H_1\otimes H_1+\cdots+H_n\otimes H_n), \ \text{ with } \ \Sigma:=\frac{\beta^2}{\lambda^2}+\lambda^4\xi^2-\lambda^2\xi(2\sigma-\alpha)-\beta(2\delta-\eta).$$

It is then also clear that $\Lambda\mathcal{R}=0$. Therefore we have a 5-parameter family of naturally reductive spaces on $G\times G$. We have $\mathcal{R}=0$ if and only if $\Sigma=0$; This includes some particular cases like $(c=1,b=1)$, $(a=1,d=1)$ and $(c=d,a=b)$. In this case, $\mathfrak{hol}(\nabla)=0$. If $\Sigma\neq 0$ then $\mathfrak{hol}(\nabla)=[\mathfrak{g},\mathfrak{g}]$. For all parameters, $\delta^g\Omega=0$, so the almost Hermitian structure is of type $\mathcal{W}_1\oplus\mathcal{W}_3$. $\square$

References

[Ag06] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) 42, 5–84 (2006). With an appendix by Mario Kassuba.

[ACFH15] I. Agricola, S. Chiossi, T. Friedrich, J. Höll, *Spinorial description of* SU(3)*- and* G_2*-manifolds*, J. Geom. Phys. 98 (2015), 535-555.

[AFF15] I. Agricola, A. C. Ferreira, Th. Friedrich, *The classification of naturaturally reductive homogeous spaces in dimensions* $n \leq 6$, Differential Geom. Appl. 39 (2015), 59-92.

[AFS15] I. Agricola, A. C. Ferreira, R. Storm, *Quaternionic Heisenberg groups as naturally reductive homogeneous spaces*, Int. J. of Geometric Methods in Modern Physics 12 (2015), article ID 1560007.

[AF10] I. Agricola, Th. Friedrich, *A note on flat metric connections with antisymmetric torsion*, Differ. Geom. Appl. 28 (2010), 480-487.

[AS58] W. Ambrose, I. M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. 25, 647-669 (1958).

[AZ79] J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Memoirs of Amer. Math. Soc. 18 (1979).

[CSch26a] É. Cartan and J. A. Schouten, *On the Geometry of the group manifold of simple and semisimple groups*, Proc. Amsterdam 29 (1926), 803–815.

[Ch16] I. Chrysikos, *Invariant connections with skew-torsion and* ∇*-Einstein manifolds*, J. Lie Theory 26 (2016), 11–48.

[Do62] P. Dombrowski, *On the geometry of tangent bundles*, J. reine angew. Math., 210 (1962), 73–88.

[Go85] C. Gordon, *Naturally reductive homogeneous Riemannian manifolds*, Can. J. Math. 37 (1985), 467–487.

[FI02] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian Journ. Math. 6 (2002), 303-336.

[GH80] A. Gray, L. M. Hervella, *The Sixteen Classes of Almost Hermition Manifolds and Their Linear Invariants*, Ann. Mat. Pura Appl., IV. Ser. 123, 35-58 (1980).

[KN69] S. Kobayashi and K. Nomizu, *Foundations of differential geometry II*, Wiley Classics Library, Wiley Inc., Princeton, 1969, 1996.

[K56] B. Kostant, *On differential geometry and homogeneous spaces II*, Proc. N.A.S 42 (1956), 354–357.

[Mi76] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. 21 (1976), 293–329.

[OR12] C. Olmos, S. Reggiani, *The skew-torsion holonomy theorem and naturally reductive spaces*, J. Reine Angew. Math. 664, 29-53 (2012).

[OR13] C. Olmos, S. Reggiani, *A note on the uniqueness of the canonical connection of a naturally reductive space*, Monatsh. Math. 172, 379-386 (2013).

[Sch07] N. Schoemann, *Almost Hermitian structures with parallel torsion*, J. Geom. Phys. 57 (2007), 2187-2212.

[Se86] M. Sekizawa, *On complete lifts of reductive homogeneous spaces and generalized symmetric spaces*, Czech. Math. J. 36 (1986), 516–534.

[TV83] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes Series, vol. 83, Cambridge Univ. Press, Cambridge, 1983.

[YK66a] K. Yano, S. Kobayashi, *Prolongations of tensor fields and connections to tangent bundles I – General theory*, J. Math. Soc. Japan 18 (1966), 194–210.

[YK66b] K. Yano, S. Kobayashi, *Prolongations of tensor fields and connections to tangent bundles II – Infinitesimal automorphisms*, J. Math. Soc. Japan 18 (1966), 236–246.

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