Old and New on the Exceptional Group G_2

Ilka Agricola

In a talk delivered in Leipzig (Germany) on June 11, 1900, Friedrich Engel gave the first public account of his newly discovered description of the smallest exceptional Lie group G_2, and he wrote in the corresponding note to the Royal Saxonian Academy of Sciences:

> Moreover, we hereby obtain a direct definition of our 14-dimensional simple group [G_2] which is as elegant as one can wish for. [En00, p. 73][1]

Indeed, Engel's definition of G_2 as the isotropy group of a generic 3-form in 7 dimensions is at the basis of a rich geometry that exists only on 7-dimensional manifolds, whose full beauty has been unveiled in the last thirty years.

This article is devoted to a detailed historical and mathematical account of G_2's first years, in particular the contributions and the life of Engel's almost forgotten Ph.D. student Walter Reichel, who worked out the details of this description in 1907. We will also give an introduction to modern G_2 geometry and its relevance in theoretical physics (in particular, superstring theory).

The Classification of Simple Lie Groups

In 1887 Wilhelm Killing [Kil89] succeeded in classifying those transformation groups that are rightly called *simple*: by definition, these are the Lie groups that are not abelian and do not have any nontrivial normal subgroups.

Every Lie group G has a Lie algebra $\mathfrak{g}$ (the tangent space to the group manifold at the identity), which is a vector space endowed with a skew-symmetric product, the *Lie bracket* $[\ ,\ ]$; as a purely algebraic object it is more accessible than the original Lie group G. If G happens to be a group of matrices, its Lie algebra $\mathfrak{g}$ is easily realized by matrices too, and the Lie bracket coincides with the usual commutator of matrices. In Killing's and Lie's time, no clear distinction was made between the Lie group and its Lie algebra. For his classification, Killing chose a maximal set $\mathfrak{h}$ of linearly independent, pairwise commuting elements of $\mathfrak{g}$ and constructed base vectors X_α of $\mathfrak{g}$ (indexed over a finite subset R of elements $\alpha \in \mathfrak{h}^*$, the *roots*) on which all elements of $\mathfrak{h}$ act diagonally through $[\ ,\ ]$:

$$[H, X_\alpha] = \alpha(H)\, X_\alpha \text{ for all } H \in \mathfrak{h}.$$

In order to avoid problems when doing so he chose the complex numbers $\mathbb{C}$ as the ground field. The dimension of the maximal abelian subalgebra $\mathfrak{h}$ (also called, somehow wrongly, a *Cartan subalgebra*) is the *rank* of the Lie algebra. It is a general fact that all roots $\alpha \neq 0$ appear only once. If we write $\mathfrak{g}_\alpha := \mathbb{C} \cdot X_\alpha$ (these are the *root spaces*), we obtain a decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

and vectors in $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ for two roots α, β satisfy an extremely easy multiplication rule: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta$ is again a root, while otherwise it is zero.

Two families of complex simple Lie algebras were well-known at that time:

(1) the Lie algebras $\mathfrak{so}(n, \mathbb{C})$ consisting of skew-symmetric complex matrices, which are the Lie algebras of the orthogonal groups $SO(n, \mathbb{C})$ ($n = 3$ or $n \geq 5$),

(2) the Lie algebras $\mathfrak{sl}(n, \mathbb{C})$ consisting of trace-free matrices, which are the Lie algebras of the groups $SL(n, \mathbb{C})$ of matrices of determinant one ($n \geq 2$).

Ilka Agricola is head of a Junior Research Group funded by the Volkswagen Foundation at Humboldt-Universität zu Berlin. Her email address is agricola@mathematik.hu-berlin.de.

[1] *In Engel's own words: "Zudem ist hiermit eine direkte Definition unserer vierzehngliedrigen einfachen Gruppe gegeben, die an Eleganz nichts zu wünschen übrig lässt."*

It was Killing's original intent to prove that these were the *only* simple complex Lie algebras.[2] In fact, there exists a third family of simple algebras, namely the Lie algebras $\mathfrak{sp}(2n, \mathbb{C})$ of the symplectic groups $\mathrm{Sp}(2n, \mathbb{C})$ for $n \geq 1$, defined as invariance groups of non-degenerate 2-forms ω on $\mathbb{C}^{2n}$:

$$\mathrm{Sp}(2n, \mathbb{C}) = \{g \in \mathrm{GL}(2n, \mathbb{C}) \,:\, \omega = g^*\omega\}.$$

Around 1886, Sophus Lie and Friedrich Engel were aware of their existence, but they had not yet appeared in print anywhere [Ha00, p. 152]. To his big surprise, in May 1887[3] Killing discovered a completely unknown complex simple Lie algebra of rank 2 and dimension 14, which is just the exceptional Lie algebra $\mathfrak{g}_2$. By October 1887, he had basically completed his classification. He discovered that besides $\mathfrak{g}_2$ and the three families mentioned above, there exist four additional exceptional simple Lie algebras. In modern notation, they are: $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, and they have dimensions 52, 78, 133, and 248 respectively.

There exist many real orthogonal Lie groups with complexification $\mathrm{SO}(n, \mathbb{C})$—namely all orthogonal groups $\mathrm{SO}(p, q)$ associated with scalar products with indefinite signature (p, q) such that $p + q = n$; they are called *real forms* of $\mathrm{SO}(n, \mathbb{C})$ (similarly for the Lie algebras), and it is an easy fact that $\mathrm{SO}(p, q)$ is compact only for $q = 0$. Just as well, the complex Lie algebra $\mathfrak{g}_2$ with complex Lie group G_2 has two real forms that we are going to denote by $\mathfrak{g}_2^c$ and $\mathfrak{g}_2^*$; of their (simply connected) Lie groups G_2^c and G_2^*, only the former is compact.

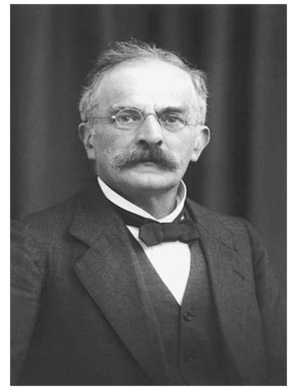
Without doubt, the classification of complex simple Lie algebras is one of the outstanding results of nineteenth century mathematics (this was not Killing's point of view, however: his original aim had been a classification of *all real Lie algebras*, he was unsatisfied with his own exposition and the incompleteness of results, so that he would not have published his results without strong encouragement from Friedrich Engel). Indeed, Killing's formidable work contains some gaps and mistakes:[4] in his thesis (1894), Élie Cartan gave a completely revised and polished presentation of the classification [Ca94], which has therefore become the standard reference for the result.

[2] *Letter from W. Killing to Fr. Engel, April 12, 1886; see [Ha00, p. 153].*

[3] *Letter from W. Killing to Fr. Engel, May 23, 1887; see [Ha00, p. 161].*

[4] *For example, he had found two exceptional Lie algebras of dimension 52 and overlooked that they are isomorphic, and his classification was based on three main theorems whose statements and proofs were partially wrong.*

Élie Cartan (1869–1951) in the year 1904.



Friedrich Engel (1861–1941) around 1922.

(Photo sources: Cartan, *Oeuvres Complètes, vol. II;* Engel, University Archive Greifswald.)

First Results on G_2

We can only conjecture how Killing and his contemporaries felt about the exceptional Lie algebras—as disturbances to the symmetry or as exotic and unique objects. But since Lie theory as a whole was developed in these times, they were investigated *too*, but not with high priority. G_2 was the first—and, for rather a long time, the only—Lie group for which further results were obtained. This is natural for dimensional considerations, but we will see later that it has also much deeper reasons.

From the weight lattice, which one obtains automatically during the classification, one can easily determine the lowest dimensional representation of any simple Lie algebra. This Cartan did in the last section of his thesis, and he rightly observed that $\mathfrak{g}_2$ admits a 7-dimensional complex representation, which furthermore possesses a symmetric nondegenerate $\mathfrak{g}_2$-invariant bilinear form [Ca94, p. 146]:

$$\beta := x_0^2 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

This scalar product has real coefficients, hence can be interpreted over the reals as well; in that case it has signature $(4, 3)$, and one can understand Cartan's result as giving a real representation of the noncompact form $\mathfrak{g}_2^*$ inside $\mathfrak{so}(4, 3)$.

At this stage, the question about explicit constructions of the exceptional Lie algebras becomes pressing. Élie Cartan and Friedrich Engel obtained the first breakthrough, published in two simultaneous notes to the Académie des Sciences de Paris [Ca93], [En93]: For every point $a \in \mathbb{C}^5$, consider the 2-plane π_a in the tangent space $T_a\mathbb{C}^5$ that is the zero set of the Pfaffian system

$$\begin{aligned} dx_3 &= x_1\, dx_2 - x_2\, dx_1, \\ dx_4 &= x_2\, dx_3 - x_3\, dx_2, \\ dx_5 &= x_3\, dx_1 - x_1\, dx_3. \end{aligned}$$

The 14 vector fields on $\mathbb{C}^5$ whose local flows map the planes π_a to each other satisfy the commutator relations of the Lie algebra $\mathfrak{g}_2$. Both authors then gave in the same papers a second geometric realization of $\mathfrak{g}_2$: Engel derived it from the first by a contact transformation, while Cartan identified $\mathfrak{g}_2$ as the symmetries of the solution space of the system of second order partial differential equations[5] ($f = f(x, y)$)

$$f_{xx} = \frac{4}{3}(f_{yy})^3, \ f_{xy} = (f_{yy})^2.$$

Both viewed their second realization as being different from the one through the Pfaffian system. Of course, this is correct: stated in modern terms, the complex Lie group G_2 has two non-conjugate 9-dimensional parabolic subgroups P_1 and P_2, and G_2 acts on the two compact homogeneous spaces $M_i^5 := G_2/P_i$, $i = 1, 2$. It is a detail that Engel and Cartan did not describe the $\mathfrak{g}_2$ action on the full spaces M_i^5, but rather on an open subset; this was the common way at that time.

Let us have a closer look at these two homogeneous spaces. For this, we need the lattice inside $\mathfrak{h}^*$ spanned by the 12 roots of $\mathfrak{g}_2$, the *root lattice*. It is the usual hexagonal planar lattice, in which the roots are denoted by arrows:

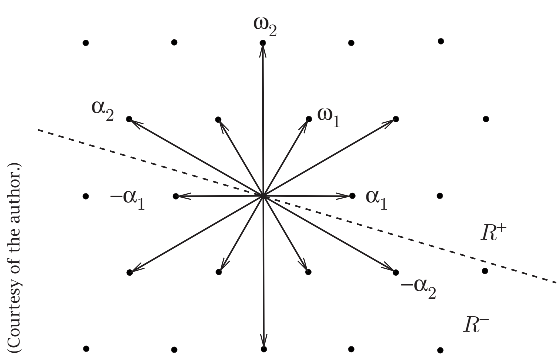


Figure 1. The $\mathfrak{g}_2$ root system and lattice.

The $\mathfrak{g}_2$ root system is the only one in which two roots include an angle of $\pi/6$, indicating its exceptional standing among all root systems. The roots above respectively below the dashed line are called *positive* respectively *negative roots*, and $R = R^+ \cup R^-$. The two positive roots marked α_1 and α_2 alone already generate the lattice; they are called *simple roots*. Keeping in mind the root system with the multiplication rule for root spaces $\mathfrak{g}_\alpha$ stated before, one sees that the direct sum of $\mathfrak{h}$ (corresponding, loosely speaking, to the origin), the six positive root spaces, and the root space of one negative of a simple root, span a subalgebra:

$$\mathfrak{p}_i := \mathfrak{h} \oplus \mathfrak{g}_{-\alpha_i} \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha.$$

Subalgebras of this kind are called *parabolic subalgebras*. The 9-dimensional groups P_1 and P_2 above are now exactly the subgroups of G_2 with Lie algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$. By general results, the space G_2/P_i is a compact homogeneous variety, and it can be realized in the projectivization of the representation space V_i with highest weight ω_i (see figure) as the G_2 orbit of some distinguished vector v_i; but ω_1 generates the 7-dimensional representation (spanned by the six short roots and zero with multiplicity one), while ω_2 is the highest weight of the adjoint representation (spanned by all roots and zero with multiplicity two). Hence, we obtain

$$\begin{aligned} M_1^5 = G_2/P_1 &= \overline{G_2 \cdot [v_1]} \subset \mathbb{P}(\mathbb{C}^7) = \mathbb{CP}^6, \\ M_2^5 = G_2/P_2 &= \overline{G_2 \cdot [v_2]} \subset \mathbb{P}(\mathfrak{g}_2) = \mathbb{CP}^{13}. \end{aligned}$$

The first space M_1^5 is thus a quadric in $\mathbb{CP}^6$; we will come back to the second space later.

Let us look again at the real situation. There are two real 9-dimensional subgroups P_i^* inside the noncompact real form G_2^* corresponding to the complex parabolic groups $P_i \subset G_2$; but they have no counterparts in the compact Lie group G_2^c (roughly speaking, $\mathfrak{g}_2^c \subset \mathfrak{so}(7)$ consists of skew-symmetric matrices, while parabolics are always upper triangular): a maximal subgroup of G_2^c is isomorphic to SU(3) and thus 8-dimensional. Hence a geometric realization of the compact form G_2^c is still missing.

G_2 and 3-forms in Seven Variables

In his talk on June 11, 1900, in Leipzig, Friedrich Engel presented some results on the complex Lie group G_2 that finally led to the missing realization of its compact form G_2^c. Engel's geometric insight into the geometry of $M_1^5 \subset \mathbb{CP}^6$ was so good that he realized that it can be written as the zero set of an equation depending solely on the coefficients of a generic 3-form in seven variables [En00, p. 220].

By a *generic p-form* we mean an element $\omega \in \Lambda^p(\mathbb{C}^n)^*$ with open $\mathrm{GL}(n, \mathbb{C})$ orbit. For dimensional reasons, $n^2 \geq \binom{n}{p}$ is a necessary condition for the existence of generic p-forms; it holds for all n if $p = 2$, but only for $n \leq 8$ when $p = 3$, and, indeed, generic p-forms do exist for these values. The isotropy group of a differential form (or, for that matter, any tensor) consists of all group elements leaving the form invariant,

$$G_\omega := \{A \in \mathrm{GL}(n, \mathbb{C}) \, : \, \omega = A^*\omega\}.$$

[5] *In 1910, Élie Cartan returned to his description of G_2^* by Pfaffian systems and differential equations* [Ca10]*; a modern treatment and further investigation can be found in the worthwhile article by P. Nurowski* [Nu05]. *It is quite remarkable that the thesis of yet another of Engel's students plays a decisive role here (Karl Wünschmann, Greifswald, 1905).*

For a generic 3-form in dimension 7, its dimension is

$$\dim G_\omega = \dim \mathrm{GL}(7,\mathbb{C}) - \dim \Lambda^3(\mathbb{C}^7)^* = 14.$$

Friedrich Engel observed that all generic 3-forms are equivalent under $\mathrm{GL}(7,\mathbb{C})$ and proved the following theorem:

Theorem 1 (F. Engel, 1900)**.** *There exists exactly one* $\mathrm{GL}(7,\mathbb{C})$ *orbit of generic complex* 3*-forms* [En00, p. 74]. *One such generic form is given by*

$$\omega_0 := (e_1e_4 + e_2e_5 + e_3e_6)e_7 - 2e_1e_2e_3 + 2e_4e_5e_6.$$

For every generic complex 3*-form* $\omega \in \Lambda^3(\mathbb{C}^7)^*$*, the following holds:*

(1) *The isotropy group of* ω *is isomorphic to the simple Lie group* G_2 [En00, p. 73]*;*

(2) ω *defines a non-degenerate symmetric bilinearform* β_ω [En00, p. 222] *that is cubic in the coefficients of* ω*, and the quadric* M_1^5 *is its isotropic cone in* $\mathbb{CP}^6$*. In particular, every isotropy group* G_ω *is contained in some* $\mathrm{SO}(7,\mathbb{C})$*.*

(3) *There exists a* G_2*-invariant polynomial* $\lambda_\omega \neq 0$ *of degree* 7 *in the coefficients of* ω [En00, p. 231].

In fact, Engel had already conjectured in a letter to Killing of April 1886 that the isotropy group of a 3-form might be a simple 14-dimensional group, but apparently neither he nor Killing had pursued this idea at that time.[6] In modern notation, we can define β_ω through [Br87]

$$\beta_\omega(X,Y) := (X \lrcorner\, \omega) \wedge (Y \lrcorner\, \omega) \wedge \omega,$$

which is a symmetric bilinear form with values in the one-dimensional vector space $\Lambda^7(\mathbb{C}^7)^*$. Over the reals, it can be turned into a true real-valued scalar product g_ω after taking an additional square root [Br87], [Hi00]. Geometrically, this means that every generic 3-form on a real 7-dimensional manifold induces a (pseudo)-Riemannian metric. An easy dimension count shows that the isotropy group of a generic 3-form can be a subset of $\mathrm{SO}(n,\mathbb{C})$ only for $n = 7, 8$.

Since β_ω is cubic in ω, its determinant is a polynomial of degree 21 in the ω coefficients; Engel understood that it is the third power of a degree 7 element λ_ω, and its non-vanishing is equivalent to the nondegeneracy of β_ω.

Engel's arguments still hold over the reals, as long as the isotropic cone does not degenerate completely. For the 3-form ω_0 cited above, g_{ω_0} is a real scalar product on $\mathbb{R}^7$ with signature $(4,3)$ [En00, p. 64]. In particular, there exists exactly one $\mathrm{GL}(7,\mathbb{R})$ orbit of real generic 3-forms $\omega \in \Lambda^3(\mathbb{R}^7)^*$ with non-degenerate isotropic cone for g_ω. Its isotropy group is again isomorphic to the real noncompact real form $G_2^* \subset \mathrm{SO}(4,3)$.

[6] *Letter from Fr. Engel to W. Killing, April 8, 1886; see [Ha00, p. 152].*

In the same article, Engel invested a lot of energy in a description of the second homogeneous space $M_2^5 \subset \mathbb{P}(\mathfrak{g}_2)$ through the coefficients of ω. For this, he used a symbolic method for invariants of alternating forms that was communicated to him by Eduard Study; however, Study's formalism is not in use anymore, hence his computations are rather hard to follow. Today, we know that $M_2^5 = G_2/P_2 \subset \mathbb{CP}^{13}$ is a rather complicated projective algebraic variety: it has degree 18 and its complete intersection with three hyperplanes is a K3 surface of genus 10 [Bor83]. For a geometric description of G_2/P_2 in terms of ω, observe that the 21-dimensional representation $\Lambda^2\mathbb{C}^7$ splits under G_2 into $\mathfrak{g}_2 \oplus \mathbb{C}^7$, hence G_2/P_2 is a subvariety of $\mathbb{P}(\Lambda^2\mathbb{C}^7)$ as well. By the Plücker embedding, the 14-dimensional Grassmann variety $G(2,7)$ of 2-planes in $\mathbb{C}^7$ lies in $\mathbb{P}(\Lambda^2\mathbb{C}^7)$. Now, G_2/P_2 is just the intersection of $G(2,7)$ with $\mathbb{P}(\mathfrak{g}_2)$ inside $\mathbb{P}(\Lambda^2\mathbb{C}^7)$. As a subvariety of $G(2,7)$, G_2/P_2 consists precisely of those 2-planes $\pi \subset \mathbb{C}^7$ on which β_ω and ω both degenerate (see [LM03] for a modern account), i. e., such that

$$\pi \lrcorner\, \beta_\omega = 0, \quad \pi \lrcorner\, \omega = 0.$$

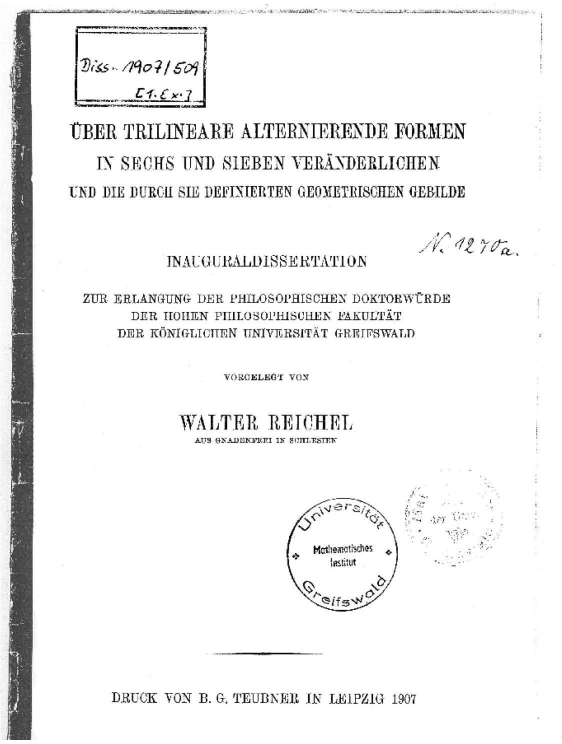
It was the realization of G_2 presented in Theorem 1 that led Friedrich Engel to the comment cited in the introduction. Besides its elegance, Theorem 1 has far-reaching consequences for modern differential geometry (see last section); furthermore, it will provide the missing realization of G_2^c, as is explained now.

Walter Reichel and the Invariants of G_2

While a professor at Greifswald University (1904-1913), Friedrich Engel turned again to this topic and assigned to his Ph. D. student Walter Reichel the task of computing a complete system of invariants for complex 3-forms in six and seven dimensions in Study's formalism. The thesis defended by Reichel in 1907 indeed contains the detailed description of the invariants and relations among them as well as normal forms of all 3-forms under the action of $\mathrm{GL}(7,\mathbb{C})$. The vanishing of λ_ω for non-generic forms and the drop of rank of the bilinear form β_ω play an essential role here.

Furthermore, Walter Reichel described the isotropy algebra $\mathfrak{g}_\omega$ of *any* generic 3-form ω directly through its coefficients [Rei07, p. 48], whereas Friedrich Engel had only computed it for one representative. Over the complex numbers, this makes no difference; but it turns out that when passing to real numbers, the *one* orbit of complex, $\mathrm{GL}(7,\mathbb{C})$ equivalent 3-forms splits into *two* orbits of real generic $\mathrm{GL}(7,\mathbb{R})$ equivalent 3-forms. If one interprets the scalar product β_ω as a real one, it turns out to have signature $(4,3)$ on one orbit and signature $(7,0)$ on the other orbit. It does not come as a surprise that the isotropy

(Courtesy University Archive Greifswald.)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IN SECHS UND SIEBEN VERÄNDERLICHEN
UND DIE DURCH SIE DEFINIERTEN GEOMETRISCHEN GEBILDE

N. 1270a.

INAUGURALDISSERTATION

ZUR ERLANGUNG DER PHILOSOPHISCHEN DOKTORWÜRDE
DER HOHEN PHILOSOPHISCHEN FAKULTÄT
DER KÖNIGLICHEN UNIVERSITÄT GREIFSWALD

VORGELEGT VON

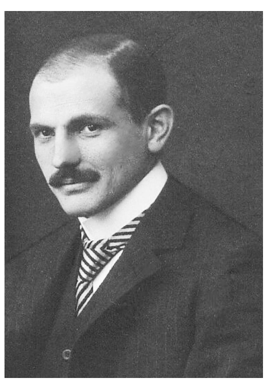
WALTER REICHEL
AUS GNADENFREI IN SCHLESIEN

Universität Mathematisches Institut Greifswald

DRUCK VON B. G. TEUBNER IN LEIPZIG 1907

Figure 2. Title page of W. Reichel's thesis.

(Photos courtesy of Irmtraut (née Reichel) Schiller, Bremen.)



Walter Reichel (1883–1918) in November 1914.

W. Reichel (undated); one can recognize a portrait of Kant on the desk.

group $\mathfrak{g}_\omega$ is isomorphic to $\mathfrak{g}_2^* \subset \mathfrak{so}(4,3)$ in the first case and isomorphic to $\mathfrak{g}_2^c \subset \mathfrak{so}(7)$ in the second case.

Thus, Reichel obtained a uniform geometric description of both real forms of G_2. Unfortunately, the result was slowly forgotten afterwards; J. A. Schouten described the normal forms of 3-forms on $\mathbb{C}^7$ in 1931 by simpler methods (without invariants, mainly by reduction to smaller dimensions) and observed that Walter Reichel had missed two out of the nine normal forms [Sch31]. Based on these results, Gurevich solved the problem in dimension 8 [Gu35]. Up to our knowledge, the next authors to cite Reichel's thesis again are E. B. Vinberg and A. G. Elashvili in 1978, who worked out the details of the extremely involved case $n = 9$ [VE78].

It is well known that G_2^c is the automorphism group of the octonians $\mathbb{O}$. Élie Cartan included this as a comment in his long article on complex numbers and their generalizations from 1908 [Ca08, p. 467] (see also [Ca14, p. 298]), but apparently never returned to this topic. This approach to exceptional Lie groups became popular through the work of Hans Freudenthal, starting with the article [Fr51], and made the memory of the 3-form approach vanish. In fact, these descriptions are equivalent (a third equivalent description is through so-called "vector cross-products"), as is explained with great care in the article by J. Baez [Ba02, p. 37-39].

The Mathematician Walter Reichel

Whereas the life and work of all mathematicians mentioned up to here are well known, virtually nothing was known about Walter Reichel, despite the fact that his thesis has been cited widely in recent years. The 100th anniversary of his thesis last year was a further motivation to investigate his story.

Walter Reichel was born on November 3, 1883, in a little Silesian village then called Gnadenfrei (now Piława Górna, Poland). This village had been founded by members of the Moravian Unity, of which Reichel's father was deacon and, later, bishop. The Moravian Unity, or Unitas Fratrum (Unity of Brethren), emerged in the middle of the fifteenth century from the Bohemian Reformation Movement around Jan Hus (1369–1415) and was renewed in the early eighteenth century in Herrnhut (Saxony, not far from the Czech and Polish borders), where the management of its European branch and its archive are still hosted today. The history of the Reichel family is closely linked to the *Brüdergemeine*, as the Moravian Unity is called in German.

In his handwritten CV (which can be found in his Ph. D. files at Greifswald University), Walter Reichel describes how he went to school first in his home village, followed by four years at the "Pädagogium" in Niesky (another town founded by the Unitas Fratrum, close to Herrnhut) and three years at the Gymnasium in Schweidnitz (now Świdnica, Poland), from where he received his high school degree ("Reifezeugnis") at Easter 1902. He then studied mathematics, physics, and philosophy at the Universities of Greifswald, then Leipzig, Halle, and again, Greifswald.

Among others, he attended lectures by Friedrich Engel and Theodor Vahlen (in Greifswald); by Carl

The "Old Pädagogium" in Niesky, now a public library, built in 1741 as the first parish house of the newly founded community in Niesky. Between 1760 and 1945 it was used as an advanced boarding school.

Neumann (in Leipzig), who formulated the Neumann boundary condition in analysis and founded the *Mathematische Annalen* together with Alfred Clebsch; by Georg Cantor and Felix Bernstein (in Halle), to whom we owe the foundations of set theory and the Cantor-Bernstein-Schröder Theorem in logic; by the theoretical physicist Gustav Mie (in Greifswald), who made important contributions to electromagnetism and general relativity; by the experimental physicist Friedrich Ernst Dorn (in Halle), who discovered the gas Radon in 1900. Moroever, he took courses in philosophy, chemistry, zoology, and art history.

In July 1907, Walter Reichel passed the examination for high school teachers in "pure and applied mathematics, physics and philosophical propaedeutics" with distinction. He spent one year as teacher-in-training in Görlitz, and was then appointed in Fall 1908 at the "Realprogymnasium" in Sprottau (now Szprotawa, Poland). In April 1914 he moved for an "Oberlehrer" position to Schweidnitz (now Świdnica, Poland). With the beginning of the First World War he was drafted, and he died in France on March 30, 1918. He has no grave, but an inscription on the WWI memorial on the "God's acre" of the Moravian community in Niesky commemorates his death.

Walter Reichel married Gertrud, née Müller (1889–1956) in 1909. They had three sons (born 1910, 1913, and 1916), who left no children, and a daughter (born March 11, 1918). After the first World War, Reichel's widow moved with her children to Niesky, where she was supported by the Moravian Unity. For many years, she accommodated pupils of the "Pädagogium" who did not live in the boarding school's dormitories. Walter Reichel's daughter Irmtraut Schiller now lives in Bremen and

(Photos this page courtesy of I. Agricola and T. Friedrich.)

Memorial stone on the "God's acre" in Niesky.

Detail of the inscription on the memorial stone. Walter Reichel's name and date of death ("30.3.18") are in the second to last line; the stone has been damaged and repaired above his name.

has three children; one of her granddaughters is a teacher of mathematics.

G_2 Geometry in Dimension 7

The classical symmetry approach to differential geometry is based on the notion of the *isometry group* of a Riemannian manifold, i.e., the group of all transformations acting on the manifold that preserve the metric. In the twentieth century, the concept of *(Riemannian) holonomy group* became fundamental to Riemannian geometry: it is the group generated by all parallel transports along closed null-homotopic loops on the manifold. Here, parallel transport is understood with respect to the Levi-Civita connection

∇^g, i. e., the straightforward—but not the only possible—generalization of the directional derivative of vectors. Marcel Berger's Holonomy Theorem from 1955 states that for an irreducible non-symmetric manifold, the holonomy group is either $SO(n)$ or from a finite list—and G_2^c is the only exceptional Lie group on that list. In this case, the manifold is necessarily 7-dimensional, the holonomy group G_2^c acts on the tangent bundle by its 7-dimensional real representation, and the manifold would be called a *parallel* or *integrable* G_2^c*-manifold*. The argument of the proof is basically reduced to the question of which compact Lie groups admit transitive sphere actions, and this is the case for G_2^c on S^6, thought of not as a symmetric space but rather as the homogeneous space $G_2^c/SU(3)$.

Since Berger's Holonomy Theorem lists only the *possible* holonomy groups without actually constructing manifolds admitting them, his classification result was not an end point, but rather a research program asking for a more detailed geometric investigation (and a long breath); in particular, no Riemannian manifolds with holonomy group G_2^c were known. In 1966, Edmond Bonan observed in a note preceding his thesis (supervised by André Lichnerowicz) that manifolds with G_2^c holonomy admit a global 3-form ω with $\nabla^g\omega = 0$, and, in consequence, have to be Ricci flat—a very restrictive geometric condition that intrigued some contemporaries [Bon66]. This 3-form ω is of course precisely the form whose stabilizer is G_2^c as described by Engel and Reichel, but had already been forgotten in this time. For his work, Edmond Bonan had started from Cartan's description of G_2^c as the automorphism group of the octonians, and saw how to derive an invariant 3-form from its multiplication law.

The credit for having made the most creative use of G_2^c and its defining 3-form in differential geometry goes without doubt to Alfred Gray. Since the 1960s he had investigated vector cross products and their geometric properties in a series of papers. In 1971 he had the radical idea of weakening the classical holonomy concept to cover interesting manifolds that do not appear in Berger's list. In particular, he defined *nearly parallel* G_2^c*-manifolds*: they have structure group G_2^c, but instead of being parallel, the 3-form ω satisfies the differential equation

$$d\omega = \lambda * \omega$$

for a real constant $\lambda \neq 0$, and they are Einstein with strictly positive scalar curvature. Later, he proved together with Marisa Fernández that there are in fact four basic classes of G_2^c-manifolds depending on the possible nature of the tensor $\nabla^g\omega$ [FG82]. Maybe even more importantly, they initiated the (still-ongoing) construction of many interesting examples—ranging from $S^7 = Spin(7)/G_2^c$ to the Allow-Wallach spheres $SU(3)/S^1$, from extensions of Heisenberg groups to clever non-homogeneous examples. Since then, these *non-integrable geometries* (not only for G_2^c, but also for contact structures, almost Hermitian structures, 8-dimensional Spin(7)-structures, etc.) have been studied intensively [Ag06]. Today, the main philosophy is that for many Riemannian geometries (M, g) defined by tensors that are not ∇^g-parallel, it is possible to replace the Levi-Civita connection by a more suitable metric connection ∇ with skew-symmetric torsion $T \in \Lambda^3(M)$ (the *characteristic connection*)

$$g(\nabla_X Y, Z) := g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z)$$

such that the object becomes parallel, and the holonomy group of this new connection plays the role of the classical Riemannian holonomy group. For example, a G_2^c-manifold (M, g, ω) admits a characteristic connection if and only if there is a vector field β such that $\delta(\omega) = -\beta \lrcorner \omega$ (this excludes one of the four basic types), and its torsion is then given by [FI02]

$$T = - * d\omega - \frac{1}{6}(d\omega, *\omega)\, \omega + *(\beta \wedge \omega).$$

For integrable G_2^c geometries, the first breakthrough was obtained a few years after Gray's work. In 1987 and 1989, Robert Bryant and Simon Salamon succeeded in constructing local complete metrics with Riemannian holonomy G_2^c ([Br87], [BrSa89]). It was only in 1996—more than forty years after Berger's original paper—that Dominic Joyce was able to show the existence of *compact* Riemannian 7-manifolds with Riemannian holonomy G_2^c [Joy00]: this comes down to proving the existence of solutions of nonlinear elliptic partial differential equations on compact manifolds by very difficult and involved analytical methods.

One distinguished property of G_2^c is still missing in our discussion—this is the one that makes it attractive for mathematical physics. The group G_2^c can be lifted to the universal covering Spin(7) of SO(7), and Spin(7) has an 8-dimensional irreducible real representation Δ_7, the *spin representation*, that decomposes under $G_2^c \subset Spin(7)$ into the trivial and the 7-dimensional representation. Thus, a 7-dimensional spin manifold endowed with a connection ∇ has a ∇-parallel spinor field if and only if the holonomy of ∇ lies inside G_2^c, and G_2^c is the isotropy group of a generic spinor. In fact, any spinor field defines a global 3-form and vice versa, so this last characterization of G_2^c is, in itself, nothing new. But it explains the intricate relation between G_2^c and spin geometry. For example, the nearly parallel G_2^c-manifolds discovered by Gray in 1971 are precisely those 7-manifolds that admit a real Killing spinor field [FK90]. More recently, superstring theory has stimulated a deep interest in 7-manifolds with integrable or non-integrable G_2^c geometry [Du02]. In this approach, a ∇-parallel spinor field is interpreted as a supersymmetry transformation: by tensoring with a spinor field, bosonic particles can be transformed into fermionic particles (and vice versa). The torsion of ∇ (if present) is related to the B-field, a higher order version of the classical field strength of Yang-Mills theory (which would be a 2-form and not a 3-form, of course).

The story of G_2 and its relatives is far from concluded. The algebraic foundations for the many lines of development sketched in this article were laid more than a hundred years ago by Friedrich Engel and his student Walter Reichel in a work of remarkable mathematical insight.

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References

[Ag06] I. Agricola, The Srní lectures on non-integrable geometries with torsion, *Arch. Math.* **42** (2006), 5–84. With an appendix by M. Kassuba.

[Ba02] John C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002), 145–205.

[Bon66] E. Bonan, Sur les variétés riemanniennes à groupe d'holonomie G_2 ou Spin(7), *C. R. Acad. Sc. Paris* **262** (1966), 127–129.

[Bor83] C. Borcea, Smooth global complete intersections in certain compact homogeneous complex manifolds, *J. Reine Angew. Math.* **344** (1983), 65–70.

[Br87] R. Bryant, Metrics with exceptional holonomy, *Ann. of Math.* **126** (1987), 525–576.

[BrSa89] R. Bryant and S. Salamon, On the construction of some complete metrics with exceptional holonomy, *Duke Math. J.* **58** (1989), 829–850.

[Ca93] É. Cartan, Sur la structure des groupes simples finis et continus, *C. R. Acad. Sc.* **116** (1893), 784–786.

[Ca94] ———, *Sur la structure des groupes de transformations finis et continus*, thèse, Paris, Nony, 1894.

[Ca08] ———, *Nombres Complexes*, Encyclop. Sc. Math., édition française, **15**, 1908, 329–468, d'après l'article allemand de E. Study.

[Ca10] ———, Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, *Ann. Éc. Norm.* **27** (1910), 109–192.

[Ca14] ———, Les groupes réels simples finis et continus, *Ann. Éc. Norm.* **31** (1914), 263–355.

[Du02] M. J. Duff, *M-theory on manifolds of G_2-holonomy: The first twenty years*, hep-th/0201062.

[En93] F. Engel, Sur un groupe simple à quatorze paramètres, *C. R. Acad. Sc.* **116** (1893), 786–788.

[En00] ———, Ein neues, dem linearen Komplexe analoges Gebilde, *Leipz. Ber.* **52** (1900), 63–76, 220–239.

[FG82] M. Fernández and A. Gray, Riemannian manifolds with structure group G_2, *Ann. Mat. Pura Appl.* **132** (1982), 19–45.

[Fr51] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Utrecht: Mathematisch Institut der Rijksuniversiteit (1951), 52 S.

[FI02] T. Friedrich and S. Ivanov, Parallel spinors and connections with skew-symmetric torsion in string theory, *Asian Jour. Math.* **6** (2002), 303–336.

[FK90] T. Friedrich and I. Kath, 7-dimensional compact Riemannian manifolds with Killing spinors, *Comm. Math. Phys.* **133** (1990), 543–561.

[Gra71] A. Gray, Weak holonomy groups, *Math. Z.* **123** (1971), 290–300.

[Gu35] G. B. Gurevich, Classification des trivecteurs ayant le rang huit, *C. R. Acad. Sci. URSS* **2** (1935), 353–356.

[Ha00] T. Hawkins, *Emergence of the theory of Lie groups*, Springer Verlag, 2000.

[Hi00] N. Hitchin, The geometry of three-forms in six and seven dimensions, *J. Diff. Geom.* **55** (2000), 547–576.

[Joy00] D. Joyce, *Compact manifolds with special holonomy*, Oxford Science Publ., 2000.

[Kil89] W. Killing, Die Zusammensetzung der stetigen endlichen Transformationsgruppen II, *Math. Ann.* **33** (1889), 1–48.

[LM03] J. Landsberg and L. Manivel, On the projective geometry of rational homogeneous varieties, *Comment. Math. Helv.* **78** (2003), 65–100.

[Nu05] P. Nurowski, Differential equations and conformal structures, *Jour. Geom. Phys.* **55** (2005), 19–49.

[Rei07] W. Reichel, *Über trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde*, Druck von B.G. Teubner in Leipzig 1907, Dissertation an der Universität Greifswald.

[Sch31] J. A. Schouten, Klassifizierung der alternierenden Größen dritten Grades in 7 Dimensionen, *Rend. Circ. Mat. Palermo* **55** (1931), 137–156.

[VE78] E. B. Vinberg and A. G. Elashvili, Classification of trivectors of a 9-dimensional space, *Sel. Math. Sov.* **7** (1988), 63–98. Translated from Tr. Semin. Vektorn. Tensorm. Anal. Prilozh. Geom. Mekh. Fiz. **18** (1978), 197–233.