SOLVMANIFOLDS WITH INTEGRABLE AND NON-INTEGRABLE G_2 STRUCTURES

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Abstract. We show that a 7-dimensional non-compact Ricci-flat Riemannian manifold with Riemannian holonomy G_2 can admit non-integrable G_2 structures of type $\mathbb{R}\oplus\mathcal{S}^2_0(\mathbb{R}^7)\oplus\mathbb{R}^7$ in the sense of Fernández and Gray. This relies on the construction of some G_2 solvmanifolds, whose Levi-Civita connection is known to give a parallel spinor, admitting a 2-parameter family of metric connections with non-zero skew-symmetric torsion that has parallel spinors as well. The family turns out to be a deformation of the Levi-Civita connection. This is in contrast with the case of compact scalar-flat Riemannian spin manifolds, where any metric connection with closed torsion admitting parallel spinors has to be torsion-free.

1. Introduction

The study and explicit construction of Riemannian metrics with holonomy G_2 on non-compact manifolds of dimension seven (called metrics with *parallel* or *integrable* G_2 structure) has been an exciting area of differential geometry since the pioneering work of Bryant and Salamon in the second half of the eighties (cf. [Br87], [BrS89] and [Sa89]). Mathematical elegance aside, these metrics have turned out to be an important tool in superstring theory, since they are exact solutions of the common sector of type II string equations with vanishing B field.

Independently of this development, the past years have shown that non-integrable geometric structures such as almost hermitian manifolds, contact structures or non-integrable G_2 and $Spin(7)$ structures can be treated successfully with the powerful machinery of metric connections with skew-symmetric torsion (see for example [FrIv02], [Agr03], [AgFr04] and the literature cited therein). In physical applications, this torsion is identified with a non-vanishing B field ([Str86], [GMW03] and many more). The interaction between these research lines was up to now limited to "cone-type arguments", i. e. a non-integrable structure on some manifold was used to construct an integrable structure on a higher dimensional manifold (like its cone, an so on). A natural question is thus whether the same Riemannian manifold (M, g) can carry structures of both type *simultaneously.* This appears to be a remarkable property. For example the projective space $\mathbb{CP}^3$ with the well-known Kähler-Einstein structure and the nearly Kähler one inherited from triality does not satisfy this requirement. The metric underlying the nearly-Kähler structure is not the Fubini–Study one in fact, cf. [ES85] and also [BFGK91].

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A spinor which is parallel with respect to a metric connection ∇ (with or without torsion) forces its holonomy to be a subgroup of the stabiliser of an algebraic spinor, and this is precisely G_2 in dimension 7. In this particular dimension furthermore, the converse statement also holds. The problem can therefore be reformulated as follows:

Question. *Are there 7-dimensional Riemannian manifolds with a parallel spinor for the Levi-Civita connection (rendering them Ricci-flat, in particular) also admitting a covariantly constant spinor for some other metric connection with skew-symmetric torsion? If yes, can the torsion connection be deformed into the Levi-Civita connection in such a way of preserving the parallel spinor?*

From the high energy physics' point of view a parallel spinor is interpreted as a supersymmetry transformation. Hence the physical problem behind the Question (which in fact motivated our investigations) is really whether a free "vacuum solution" can also carry a non-vacuum supersymmetry, and how the two are related.

The case of a compact Riemannian manifold was treated in [AgFr04]. There, as a main application of the "rescaled Schrödinger-Lichnerowicz formula", one showed a rigidity theorem for compact manifolds of non-positive scalar curvature. More precisely,

Theorem. *Suppose (M^n, g, T) is a compact, Riemannian spin manifold of non-positive scalar curvature, $\mathrm{Scal}^g \leqslant 0$, and the 4-form dT acts on spinors as a non-positive endomorphism. Then if there exists a solution $\psi \neq 0$ of the equation*

$$\nabla^T_X \psi \ := \ \nabla^g_X \psi + (X \lrcorner\, T) \cdot \psi \ = \ 0 \ ,$$

then the 3-form and the scalar curvature vanish, $T = 0 = \mathrm{Scal}^g$, and ψ is parallel with respect to the Levi-Civita connection.

This applies, in particular, to Calabi-Yau and Joyce manifolds. These are compact, Riemannian Ricci-flat manifolds of dimension $n = 6, 7, 8$ with (at least) one LC-parallel spinor field; under mild assumptions on the derivative of the torsion form T, they do not admit parallel spinors for any metric connection with $T \neq 0$. Since these manifolds have not been realized in any geometrically explicit way so far, harmonic or closed forms are the natural candidates to be torsion forms on them.

The present paper deals with the non-compact case. Gibbons et al. produced non-complete metrics with holonomy G_2 in [GLPS02]. Those metrics have the interesting feature, among others, of admitting a 2-step nilpotent isometry group N acting on orbits of codimension one. By [ChF05] such metrics are locally conformal to homogeneous metrics on rank-one solvable extension of N, and the induced $SU(3)$ structure on N is half-flat. In the same paper all half-flat $SU(3)$ structures on 6-dimensional nilpotent Lie groups whose rank-one solvable extension is endowed with a conformally parallel G_2 structure were classified. There are exactly six instances, which we considered in relation to the problem posed. It turns out that one of these manifolds provides a positive answer to both questions (Theorem 4.1), hence becoming the most interesting. The wealth of parallel spinors this manifold admits is organised into a continuous family parametrised by the real projective line, plus a bunch of 'isolated' instances. To achieve this we proved a sort of 'reduction' result that allows to assume the spinors have an extremely simple block form (Theorem 3.1). The Lie algebra associated to this solvmanifold has non-vanishing Lie brackets

$$\begin{aligned}
&[e_i, e_7] = -\tfrac{3}{5} m e_i, \ i = 1, 2, 5, \\
&[e_j, e_7] = -\tfrac{6}{5} m e_j, \ j = 3, 4, 6, \\
&[e_1, e_5] = \tfrac{2}{5} m e_3, \ [e_2, e_5] = \tfrac{2}{5} m e_4, \ [e_1, e_2] = \tfrac{2}{5} m e_6.
\end{aligned}$$

The homogeneous metric it bears can be also seen as a G_2 metric on the product $\mathbb{R}\times\mathbb{T}$, where $\mathbb{T}$ is the total space of a T^3-bundle over another 3-torus.

Four metrics of the six only carry integrable G_2 structures (Theorem 5.1), thus reproducing the pattern of the compact situation, whilst the remaining one (example (4)) is singled out by complex solutions, a proper interpretation for which is still lacking (Theorem 6.1). Nevertheless, all the G_2-metrics generated by these examples have a physical relevance [GLPS02, LT05].

2. General set-up

The starting point of the present analysis is the classification of conformally parallel G_2-manifolds on solvable Lie groups of [ChF05], whose results we briefly summarise. We shall adopt a similar notation, except that the 3-forms $\psi^{\pm}$ have become $\eta^{\pm}$, the conformal constant m has changed sign to $-m$, merely for aesthetic reasons, and the extension coefficients are now denoted by capital C's. We shall also not distinguish between vectors and covectors.

2.1. **Round-up on G_2 solvable extensions.** Consider a six-dimensional nilpotent Lie group N with Lie algebra $\mathfrak{n}$ endowed with an invariant $SU(3)$ structure (ω,η^+), i.e. non-degenerate 2- and 3-forms with stabilisers $Sp(6,\mathbb{R})$ and $SL(3,\mathbb{C})$ respectively. These define a Riemannian metric with orthonormal basis $e_1,\dots,e_6$ and an orthogonal almost complex structure J. Recall that $\mathrm{ad}_U(V)=[U,V]$ gives the adjoint representation of a Lie algebra $\mathfrak{g}$. Pick the rank-one metric solvable extension $\mathfrak{s}:=\mathfrak{n}\oplus\mathbb{R}e_7$, with $e_7\perp\mathfrak{n}$ a unit element, defined by ad_{e_7} as non-singular self-adjoint derivation. The Lie bracket and inner product on $\mathfrak{s}$ are, when restricted to ${e_7}^{\perp}$, precisely those of $\mathfrak{n}$.

One is actually entitled to assume that there exists a unitary basis $(e_1,\dots,e_6)$ on $\mathfrak{n}$ consisting of eigenvectors of the derivation ad_{e_7} with non-zero real eigenvalues $C_1,\dots,C_6$. In addition, all eigenvalues are positive integers without common divisor, up to a rescaling of e_7 [H98, Wi03]. Relatively to this basis of $\mathfrak{n}$, the hermitian geometry of N is prescribed by

$$\omega \;=\; e_{14}-e_{23}+e_{56},\qquad \eta^+ + i\eta^- = (e_1+ie_4)\wedge(e_2-ie_3)\wedge(e_5+ie_6),$$

The (non-integrable) G_2 structure inducing g

$$\varphi \;:=\; \omega\wedge e_7+\eta^+ \;=\; e_{147}-e_{237}+e_{567}+e_{125}+e_{136}+e_{246}-e_{345}$$

on the solvable Lie group S corresponding to $\mathfrak{s}$ is conformally parallel if and only if $\mathfrak{n}$ is isomorphic to one of the following:

(1) $(0,0,e_{15},0,0,0)$,
(2) $(0,0,e_{15},e_{25},0,e_{12})$,
(3) $(0,0,e_{15}-e_{46},0,0,0)$,
(4) $(0,e_{45},-e_{15}-e_{46},0,0,0)$,
(5) $(0,e_{45},e_{46},0,0,0)$,
(6) $(0,e_{16}+e_{45},e_{15}-e_{46},0,0,0)$.

The notation for Lie algebras is the usual differential one: in (2) for instance, e_{15} means $e_1\wedge e_5$ and the only non-vanishing Lie brackets on $\mathfrak{n}$ are $[e_1,e_5]=-e_3$, $[e_2,e_5]=-e_4$, $[e_1,e_2]=-e_6$. Throughout this article, the numeration shall respect the previous list.

So the central issue here is the interplay of:

(i) the 6-dimensional manifold (N,ω,η^+);

(ii) the geometry of S associated to the metric g conformal to a parallel one $\tilde{g}$;
(iii) the Ricci-flat metric $\tilde{g}$ on S obtained by conformal change.

We are mainly interested in the last structure, that is to say in the incomplete metric $\tilde{g}$ with Riemannian holonomy contained in G_2. We will show that in certain cases $\tilde{g}$ is induced by another G_2 structure, whose kind we describe. This helps to explain how this non-integrable reduction is related to an integrable G_2 structure.

As a matter of fact, this is the expression for the integrable G_2 structure on $(S, \tilde{g})$ with respect to its (new) orthonormal basis as well. It is known that φ defines a $\nabla^{\tilde{g}}$-parallel spinor Ψ by

$$\varphi(X,Y,Z) \;=\; \tfrac{1}{4}\langle X\cdot Y\cdot Z\cdot\Psi,\Psi\rangle, \tag{2.1}$$

where dots denote Clifford multiplication and $\langle\,,\rangle$ is the scalar product in the spinor bundle. The constant $1/4$ is arbitrary.

In terms of the seven-dimensional spin representation Δ_7 used in [AgFr04] (explicitly given in Section 3), the spinor Ψ of (2.1) has components

$$\Psi \;=\; (0,0,0,0,1,1,-1,1). \tag{2.2}$$

Since Δ_7 is the complexification of a real representation, we assume all spinors to be real, unless stated otherwise.

2.2. **Classification of G_2 structures.** The various G_2-properties of 7-manifolds (S,φ) can be studied using the approach of Fernández and Gray [FG82], i.e. describing algebraically the four irreducible G_2-representations $\mathcal{T}_i$ of the intrinsic torsion space

$$T^*S\otimes\mathfrak{g}_2^{\perp} = \bigoplus_{i=1}^{4}\mathcal{T}_i \cong \mathbb{R}\oplus\mathfrak{g}_2\oplus\mathcal{S}^2_0\mathbb{R}^7\oplus\mathbb{R}^7. \tag{2.3}$$

The first summand is merely spanned by φ, the second denotes the adjoint representation of G_2, whilst the third is the space of symmetric tensors on $\mathbb{R}^7$ with no trace. The corresponding components $(\tau_1,\tau_2,\tau_3,\tau_4)$ of the intrinsic torsion are uniquely defined differential forms such that

$$d\varphi \;=\; \tau_1{*\varphi} + 3\,\tau_4\wedge\varphi + {*\tau_3}, \qquad \delta\varphi = -4{*}(\tau_4\wedge{*\varphi}) + {*}(\tau_2\wedge\varphi), \tag{2.4}$$

with $\delta = -{*d*}$ the codifferential of forms, see [Br03]. For instance τ_1 and the Lee form τ_4 are given by

$$\tau_1 = g(d\varphi,{*\varphi})/7 \quad\text{and}\quad \tau_4 = -{*}({*d\varphi}\wedge\varphi)/12.$$

Is is moreover known that $\tau_2 = 0$ is equivalent to the existence of an affine connection $\tilde{\nabla}$ with skew-symmetric torsion such that $\tilde{\nabla}\varphi = 0$ [FrIv02].

What we mean by the ubiquitous and often abused terms *integrable* (or *parallel*) and *non-integrable* is

φ is an integrable G_2 structure $\iff$ $\tau_i = 0$ for $i = 1,2,3,4$.

φ is non-integrable $\iff$ one of the τ_i's at least survives, in which case the type of φ is described by the non-zero summands in (2.3).

This terminology is consistent with the landscape of general geometric structures described in [Fr02].

For example, a cosymplectic G_2 structure φ is characterised by the equation $d{*\varphi} = 0$, so it is non-integrable and has type $\mathcal{T}_1\oplus\mathcal{T}_3\cong\mathbb{R}\oplus\mathcal{S}^2_0\mathbb{R}^7$. The G_2 structure of the previous page instead has type $\mathcal{T}_4$, as all τ_i's are zero except $\tau_4 = me_7$.

Example	$\mathfrak{n}$ isomorphic to	eigenvalue type of ad_{e_7}
(1)	$(0, 0, e_{15}, 0, 0, 0)$	$(2m/3, m, 4m/3, m, 2m/3, m)$
(2)	$(0, 0, e_{15}, e_{25}, 0, e_{12})$	$(3m/5, 3m/5, 6m/5, 6m/5, 3m/5, 6m/5)$
(3)	$(0, 0, e_{15} - e_{46}, 0, 0, 0)$	$(3m/4, m, 3m/2, 3m/4, 3m/4, 3m/4)$
(4)	$(0, e_{45}, -e_{15} - e_{46}, 0, 0, 0)$	$(4m/5, 6m/5, 7m/5, 3m/5, 3m/5, 4m/5)$
(5)	$(0, e_{45}, e_{46}, 0, 0, 0)$	$(m, 5m/4, 5m/4, m/2, 3m/4, 3m/4)$
(6)	$(0, e_{16} + e_{45}, e_{15} - e_{46}, 0, 0, 0)$	$(2m/3, 4m/3, 4m/3, 2m/3, 2m/3, 2m/3)$

TABLE 1. The eigenvalue types and the underlying nilpotent Lie algebras $\mathfrak{n}$.

2.3. **The Levi-Civita connection.** Let us sketch how one computes the torsion-free connection. Denoting by $\hat{d}$ and d the exterior differentials on N and S, the Maurer–Cartan equations for $\mathfrak{s} = \mathfrak{n} + \mathbb{R}e_7$ have the form

$$de_j \;=\; \hat{d}e_j + C_j e_{j7} \;\text{ for } j = 1, \dots, 6 \;\text{ and } de_7 \;=\; 0.$$

The constant m is real and positive, and it is important to remark that each example is distinguished by a unique set of eigenvalues, as shown in Table 1. A routine application of the Koszul formula yields the expression of ∇^g on S with respect to its orthonormal basis $(e_1, \dots, e_7)$. For instance it is not hard to see that

$$\nabla^g_{e_i} e_7 \;=\; C_i e_i, \quad \nabla^g_{e_i} e_i \;=\; -C_i e_i, \;\forall i \neq 7, \quad \nabla^g_{e_7} e_7 \;=\; 0.$$

The new metric $\tilde{g} = e^{2f} g$ is determined by $df = me_7$. The modified Levi-Civita connection can be computed through

$$\nabla^{\tilde{g}}_X Y \;=\; \nabla^g_X Y + df(X)Y + df(Y)X - g(X, Y)\mathrm{grad}\, f,$$

so in particular

$$\begin{aligned} \nabla^{\tilde{g}}_{e_i} e_7 &= (C_i - 1)e_i, \quad \nabla^{\tilde{g}}_{e_7} e_i = me_i, \\ \nabla^{\tilde{g}}_{e_i} e_i &= (1 - C_i)e_i, \quad \nabla^{\tilde{g}}_{e_7} e_7 = me_7 \end{aligned} \tag{2.5}$$

for all $i \neq 7$. The expression for the covariant derivatives of the orthonormal basis $\tilde{e}_i := e^{-f} e_i$ of $\tilde{g}$ can eventually be lifted to the spinor bundle. We shall write e_i instead of $\tilde{e}_i$ when no confusion arises. Therefore

Lemma 2.1. *The derivatives of all vectors on $\mathfrak{n}$ in the seventh direction are zero*

$$\nabla^{\tilde{g}}_{e_7} U = 0 \quad \textit{for all } U \in \mathfrak{n}.$$

Proof. This follows at once by conformally changing the relations in the second column of (2.5). □

This will come handy in the next Section.

3. Reduction theorem for potential solutions

Now we investigate whether the solvable Lie group $(S, \tilde{g})$ admits a parallel spinor for another metric connection with skew-symmetric torsion $T = \sum c_{\alpha\beta\gamma}\, e_{\alpha\beta\gamma}$. Instead of taking the most general 3-form in dimension seven which has 35 summands, we will make the Ansatz that T be a linear combination of the simple forms appearing in η^+, η^-

and $\omega \wedge e_7$. Let $\Lambda^3_{11}(S)$ denote the subspace of $\Lambda^3(S)$ they span. Throughout the treatise we shall take the spin representation Δ_7 used in [AgFr04, BFGK91]:

$$\begin{aligned} e_1 &= +E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46}, \\ e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48}, \\ e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= +E_{14} - E_{23} - E_{58} + E_{67}, \\ e_7 &= +E_{12} - E_{34} - E_{56} + E_{78}, \end{aligned}$$

where E_{ij} stands for the endomorphism of $\mathbb{R}^7$ sending e_i to e_j, e_j to $-e_i$ and everything else to zero. Assuming then that the torsion looks like this

$$\begin{aligned} T \;=\; & c_{125}\, e_{125} + c_{136}\, e_{136} + c_{246}\, e_{246} + c_{345}\, e_{345} + \\ & c_{126}\, e_{126} + c_{346}\, e_{346} + c_{135}\, e_{135} + c_{245}\, e_{245} + \\ & c_{147}\, e_{147} + c_{567}\, e_{567} + c_{237}\, e_{237}, \end{aligned}$$

and denoting by $\lrcorner$ the interior product, one infers that Clifford multiplication by $e_i \lrcorner T$ has – as an endomorphism – the block structure $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$ for any i. This is particularly interesting when $i = 7$ in the light of Lemma 2.1. It allows one to determine the structure of elements in $\ker(\nabla^{\tilde{g}}_{e_7} + e_7 \lrcorner T) = \ker(e_7 \lrcorner T)$ without too much effort. Clearly only the coefficients c_{147}, c_{237} and c_{567} of T are involved.

Reduction Theorem 3.1. *For $T \in \Lambda^3_{11}(S)$ a non-trivial element annihilated by $e_7 \lrcorner T$ is a linear combination of* upper block *reduced forms*

(A) $\psi = (a, b, c, d, 0, 0, 0, 0)$ *with* $c_{567} = 0$, $c_{147} = -c_{237}$,

(B) $\psi = (a, b, -\varepsilon\, a, \varepsilon\, b, 0, 0, 0, 0)$ *and* $c_{147} = -c_{237} + \varepsilon c_{567}$ *with* $\varepsilon = \pm 1$,

or lower block *reduced forms*

(C) $\psi = (0, 0, 0, 0, e, f, g, h)$ *with* $c_{567} = 0$, $c_{147} = +c_{237}$,

(D) $\psi = (0, 0, 0, 0, e, f, \varepsilon\, e, -\varepsilon\, f)$ *and* $c_{147} = +c_{237} + \varepsilon c_{567}$ *with* $\varepsilon = \pm 1$.

Remark 3.1. Notice that the cases are not mutually exclusive: for example if $c_{567} = 0$, (B) is a special case of (A) as (D) is of (C).

In conclusion, one can always assume that a spinor has such a block structure, with the coefficients $c_{147}, c_{237}, c_{567}$ subjected to one addtional linear constraint.

4. Families of real solutions

The solvable extension of Example (2) is equipped with a Ricci-flat metric with Riemannian holonomy equal to G_2, implying that there exists a unique $\nabla^{\tilde{g}}$-parallel spinor ψ. In terms of the endomorphisms E_{ij}, the Levi-Civita connection on the tangent bundle has components

$$\begin{aligned} \nabla^{\tilde{g}}_{e_1} &= -\tfrac{1}{5} m e^{-f} (2E_{17} + E_{35} - E_{26}), & \nabla^{\tilde{g}}_{e_2} &= -\tfrac{1}{5} m e^{-f} (E_{16} + 2E_{27} + E_{45}), \\ \nabla^{\tilde{g}}_{e_3} &= -\tfrac{1}{5} m e^{-f} (E_{15} - E_{37}), & \nabla^{\tilde{g}}_{e_4} &= -\tfrac{1}{5} m e^{-f} (E_{25} - E_{47}), \\ \nabla^{\tilde{g}}_{e_5} &= -\tfrac{1}{5} m e^{-f} (E_{13} + E_{24} + 2E_{57}), & \nabla^{\tilde{g}}_{e_6} &= -\tfrac{1}{5} m e^{-f} (E_{12} - E_{67}), \end{aligned}$$

and $\nabla^{\tilde{g}}_{e_7} = 0$.

Let us now study the existence of solutions $\psi \neq 0$ of the equation $\nabla^T_{e_i} \psi = 0$, where by definition

$$\nabla^T_{e_i} \psi = \nabla^{\tilde{g}}_{e_i} \psi + (e_i \lrcorner T) \cdot \psi. \tag{4.1}$$

Theorem 4.1. *The equation $\nabla^T\psi = 0$ admits precisely 7 solutions for some $T \in \Lambda^3_{11}(S)$, namely:*

a) *A two-parameter family of pairs $(T_{r,s}, \psi_{r,s})$ such that $\nabla^{T_{r,s}}\psi_{r,s} = 0$;*

b) *Six 'isolated' solutions occuring in pairs, $(T_i^\varepsilon, \psi_i^\varepsilon)$ for $i = 1, 2, 3$ and $\varepsilon = \pm$.*

All G_2 structures admit one parallel spinor, and for

$|r| \neq |s|$: *$\varphi_{r,s}$ is of general type $\mathbb{R} \oplus \mathcal{S}^2_0\mathbb{R}^7 \oplus \mathbb{R}^7$,*

$r = s$: *$\varphi_{r,r}$ is parallel, the torsion $T_{r,r} = 0$ and $\psi_{r,r}$ is a multiple of ψ.*

$r = -s$: *the G_2 type of $\varphi_{-s,s}$ has no $\mathbb{R}$-term.*

Proof. By Reduction Theorem 3.1 we can treat cases (A)-(D) separately. This yields the following possibilities.

Solution a). Set

$$\lambda_{r,s} = \frac{r^2 - s^2}{2(r^2 + s^2)}, \quad \mu_{r,s} = \frac{(r-s)^2}{r^2+s^2} \tag{4.2}$$

and

$$\psi_{r,s} = (0, 0, 0, 0, r, s, -r, s).$$

The spinor $\psi_{r,s}$ is parallel with respect to the connection $\nabla^{r,s} := \nabla^{T_{r,s}}$ determined by

$$T_{r,s} = -\tfrac{1}{10}me^{-f}\left[\lambda_{r,s}(\eta^+ - 6\,e_{125}) + \mu_{r,s}(\eta^- + 3\,e_{346})\right].$$

Notice that this family of 3-forms contains no terms in e_7. Furthermore, $\lambda_{r,s} = \lambda_{cr,cs}$ and $\mu_{r,s} = \mu_{cr,cs}$ for any real constant $c \neq 0$, reflecting the fact that any multiple of $\psi_{r,s}$ is again parallel for the connection with the same torsion form. The G_2 structure corresponding to $\psi_{r,s}$ is

$$\varphi_{r,s} = rs\,\eta^+ + \frac{1}{2}(s^2 - r^2)\,\eta^- + \frac{1}{2}(s^2 + r^2)\,\omega \wedge e_7. \tag{4.3}$$

It is by now clear why taking $r = \pm s$ plays a special role, for $T_{r,s}$ and $\varphi_{r,s}$ both simplify. The type of $\varphi_{r,s}$ is determined once one computes its differential and codifferential. Recall that from the covariant derivative of a 3-form $\xi = e_{ijk}$,

$$\nabla_X(e_{ijk}) = (\nabla_X e_i) \wedge e_{jk} + e_i \wedge (\nabla_x e_j) \wedge e_k + e_{ij} \wedge (\nabla_X e_k),$$

one obtains d and δ by

$$d\xi(X_0, \ldots, X_3) = \sum_{i=0}^{3}(-1)^i(\nabla_{X_i}\xi)(X_0, \ldots, \hat{X}_i, \ldots, X_3), \qquad \delta\xi = -\sum_i e_i \lrcorner \nabla_{e_i}\xi.$$

The result of these lengthy calcultations is given in Table 2.

For $r = s$ all components τ_i of the intrinsic torsion vanish, since $\varphi_{r,s}$ is integrable. By construction τ_4 is proportional to e_7, with constant c resulting from the discussion. In general, (2.4) gives

$$\begin{aligned} d\varphi_{r,s} &= \tfrac{s^2-r^2}{2}d\eta^- + \tfrac{(s-r)^2}{2}d\eta^+ \\ \delta\varphi_{r,s} &= -\tfrac{1}{5}me^{-f}(r-s)^2\omega \quad \text{and} \quad -4{*}(c\,e_7 \wedge {*}\varphi_{r,s}) = -2c(r^2+s^2)\omega. \end{aligned}$$

This implies that $c = \frac{1}{10}me^{-f}\mu_{r,s} \neq 0$ for $r \neq s$ and τ_2 is identically zero, as one expects. As for $\tau_1 = -\frac{3}{10}me^{-f}(r^2 - s^2)(2r^2 + 2s^2 - rs)$, one sees it also vanishes for $r = -s$, since

Form	differential d	Hodge $*$	codifferential δ
e_{125}	$-\frac{6}{5}me^{-f}e_{1257}$	e_{3467}	$\frac{2}{5}me^{-f}\omega$
e_{136}	0	e_{2457}	0
e_{246}	0	e_{1357}	0
e_{345}	0	e_{1267}	0
e_{126}	$-\frac{3}{5}me^{-f}e_{1267}$	$-e_{3457}$	0
e_{135}	$-\frac{3}{5}me^{-f}e_{1357}$	$-e_{2467}$	0
e_{245}	$-\frac{3}{5}me^{-f}e_{2457}$	$-e_{1367}$	0
e_{346}	$\frac{1}{5}me^{-f}\omega^2+\frac{3}{5}me^{-f}e_{3467}$	$-e_{1257}$	0
e_{147}	$\frac{2}{5}me^{-f}e_{1257}$	e_{2356}	$-\frac{2}{5}me^{-f}e_{14}$
e_{237}	$\frac{2}{5}me^{-f}e_{1257}$	e_{1456}	$-\frac{2}{5}me^{-f}e_{23}$
e_{567}	$\frac{2}{5}me^{-f}e_{1257}$	e_{1234}	$-\frac{2}{5}me^{-f}e_{56}$

TABLE 2. Derivatives of the simple forms spanning $\Lambda^3_{11}(S)$.

e_{1257} does not appear in $*\varphi_{r,s}$. The 4-form

$$\begin{aligned} *\tau_3 \;=\; & -\tfrac{3}{5}m(s-r)^2e_{1257}+\tfrac{3}{10}m(s^2-r^2)(\eta^-+2e_{346})\wedge e_7+\tfrac{1}{10}m(s^2-r^2)\omega^2+\\ & \tfrac{3}{10}m(s^2-r^2)(2s^2+2r^2-sr)\big(-rs\eta^-\wedge e_7+\tfrac{s^2-r^2}{2}\,\eta^+\wedge e_7-\tfrac{s^2+r^2}{4}\,\omega^2\big)-\\ & \tfrac{3}{10}m\,\tfrac{(r-s)^2}{s^2+r^2}\big(rs\,\eta^+\wedge e_7+\tfrac{(s^2-r^2)}{2}(\eta^+-2e_{126})\wedge e_7\big)\end{aligned}$$

is never zero for $r\neq s$, instead.

Solution b). The isolated solutions occur in pairs labelled $\pm$, basically corresponding to the choice of sign for ε in the Reduction Theorem. The first couple consists of the spinors

$$\psi_1^+=(0,1,0,-1,0,0,0,0)\quad\text{and}\quad\psi_1^-=(1,0,1,0,0,0,0,0)$$

(denoted ψ_1^ε with $\varepsilon=\pm$) and the 3-forms

$$T_1^\varepsilon \;=\; -\frac{me^{-f}}{10}\left[\frac{\varepsilon}{2}(\eta^++4\,e_{125}-2e_{246})+\frac{1}{3}(\eta^--2e_{135}-e_{346})-\frac{2\varepsilon}{3}(\omega-e_{23})\wedge e_7\right].$$

The additional relation on the c_{ij7}'s reads $c_{147}=-c_{567}-c_{237}$. Via equation (2.1) the characteristic form is

$$2\,\varphi_1^\varepsilon \;=\; \varepsilon(e_{126}+e_{135}-e_{245}+e_{346})-e_{147}-e_{567}-e_{237}.$$

The second pair of solutions gives spinors

$$\psi_2^+=(0,1,0,1,0,0,0,0)\quad\text{and}\quad\psi_2^-=(1,0,-1,0,0,0,0,0),$$

together with the torsion

$$T_2^\varepsilon \;=\; -\frac{me^{-f}}{10}\left[\frac{\varepsilon}{2}(\eta^++4\,e_{125}-2e_{136})+\frac{1}{3}(\eta^-+2e_{245}-e_{346})-\frac{2\varepsilon}{3}(\omega+e_{14})\wedge e_7\right].$$

The underlying relation is $c_{147} = c_{567} - c_{237}$. The characteristic 3-form is

$$2\,\varphi_2^\varepsilon \;=\; \varepsilon(-e_{126} + e_{135} - e_{245} - e_{346}) + e_{147} - e_{567} + e_{237}.$$

For the last pair, the spinors are lower block

$$\psi_3^+ = (0,0,0,0,1,0,1,0) \quad \text{and} \quad \psi_3^- = (0,0,0,0,0,1,0,-1).$$

The torsion 3-form is then

$$T_3^\varepsilon \;=\; -\frac{me^{-f}}{10}\left[\frac{1}{2}(\eta^+ + 4\,e_{125} + 2e_{345}) + \frac{1}{3}(\eta^- - 2e_{126} - e_{346}) - \frac{2\varepsilon}{3}(\omega + e_{56})\wedge e_7\right].$$

In this case the equation $c_{147} = c_{567} + c_{237}$ holds. Now the characteristic 3-form is

$$2\,\varphi_3^\varepsilon \;=\; \varepsilon(e_{126} + e_{135} + e_{245} - e_{346}) - e_{147} + e_{567} + e_{237}.$$

In all cases it is not hard to check that φ_i^ε have type $\mathbb{R}\oplus\mathcal{S}_0^2(\mathbb{R}^7)\oplus\mathbb{R}^7$. □

Remark 4.1. The family of G_2 structures (4.3) depends upon the two homogeneous parameters (4.2), or if one prefers on the projective coordinate $w = r/s$. In fact $\lambda = \lambda_{r,s}, \mu = \mu_{r,s}$ lie on the ellipse $(\mu-1)^2 + 4\lambda^2 - 1 = 0$ in the (λ,μ)-plane. The extremal points $w = \infty, 0$ correspond to $\varphi_{r,0} = \frac{r^2}{2}(-\eta^- + \omega\wedge e_7)$ and $\varphi_{0,s} = \frac{s^2}{2}(+\eta^- + \omega\wedge e_7)$, where η^+ is missing. Similarly, the origin of $\mathbb{R}^2$ is $\varphi_{r,r} = r^2(\eta^+ + \omega\wedge e_7)$ whilst $w = -1$ produces the form $\varphi_{r,-r} = r^2(-\eta^+ + \omega\wedge e_7)$, and the roles of $\eta^\pm$ are swapped. It is interesting perhaps to notice that each w on the conic $\mathbb{RP}^1$ corresponds to a specific choice of 3-form in the canonical bundle of N, and does not touch significantly the term $\omega\wedge e_7$.

5. The other examples

The solvmanifolds extending numbers (1), (3), (5), and (6) admit no non-trivial solutions to (4.1), whereas (4) yields only *complex* solutions. We quickly gather the results, writing in particular the Levi-Civita connection.

Example (1). In many respects, this example is the closest to the Riemannian flat case $\mathbb{R}^7$. Although trivially Ricci-flat, Euclidean space admits no parallel spinors for a connection with non-vanishing skew-symmetric torsion [AgFr04]. Here a similar result holds. The Riemannian holonomy reduces to $SU(2)\subset G_2$, and only three components of the LC connection survive, precisely

$$\nabla^{\tilde g}_{e_1} = -\tfrac{1}{3}me^{-f}(E_{17}+E_{35}),\quad \nabla^{\tilde g}_{e_3} = -\tfrac{1}{3}me^{-f}(E_{15}-E_{37}),\quad \nabla^{\tilde g}_{e_5} = -\tfrac{1}{3}me^{-f}(E_{13}+E_{57}),$$

and the four $\nabla^{\tilde g}$-parallel spinors are

$$(1,1,0,0,0,0,0,0),\quad (0,0,-1,1,0,0,0,0),\quad (0,0,0,0,1,1,0,0),\quad (0,0,0,0,0,0,-1,1).$$

Example (3). The Levi-Civita connection on the tangent bundle is given by

$$\begin{aligned}
\nabla^{\tilde g}_{e_1} &= -\tfrac{me^{-f}}{4}(E_{17}+E_{35}), & \nabla^{\tilde g}_{e_2} &= 0,\\
\nabla^{\tilde g}_{e_3} &= -\tfrac{me^{-f}}{4}(E_{15}-2E_{37}-E_{46}), & \nabla^{\tilde g}_{e_4} &= -\tfrac{me^{-f}}{4}(-E_{36}+E_{47}),\\
\nabla^{\tilde g}_{e_5} &= -\tfrac{me^{-f}}{4}(E_{13}+E_{57}), & \nabla^{\tilde g}_{e_6} &= -\tfrac{me^{-f}}{4}(E_{34}+E_{67}).
\end{aligned}$$

It has holonomy group $SU(3)$, so Ψ of (2.2) pairs up with a second LC-parallel spinor $(1,1,1,-1,0,0,0,0)$.

Example (5). The Levi-Civita connection is given by

$$\begin{array}{llll}\nabla^{\tilde g}_{e_1} = 0, & \nabla^{\tilde g}_{e_2} &=& -\frac{me^{-f}}{4}(-E_{27}-E_{45}),\\ \nabla^{\tilde g}_{e_3} = -\frac{me^{-f}}{4}(-E_{37}-E_{46}), & \nabla^{\tilde g}_{e_4} &=& -\frac{me^{-f}}{4}(-E_{25}-E_{36}+2E_{47}),\\ \nabla^{\tilde g}_{e_5} = -\frac{me^{-f}}{4}(E_{24}+E_{57}), & \nabla^{\tilde g}_{e_6} &=& -\frac{me^{-f}}{4}(E_{34}+E_{67}).\end{array}$$

The holonomy is $SU(3)$, hence there exists another $\nabla^{\tilde g}$-parallel spinor besides Ψ, namely $(-1,1,1,1,0,0,0,0)$.

Example (6). The Levi-Civita connection on the tangent bundle is given by

$$\begin{array}{llll}\nabla^{\tilde g}_{e_1} = -\frac{me^{-f}}{6}(2E_{17}+E_{35}-E_{26}), & \nabla^{\tilde g}_{e_2} &=& -\frac{me^{-f}}{6}(-E_{16}-2E_{27}-E_{45}),\\ \nabla^{\tilde g}_{e_3} = -\frac{me^{-f}}{6}(E_{15}-2E_{37}-E_{46}), & \nabla^{\tilde g}_{e_4} &=& -\frac{me^{-f}}{6}(-E_{25}-E_{36}+2E_{47}),\\ \nabla^{\tilde g}_{e_5} = -\frac{me^{-f}}{6}(E_{13}+E_{24}+2E_{57}), & \nabla^{\tilde g}_{e_6} &=& -\frac{me^{-f}}{6}(-E_{12}+E_{34}+2\,E_{67}),\end{array}$$

This manifold has full holonomy G_2. Then again

Theorem 5.1. *Let $(S,\tilde g)$ be one of the solvmanifolds* (1), (3), (5), *or* (6). *If there exists a non-zero spinor ψ solving $\nabla^T\psi = 0$ for some $T\in\Lambda^3_{11}(S)$, then $T=0$ and ψ is a linear combination of the given $\nabla^{\tilde g}$-parallel spinors.*

Proof. By the Reduction Theorem one can assume that ψ has a block structure. Considering cases (A)-(D) separately tells that there are no solutions except for $T=0$. □

6. Complex solutions

The Riemannian connection of the manifold (4) reads

$$\begin{array}{llll}\nabla^{\tilde g}_{e_1} = -\frac{me^{-f}}{5}(E_{17}+E_{35}), & \nabla^{\tilde g}_{e_2} &=& -\frac{me^{-f}}{5}(-E_{27}-E_{45}),\\ \nabla^{\tilde g}_{e_3} = -\frac{me^{-f}}{5}(E_{15}-2E_{37}-E_{46}), & \nabla^{\tilde g}_{e_4} &=& -\frac{me^{-f}}{5}(-E_{25}-E_{36}+2E_{47}),\\ \nabla^{\tilde g}_{e_5} = -\frac{me^{-f}}{5}(E_{13}+E_{24}+2E_{57}), & \nabla^{\tilde g}_{e_6} &=& -\frac{me^{-f}}{5}(E_{34}+E_{67}).\end{array}$$

There are similarities with the Levi-Civita expression relative to example (2), although the two solvmanifolds are not isometric.

It is rather curious to be in presence of *complex* solutions. Though one is usually interested in real spinors and differential forms, complex coefficients might as well be relevant for other considerations. As in proof of Theorem 4.1, by the reduction process of 3.1 we can consider the occurring cases one by one.

Theorem 6.1. *Let $(S,\tilde g)$ be the solvmanifold of example* (4). *If there exists a non-zero spinor ψ satisfying Equation* (4.1) *for some $T\in\Lambda^3_{11}(S)$ and all $i=1,\dots,7$, then:*

(a) *ψ is a multiple of $\big(1+2i\varepsilon\sqrt2,\ 3,\ 1+2i\varepsilon\sqrt2,\ -3,0,0,0,0\big)$ and*

$$\begin{aligned}T &= \tfrac23[-2e_{126}+e_{135}-4e_{245}+e_{346}]+i\varepsilon\sqrt2\,[e_{125}+e_{136}+e_{246}+e_{345}]\\ &+ \tfrac23\,i\varepsilon\sqrt2\,[-e_{147}-e_{567}+2e_{237}], \quad \textit{or}\end{aligned}$$

(b) *ψ is a multiple of $\big(3,\ -1+2i\varepsilon\sqrt2,\ -3,\ -1+2i\varepsilon\sqrt2,0,0,0,0\big)$ and*

$$\begin{aligned}T &= \tfrac23[e_{126}-e_{135}+4e_{245}-2e_{346}]+i\varepsilon\sqrt2\,[-e_{125}+e_{136}+e_{246}-e_{345}]\\ &+ \tfrac23\,i\varepsilon\sqrt2\,[-e_{147}+e_{567}+2e_{237}], \quad \textit{or}\end{aligned}$$

(c) *ψ is a multiple of* $\big(0,0,0,0,\ 1+2i\varepsilon\sqrt{2},\ 3,\ 1+2i\varepsilon\sqrt{2},\ -3\big)$ *and*

$$\begin{aligned} T &= \tfrac{2}{3}\,[e_{126}-2e_{135}+4e_{245}-e_{346}]+i\varepsilon\sqrt{2}\,[e_{125}+e_{136}-e_{246}-e_{345}] \\ &+ \tfrac{2}{3}\,i\varepsilon\sqrt{2}\,[e_{147}-e_{567}+2e_{237}]. \end{aligned}$$

Above ε is 1 or -1 and stems from the solution of a quadratic equation.

Remark 6.1. In Strominger's model of superstring theory ([Str86], [FrIv02]), the contraction $T(i,j) := \sum_{m,n} T_{imn}T_{jmn}$ appears as a relevant term, essentially the torsion contribution to the Ricci tensor. A question of interest is then whether the term is real for the complex solutions above. Now $T(i,j)$ is a real number, possibly zero, apart when $e_i = \pm J(e_j)$

(a) $T(1,4) = T(2,3) = T(5,6) = -8/3\sqrt{2}i\varepsilon$

(b) $T(1,4) = -T(2,3) = T(5,6) = 8/3\sqrt{2}i\varepsilon$

(c) $T(1,4) = -T(2,3) = -T(5,6) = 8/3\sqrt{2}i\varepsilon$.

There seems to be no physical meaning for these solutions in the models currently under investigation.

It is tempting to pursue the same analysis without the assumption that *all* coefficients C_j of the solvable extension S be non-zero, which is important only in connection to the existence of Einstein metrics on S [H98]. With hindsight, we reasonably expect to find metrics with holonomy strictly contained in G_2, so the developed technique might furnish many parallel spinors.

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