A NOTE ON FLAT METRIC CONNECTIONS WITH ANTISYMMETRIC TORSION

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ABSTRACT. In this short note we study flat metric connections with antisymmetric torsion $T \neq 0$. The result has been originally discovered by Cartan/Schouten in 1926 and we provide a new proof not depending on the classification of symmetric spaces. Any space of that type splits and the irreducible factors are compact simple Lie group or a special connection on S^7. The latter case is interesting from the viewpoint of G_2-structures and we discuss its type in the sense of the Fernandez-Gray classification. Moreover, we investigate flat metric connections of vectorial type.

1. INTRODUCTION

Consider a complete Riemannian manifold (M^n, g, ∇) endowed with a metric connection ∇. The torsion T of ∇, viewed as a $(3,0)$ tensor, is defined by

$$T(X, Y, Z) := g(T(X, Y), Z) = g(\nabla_X Y - \nabla_Y X - [X, Y], Z).$$

Metric connections for which T is antisymmetric in all arguments, i.e. $T \in \Lambda^3(M^n)$ are of particular interest, see [Agr06]. They correspond precisely to those metric connections that have the same geodesics as the Levi-Civita connection. In this note we will investigate *flat* connections of that type.

The observation that any simple Lie group carries in fact two flat connections, usually called the $(+)$- and the $(-)$-connection, with torsion $T(X, Y) = \pm[X, Y]$ is due to É. Cartan and J. A. Schouten [CSch26a] and is explained in detail in [KN69, p. 198-199]. If one then chooses a biinvariant metric, these connections are metric and the torsion becomes a 3-form, as desired. Hence, the question is whether there are any further examples of flat metric connections with antisymmetric torsion beside products of Lie groups.

The answer can be found in Cartan's work. In fact, É. Cartan and J. A. Schouten published a second joint paper very shortly after the one mentioned above, [CSch26b]. There is only one additional such geometry, realized on S^7. Their proof that no more cases can occur is by diligent inspection of some defining tensor fields.

Motivated from the problem when the Laplacian of a Riemannian manifold can at least locally be written as a sum $\Delta = -\sum X_i \circ X_i$, d'Atri and Nickerson investigated in 1968 manifolds which admit an orthonormal frame consisting of Killing vector fields. This question is almost equivalent to the previous. In two beautiful papers, Joe Wolf picked up the question again in the early 70ies and provided a complete classification of all complete (reductive) pseudo-Riemannian manifolds admitting absolute parallelism,

thus reproving the Cartan-Schouten result by other means [W72a], [W72b]. The key observation was that the Riemannian curvature of such a space must, for three ∇-parallel vector fields, be given by $R(X,Y)Z = -[[X,Y],Z]/4$, and thus defines a Lie triple system. The proof is then reduced to an (intricate) algebraic problem about Lie triple systems, and S^7 (together with two pseudo-Riemannian siblings) appears because of the outer automorphism inherited from triality.

The main topic of this paper is to understand this very interesting result in terms of special geometries with torsion. We will give a new and elementary proof of the result not using the classification of symmetric spaces. Moreover, we describe explicitely the family of flat metric connections with antisymmetric torsion on S^7 and make the link to G_2 geometry apparent.

2. The case of skew symmetric torsion

Let (M^n, g, ∇) be a connected Riemannian manifold endowed with a flat metric connection ∇. The parallel transport of any orthonormal frame in a point will define a local orthonormal frame $e_1, \ldots e_n$ in all other points. In the sequel, no distinction will be made between vector fields and 1-forms. The standard formula for the exterior derivative of a 1-form yields

$$\begin{aligned} de_i(e_j, e_k) &= e_j \langle e_i, e_k \rangle - e_k \langle e_i, e_j \rangle - \langle e_i, [e_j, e_k] \rangle = -\langle e_i, [e_j, e_k] \rangle \\ &= -\langle e_i, \nabla_{e_j} e_k - \nabla_{e_k} e_j - T(e_j, e_k) \rangle = \langle e_i, T(e_j, e_k) \rangle. \end{aligned}$$

Hence, the torsion can be computed from the frame e_i and their differentials. As was shown by Cartan 1925, the torsion T of ∇ can basically be of 3 possible types—a 3-form, a vector, and a more difficult type that has no geometric interpretation [TV83], [Agr06]. We shall first study the case that the torsion is a 3-form. We state the explicit formula for the torsion and draw some first conclusions from the identities relating the curvatures of ∇ and ∇^g, the Levi-Civita connection. Let the flat connection ∇ be given by

$$\nabla_X Y = \nabla_X^g Y + \frac{1}{2} T(X, Y, -)$$

for a 3-form T. The general relation between Ric^g and Ric^∇ [Agr06, Thm A.1] yields for any orthonormal frame $e_1, \ldots, e_n$ that the Riemannian Ricci tensor can be computed directly from T,

$$\mathrm{Ric}^g(X, Y) = \frac{1}{4} \sum_{i=1}^n \langle T(X, e_i), T(Y, e_i) \rangle, \quad \mathrm{Scal}^g = \frac{3}{2} \|T\|^2.$$

In particular, Ric^g is non-negative, $\mathrm{Ric}^g(X, X) \geq 0$ for all X, and $\mathrm{Ric}^g(X, X) = 0$ if and only if $X \lrcorner T = 0$. The torsion form T is coclosed, $\delta T = 0$, because it coincides with the skew-symmetric part of $\mathrm{Ric}^\nabla = 0$. We define the 4-form σ_T by the formula

$$\sigma_T := \frac{1}{2} \sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T),$$

or equivalently by the formula

$$\sigma_T(X, Y, Z, V) := \langle T(X, Y), T(Z, V) \rangle + \langle T(Y, Z), T(X, V) \rangle + \langle T(Z, X), T(Y, V) \rangle.$$

Denote by $\nabla^{1/3}$ the metric connection with torsion $T/3$. Then we can formulate some properties of the Riemannian manifold and the torsion form.

Proposition 2.1. *Let ∇ be a flat metric connection with torsion $T \in \Lambda^3(M^n)$. Then*

$$3\,dT \;=\; 2\,\sigma_T\,, \quad \nabla^{1/3}T \;=\; 0\,, \quad \nabla^{1/3}\sigma_T \;=\; 0\,.$$

The covariant derivative ∇T is a 4-form and given by

$$(\nabla_V T)(X,Y,Z) \;=\; \frac{1}{3}\sigma_T(X,Y,Z,V) \quad or \quad \nabla_V T \;=\; -\frac{1}{3}\,(V \lrcorner\, \sigma_T)\,,$$

$$(\nabla^g_V T)(X,Y,Z) \;=\; -\frac{1}{6}\sigma_T(X,Y,Z,V) \quad or \quad \nabla^g_V T \;=\; \frac{1}{6}\,(V \lrcorner\, \sigma_T)\,.$$

In particular, the length $\|T\|$ and the scalar curvature are constant. The full Riemann curvature tensor is given by

$$\mathcal{R}^g(X,Y,Z,V) \;=\; -\frac{1}{6}\langle T(X,Y), T(Z,V)\rangle + \frac{1}{12}\langle T(Y,Z), T(X,V)\rangle + \frac{1}{12}\langle T(Z,X), T(Y,V)\rangle,$$

and is $\nabla^{1/3}$-parallel, $\nabla^{1/3}\mathcal{R}^g = 0$. Finally, the sectional curvature is non-negative,

$$K(X,Y) \;=\; \frac{\|T(X,Y)\|^2}{4[\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2]} \;\geq\; 0.$$

Proof. The first Bianchi identity [Agr06, Thm 2.6], [FrI02] states for flat ∇

$$(1) \qquad dT(X,Y,Z,V) - \sigma_T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z) \;=\; 0.$$

By the general formula [FrI02, Cor. 3.2] we have $3dT = 2\,\sigma_T$ for any flat connection with skew-symmetric torsion. Together with equation (1), this shows the first and second formula. The expression for the curvature follows from this and the general identity [Agr06, Thm A.1], [FrI02]

$$\mathcal{R}^g(X,Y,Z,V) \;=\; \mathcal{R}^\nabla(X,Y,Z,V) - \frac{1}{2}(\nabla_X T)(Y,Z,V)$$
$$+ \frac{1}{2}(\nabla_Y T)(X,Z,V) - \frac{1}{4}\langle T(X,Y), T(Z,V)\rangle - \frac{1}{4}\sigma_T(X,Y,Z,V).$$

Since $\nabla - \nabla^{1/3} = \frac{1}{3}T$, we obtain

$$(\nabla_V T)(X,Y,Z) - (\nabla^{1/3}_V T)(X,Y,Z) \;=\; \frac{1}{3}\,T(V,-,-)[T](X,Y,Z),$$

where $T(V,-,-)[T]$ denotes the action of the 2-form $T(V,-,-)$ on the 3-form T. Computing this action, we obtain

$$T(V,-,-)[T](X,Y,Z) \;=\; \sigma_T(X,Y,Z,V).$$

$\nabla^{1/3}T = 0$ follows now directly from the formula for ∇T. In a similar way we compute $\nabla^g T$,

$$\nabla^g_V T \;=\; \nabla_V T - \frac{1}{2}\,(V \lrcorner\, T)[T] \;=\; -\frac{1}{3}\,(V \lrcorner\, \sigma_T) + \frac{1}{2}\,(V \lrcorner\, \sigma_T) \;=\; \frac{1}{6}(V \lrcorner\, \sigma_T). \qquad \square$$

Observe that the curvature identity of the last proposition is nothing than formula (6) in [CSch26b] and the formula for $\nabla^g T$ is formula (7) in the Cartan/Schouten paper.

Corollary 2.1. *Consider a tensor field $\mathcal{T}$ being a polynomial of the torsion form T. Then we have*

$$\nabla \mathcal{T} \;=\; -2\,\nabla^g \mathcal{T}\,.$$

In particular, $\mathcal{T}$ is ∇-parallel if and only if it is ∇^g-parallel.

We derive the following splitting principle, which can again be found in [CSch26b].

Proposition 2.2. *If $M^n = M_1^{n_1} \times M_2^{n_2}$ is the Riemannian product and T is the torsion form of a flat metric connection, then T splits into $T = T_1 + T_2$, where $T_i \in \Lambda^3(M_i^{n_i})$ are 3-forms on $M_i^{n_i}$. Moreover, the connection splits,*

$$(M^n, g, \nabla) \;=\; (M^{n_1}, g_1, \nabla^1) \times (M^{n_2}, g_2, \nabla^2)$$

Proof. Consider two vectors $X \in T(M^{n_1})$, $Y \in T(M^{n_2})$. Then the sectional curvature of the $\{X, Y\}$-plane vanishes, $\mathcal{R}^g(X, Y, Y, X) = 0$. Consequently, we conclude that $X \lrcorner (Y \lrcorner T) = 0$ holds. $\qquad\square$

In the simply connected and complete case we can decompose our flat metric structure into a product of irreducible ones (de Rham decomposition Theorem). Consequently we assume from now on that M^n is a complete, simply connected and irreducible Riemannian manifold and that $T \neq 0$ is non-trivial. The ∇-parallel vector fields $e_1, \ldots, e_n$ are Killing and we immediately obtain the formulas

$$\nabla_{e_k}^g e_l \;=\; -\nabla_{e_l}^g e_k, \quad [e_k, e_l] \;=\; 2\,\nabla_{e_k}^g e_l \;=\; -T(e_k, e_l)$$

and

$$e_k\big(\langle [e_i, e_j], e_l\rangle\big) \;=\; -(\nabla_{e_k} T)(e_i, e_j, e_l) \;=\; -\frac{1}{3}\,\sigma_T(e_i, e_j, e_l, e_k).$$

In particular, $e_k\big(\langle [e_i, e_j], e_l\rangle\big)$ is totally skew-symmetric and the function $\langle [e_i, e_j], e_l\rangle$ is constant if and only if the torsion form is ∇-parallel, $\nabla T = 0$ (see [D'AN68], Lemma 3.3 and Proposition 3.7).

Proposition 2.3 (see [D'AN68], Lemma 3.4)**.** *The Riemannian curvature tensor in the frame $e_1, \ldots, e_n$ is given by the formula*

$$\mathcal{R}^g(e_i, e_j)e_k \;=\; -\frac{1}{4}\big[[e_i, e_j], e_k\big].$$

In particular, $\mathcal{R}^g(e_i, e_j)e_k$ is a Killing vector field.

Proof. We compute

$$\begin{aligned}
\langle [e_i, e_j], [e_k, e_l]\rangle &= 2\,\langle [e_i, e_j], \nabla_{e_k}^g e_l\rangle \;=\; -2\,\langle \nabla_{e_k}^g [e_i, e_j], e_l\rangle + 2e_k\big(\langle [e_i, e_j], e_l\rangle\big) \\
&= -2\,\big\langle \nabla_{[e_i, e_j]}^g e_k + \big[e_k, [e_i, e_j]\big], e_l\big\rangle + 2\,e_k\big(\langle [e_i, e_j], e_l\rangle\big) \\
&= 2\,\langle \nabla_{e_l}^g e_k, [e_i, e_j]\rangle + 2\,\big\langle \big[[e_i, e_j], e_k\big], e_l\big\rangle + 2\,e_k\big(\langle [e_i, e_j], e_l\rangle\big) \\
&= \langle [e_l, e_k], [e_i, e_j]\rangle + 2\,\big\langle \big[[e_i, e_j], e_k\big], e_l\big\rangle + 2\,e_k\big(\langle [e_i, e_j], e_l\rangle\big)
\end{aligned}$$

and we obtain the following formula

$$\langle [e_i, e_j], [e_k, e_l]\rangle \;=\; \big\langle \big[[e_i, e_j], e_k\big], e_l\big\rangle + e_k\big(\langle [e_i, e_j], e_l\rangle\big).$$

The required formula follows now from the Jacobi identity and the fact, that $e_k\big(\langle [e_i, e_j], e_l\rangle\big)$ is totally skew-symmetric,

$$\begin{aligned}
\mathcal{R}^g(e_i, e_j, e_k, e_l) &= -\frac{1}{6}\langle [e_i, e_j], [e_k, e_l]\rangle + \frac{1}{12}\,\langle [e_j, e_k], [e_i, e_l]\rangle + \frac{1}{12}\,\langle [e_k, e_i], [e_j, e_l]\rangle \\
&= -\frac{1}{6}\,\big\langle \big[[e_i, e_j], e_k\big], e_l\big\rangle - \frac{1}{6}\,e_k\big(\langle [e_i, e_j], e_l\rangle\big) + \frac{1}{12}\,\big\langle \big[[e_j, e_k], e_i\big], e_l\big\rangle \\
&\quad + \frac{1}{12}\,e_i\big(\langle [e_j, e_k], e_l\rangle\big) + \frac{1}{12}\,\big\langle \big[[e_k, e_i], e_j\big], e_l\big\rangle + \frac{1}{12}\,e_j\big(\langle [e_k, e_i], e_l\rangle\big) \\
&= -\frac{1}{4}\,\big\langle \big[[e_i, e_j], e_k\big], e_l\big\rangle. \qquad\qquad\qquad\qquad\qquad\square
\end{aligned}$$

Lemma 2.1. *Let X, Y be a pair of Killing vector fields such that $\langle X, Y \rangle$ is constant and let Z be a third Killing vector field. Then*

$$X\big(\langle Y, Z \rangle\big) \;=\; -\,Y\big(\langle X, Z \rangle\big).$$

is skew-symmetric in X, Y.

Proof. For any vector field W, we obtain

$$\langle \nabla_X^g Y, W \rangle \;=\; -\langle \nabla_W^g Y, X \rangle \;=\; -W\big(\langle X, Y \rangle\big) + \langle Y, \nabla_W^g X \rangle \;=\; -\langle \nabla_Y^g X, W \rangle,$$

i. e., $\nabla_X^g Y = -\nabla_Y^g X$. Then the result follows,

$$X\big(\langle Y, Z \rangle\big) \;=\; \langle \nabla_X^g Y, Z \rangle + \langle Y, \nabla_X^g Z \rangle \;=\; -\langle \nabla_Y^g X, Z \rangle - \langle X, \nabla_Y^g Z \rangle \;=\; -Y\big(\langle X, Z \rangle\big). \quad \square$$

Denote by $R_{ijkl} = \mathcal{R}^g(e_i, e_j, e_k, e_l)$ the coefficients of the Riemannian curvature with respect to the ∇-parallel frame $e_1, \ldots, e_n$. Since $\big[[e_i, e_j], e_k\big]$ is a Killing vector field, the latter Lemma reads as $e_m(R_{ijkl}) = -\,e_l(R_{ijkm})$. If m is one of the indices i, j, k, l, we obtain $e_m(R_{ijkl}) = 0$ immediately. Otherwise we use in addition the symmetry properties of the curvature tensor,

$$\begin{aligned}
e_1(R_{2345}) \;&=\; -\,e_5(R_{2341}) \;=\; -\,e_5(R_{1432}) \;=\; e_2(R_{1435}) \;=\; -\,e_2(R_{3541}) \\
&=\; e_1(R_{3542}) \;=\; e_1(R_{4235}).
\end{aligned}$$

Similarly one derives $e_1(R_{4235}) = e_1(R_{3425})$ and the Bianchi identity $R_{2345} + R_{4235} + R_{3425} = 0$ yields the result, $e_1(R_{2345}) = 0$. Consequently, the coefficients are constant and we proved the following

Theorem 2.1 (see [CSch26b], formula (25), [D'AN68], Theorem 3.6)**.** *The Riemannian curvature tensor $\mathcal{R}^g$ is ∇- and ∇^g-parallel. In particular,*

$$\big[\, X \lrcorner\, T \,,\, \mathcal{R}^g \,\big] \;=\; 0$$

holds for any vector $X \in T(M^n)$.

Proof. $\mathcal{R}^g$ is a polynomial depending on T. Consequently, $\nabla^g \mathcal{R}^g = 0$ implies $\nabla \mathcal{R}^g = 0$, see Corollary 2.1. Hence, the difference

$$0 \;=\; (\nabla_X - \nabla_X^g)\mathcal{R}^g \;=\; \big[\, X \lrcorner\, T \,,\, \mathcal{R}^g \,\big]$$

vanishes, too. $\qquad\square$

Corollary 2.2. *Let (M^n, g, ∇, T) be a simply connected, complete and irreducible Riemannian manifold equipped with a flat metric connection and totally skew-symmetric torsion $T \neq 0$. Then M^n is a compact, irreducible symmetric space. Its Ricci tensor is given by*

$$\mathrm{Ric}^g(X, Y) \;=\; \frac{1}{4} \sum_{i=1}^n \langle T(X, e_i), T(Y, e_i) \rangle \;=\; \frac{\mathrm{Scal}^g}{n}\, \langle X, Y \rangle \;, \quad \mathrm{Scal}^g \;=\; \frac{3}{2} \|T\|^2.$$

Since σ_T is $\nabla^{1/3}$-parallel, there are two cases. If $\sigma_T \equiv 0$, then the scalar products $\langle [e_i, e_j], e_k \rangle$ are constant, i.e., the vector fields $e_1, , \ldots, e_n$ are a basis of a n-dimensional Lie algebra. The corresponding simply connected Lie group is a simple, compact Lie group and isometric to M^n. The torsion form of the flat connection is defined by $T(e_k, e_l) = -[e_k, e_l]$ (see [KN69], chapter X).

The case $\sigma_T \not\equiv 0$ is more complicated. Since σ_T is a 4-form, the dimension of the manifold is at least four. Cartan/Schouten (1926) proved that only the 7-dimensional round

sphere is possible. A different argument has been used by D'Atri/Nickerson (1968) and Wolf (1972), namely the classification of irreducible, compact symmetric spaces with vanishing Euler characteristic. This list is very short. Except the compact, simple Lie groups most of them do not admit Killing vector field of constant length.

We provide now a new proof that does not use the classification of symmetric spaces. Consider, at any point $m \in M^n$, the Lie algebra

$$\hat{\mathfrak{g}}_T(m) \; := \; Lie\big\{ \, X \lrcorner T \, : \, X \in T_m(M^n) \, \big\} \; \subset \; \mathfrak{so}(T_m(M^n)),$$

that was introduced in [AF04] for the systematic investigation of algebraic holonomy algebras. Since T is $\nabla^{1/3}$-parallel, the algebras $\hat{\mathfrak{g}}_T(m)$ are $\nabla^{1/3}$-parallel, too.

Proposition 2.4. *Let (M^n, g, ∇, T) be a simply connected, complete and irreducible Riemannian manifold equipped with a flat metric connection and totally skew-symmetric torsion $T \neq 0$. Then the representation $(\hat{\mathfrak{g}}_T(m), T_m(M^n))$ is irreducible.*

Proof. Suppose that the tangent space splits at some point. Then any tangent space splits and we obtain a $\nabla^{1/3}$-parallel decomposition $T(M^n) = V_1 \oplus V_2$ of the tangent bundle into two subbundles. Moreover, the torsion form $T = T_1 + T_2$ splits into $\nabla^{1/3}$-parallel forms $T_1 \in \Lambda^3(V_1)$ and $T_2 \in \Lambda^3(V_2)$, see [AF04]. The subbundles V_1, V_2 are involutive and their leaves are totally geodesic submanifolds of (M^n, g). This contradicts the assumption that M^n is an irreducible Riemannian manifold. $\square$

If the Lie algebra $\hat{\mathfrak{g}}_T \subset \mathfrak{so}(n)$ of a 3-form acts irreducibly on the euclidian space, then there are two possibilities. Either the 3-form of the euclidian space satisfies the Jacobi identity or the Lie algebra coincides with the full algebra, $\hat{\mathfrak{g}}_T = \mathfrak{so}(n)$ (see [AF04], [Na07], [OR08]). The first case again yields the result that the manifold M^n is a simple Lie group (we recover the case of $\sigma_T = 0$). Otherwise the Lie algebra $\hat{g}_T$ coincides with $\mathfrak{so}(n)$ and Theorem 2.1 implies that M^n is a space of positive constant curvature, $\mathcal{R}^g = c \cdot \text{Id}$. The formula for the sectional curvature

$$K \; = \; K(X, Y) \; = \; \frac{\|T(X, Y)\|^2}{4[\|X\|^2\|Y\|^2 - \langle X, Y \rangle^2]}$$

means that the 3-form T defines a metric vector cross product. Consequently, the dimension of the sphere is seven.

Theorem 2.2 (see [CSch26b])**.** *Let (M^n, g, ∇, T) be a simply connected, complete and irreducible Riemannian manifold equipped with a flat metric connection and totally skew-symmetric torsion $T \neq 0$. If $\sigma_T = 0$, then M^n is isometric to a compact simple Lie group. Otherwise ($\sigma_T \neq 0$) M^n is isometric to S^7.*

3. THE CASE OF VECTORIAL TORSION

By definition, such a connection ∇ is given by

$$\nabla_X Y \; = \; \nabla_X^g Y + \langle X, Y \rangle V - \langle V, Y \rangle X$$

for some vector field V. The general relation between the curvature transformations for ∇ and ∇^g [Agr06, App. B, proof of Thm 2.6(1)] reduces to

$$\mathcal{R}^g(X, Y)Z \; = \; \langle X, Z \rangle \nabla_Y V - \langle Y, Z \rangle \nabla_X V + Y \langle \nabla_X V + \|V\|^2 X, Z \rangle - X \langle \nabla_Y V + \|V\|^2 Y, Z \rangle.$$

Hence, the curvature depends not only on V, but also on ∇V. This remains true when considering the Ricci tensor, which does not simplify much. However, the following claim

may be read off immediately: If $\nabla V = 0$, then M^n is a *non-compact* space of constant negative sectional curvature $-\|V\|^2$ and the divergence of the vector field V is constant, $\delta^g(V) = (n-1)\|V\|^2 = \mathrm{const} > 0$. Moreover, the integral curves of the vector field V are geodesics in M^n, $\nabla^g_V V = 0$. In [TV83], this case is discussed in detail; in particular, a flat metric connection with vectorial torsion is explicitely constructed.

For the general case ($\nabla V \neq 0$), the first Bianchi identity [Agr06, Thm 2.6] for a flat connection

$$0 = \overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(X,Y)Z = \overset{X,Y,Z}{\mathfrak{S}} dV(X,Y)Z.$$

yields and interesting consequence: For $\dim M \geq 3$, X, Y, Z can be chosen linearly independent, hence $dV = 0$ and V is locally a gradient field. Observe that a routine calculation shows that $dV(X,Y) = 0$ for all X and Y is equivalent to $\langle \nabla^g_X V, Y \rangle = \langle \nabla^g_Y V, X \rangle$, and one checks that the same property holds for ∇^g replaced by ∇. The triple (M^n, g, V) defines a Weyl structure, i.e., a conformal class of Riemannian metrics and a torsion free connection ∇^w preserving the conformal class. In general , the Weyl connection and its curvature tensor are given by the formulas

$$\nabla^w_X Y = \nabla^g_X Y + g\langle X, V \rangle Y + \langle Y, V \rangle X - \langle X, Y \rangle V,$$
$$\mathcal{R}^\nabla(X,Y)Z = \mathcal{R}^w(X,Y)Z - dV(X,Y)Z.$$

The connection ∇ with vectorial torsion is flat if and only if $dV = 0$ and the Weyl connection is flat, $\mathcal{R}^w = 0$.

Proposition 3.1. *There is a correspondence between triples (M^n, g, ∇), $n \geq 3$, of Riemannian manifolds and flat metric connections ∇ with vectorial torsion and closed, flat Weyl structures.*

In particular, if a Riemannian manifold (M^n, g), $n \geq 3$, admits a flat metric connection with vectorial torsion, then it is locally conformal flat (the Weyl tensor vanishes). Moreover, we can apply Theorem 2.1. and Proposition 2.2. of the paper [AF06]. If M^n is compact, then its universal covering splits and is conformally equivalent to $S^{n-1} \times \mathbb{R}^1$.

Let us discuss the exceptional dimension two. In this case, the curvature $\mathcal{R}^\nabla$ is completely defined by *one* function, namely $\langle \mathcal{R}^\nabla(e_1, e_2)e_1, e_2 \rangle$. Using the formula for the Riemannian curvature tensor we compute this function and then we obtain immediately

Proposition 3.2. *Let (M^2, g) be a 2-dimensional Riemannian manifold with Gaussian curvature G. A metric connection with vectorial torsion is flat if and only if*

$$G = \mathrm{div}^g(V)$$

holds. In particular, if M^2 is compact, then M^2 diffeomorphic to the torus or the Klein bottle.

4. A FAMILY OF FLAT CONNECTIONS ON S^7

4.1. **Construction.** In dimension 7, the complex Spin(7)-representation $\Delta_7^{\mathbb{C}}$ is the complexification of a real 8-dimensional representation $\kappa : \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7)$, since the real Clifford algebra $\mathcal{C}(7)$ is isomorphic to $\mathcal{M}(8) \oplus \mathcal{M}(8)$. Thus, we may identify $\mathbb{R}^8$ with the vector space Δ_7 and embed therein the sphere S^7 as the set of all spinors of

length one. Fix your favorite explicit realization of the spin representation by skew matrices, $\kappa_i := \kappa(e_i) \in \mathfrak{so}(8) \subset \mathrm{End}(\mathbb{R}^8)$, $i = 1, \ldots, 7$. We shall use it to define an explicit parallelization of S^7 by Killing vector fields. Define vector fields $V_1, \ldots, V_7$ on S^7 by

$$V_i(x) = \kappa_i \cdot x \text{ for } x \in S^7 \subset \Delta_7.$$

From the antisymmetry of $\kappa_1, \ldots, \kappa_7$, we easily deduce the following properties for these vector fields:

(1) They are indeed tangential to S^7, $\langle V_i(x), x \rangle = 0$.
(2) They are of constant length one,

$$\langle V_i(x), V_i(x) \rangle = \langle \kappa_i x, \kappa_i x \rangle = -\langle \kappa_i^2 x, x \rangle = -\langle (-1) \cdot x, x \rangle = 1.$$

(3) They are pairwise orthogonal ($i \neq j$).

The commutator of vector fields is inherited from the ambient space, hence $[V_i(x), V_j(x)] = [\kappa_i, \kappa_j](x) = 2\kappa_i\kappa_j x$ for $i \neq j$. In particular, one checks immediately that $[V_i(x), V_j(x)]$ is again tangential to S^7, as it should be. Furthermore, the vector fields $V_i(x)$ are Killing. We now define a connection ∇ on TS^7 by $\nabla V_i(x) = 0$; observe that this implies that all tensor fields with constant coefficients are parallel as well. This connection is trivially flat and metric, and its torsion is given by ($i \neq j$)

$$T(V_i, V_j, V_k)(x) = -\langle [V_i, V_j], V_k \rangle = -2\langle \kappa_i\kappa_j x, \kappa_k x \rangle = 2\langle \kappa_i\kappa_j\kappa_k x, x \rangle.$$

If k is equal to i or j, this quantity vanishes, otherwise the laws of Clifford multiplication imply that it is antisymmetric in all three indices. Observe that this final expression is also valid for $i = j$, though the intermediate calculation is not. Thus, the torsion lies in $\Lambda^3(S^7)$ as wished, and can be written as

$$(*) \qquad T(x) = 2 \sum_{i < j < k} \langle \kappa_i\kappa_j\kappa_k x, x \rangle (V_i \wedge V_j \wedge V_k)(x) .$$

Since in general $\nabla_X Y = \nabla_X^g Y + T(X, Y, -)/2$, the definition of ∇ can equivalently be described for the Levi-Civita connection ∇^g by

$$\nabla_{V_i}^g V_j = \begin{cases} \kappa_i\kappa_j x & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases} .$$

T is not ∇-parallel, as it does not have constant coefficients. Of course, the choice of the vector fields $V_1(x), \ldots, V_7(x)$ is arbitrary: they can be replaced by any other orthonormal frame $W_i(x) := A \cdot V_i(x)$ for a transformation $A \in \mathrm{SO}(7)$. However, any $A \in \mathrm{Stab}\, T \cong G_2 \subset \mathrm{SO}(7)$ will yield the same torsion and hence connection, thus we obtain a family of connections with $7 = \dim \mathrm{SO}(7) - \dim G_2$ parameters.

4.2. ∇ **as a G_2 connection.** The connection ∇ is best understood from the point of view of G_2 geometry. Recall (see [FrI02, Thm 4.8]) that a 7-dimensional Riemannian manifold (M^7, g) with a fixed G_2 structure $\omega \in \Lambda^3(M^7)$ admits a 'characteristic' connection ∇^c (i. e., a metric G_2 connection with antisymmetric torsion) if and only if it is of Fernandez-Gray type $\mathfrak{X}_1 \oplus \mathfrak{X}_3 \oplus \mathfrak{X}_4$ (see [FG82] and [Agr06, p. 53] for this notation). Furthermore, if existent, ∇^c is unique, the torsion of ∇^c is given by

$$(**) \qquad T^c = -*d\omega - \frac{1}{6}\langle d\omega, *\omega \rangle \omega + *(\theta \wedge \omega),$$

where θ is the 1-form that describes the $\mathfrak{X}_4$-component defined by $\delta^g(\omega) = -(\theta \lrcorner \omega)$.

Now, *any* generic 3-form $\omega \in \Lambda^3(S^7)$ that is parallel with respect to our connection ∇ admits ∇ as its characteristic connection, and is related to the torsion T given in $(*)$ by the general formula $(**)$. Thus, there is a large family of G_2 structures ω (namely, all generic 3-forms with constant coefficients) that induce the flat connection ∇ as their G_2 connection. Let us discuss the possible type of the G_2 structures ω inducing ∇. One sees immediately that none of these G_2 structures can be nearly parallel (type $\mathfrak{X}_1$), since T fails to be parallel. A more elaborate argument shows that they cannot even be cocalibrated (type $\mathfrak{X}_1 \oplus \mathfrak{X}_3$): by [FrI02, Thm 5.4], a cocalibrated G_2 structure on a 7-dimensional manifold is Ric^∇-flat if and only if its torsion T is harmonic. Since $H^3(S^7, \mathbb{R}) = 0$, the assertion follows. Finally, we show that the underlying G_2 structures can also not be locally conformally parallel (type $\mathfrak{X}_4$): in [AF06, Example 3.1], we showed that such a structure always satisfies $12\,\delta\theta = 6\|T\|^2 - \mathrm{Scal}^\nabla$. Since ∇ is flat, the divergence theorem implies

$$0 \;=\; 2\int_{S^7} \delta\theta \, dS^7 \;=\; \int_{S^7} \|T\|^2 \, dS^7,$$

a contradiction to $T \neq 0$. To summarize: There exists a multitude of G_2 structures $\omega \in \Lambda^3(S^7)$ that admit the flat metric connection ∇ as their characteristic connection; all these G_2 structures are of general type $\mathfrak{X}_1 \oplus \mathfrak{X}_3 \oplus \mathfrak{X}_4$.

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