# EIGENVALUE ESTIMATES FOR DIRAC OPERATORS WITH PARALLEL CHARACTERISTIC TORSION

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ABSTRACT. Assume that the compact Riemannian spin manifold $(M^n, g)$ admits a $G$-structure with characteristic connection $\nabla$ and parallel characteristic torsion ($\nabla T = 0$), and consider the Dirac operator $D^{1/3}$ corresponding to the torsion $T/3$. This operator plays an eminent role in the investigation of such manifolds and includes as special cases Kostant's "cubic Dirac operator" and the Dolbeault operator. In this article, we describe a general method of computation for lower bounds of the eigenvalues of $D^{1/3}$ by a clever deformation of the spinorial connection. In order to get explicit bounds, each geometric structure needs to be investigated separately; we do this in full generality in dimension 4 and for Sasaki manifolds in dimension 5.

## 1. Introduction

Lower bounds for the first eigenvalue of the Riemannian Dirac operator $D^g$ on a compact Riemannian spin manifold depending on the scalar curvature are well known since more than two decades (see [15], [24]). In past years, another operator of Dirac type turned out to play a crucial role in the investigation of non-integrable geometric structures as well as in several models in superstring theory. Indeed, for many of these geometries it is known that one can replace the Levi-Civita connection by a unique adapted metric connection $\nabla$ with skew-symmetric torsion $T$ preserving the geometric structure, the so-called *characteristic connection* of the geometric structure. The survey article [2] discusses these developments. The Dirac operator in question is then not merely the Dirac operator associated with $\nabla$, but the operator $D^{1/3} = D^g + \frac{1}{4}T$ corresponding to the torsion form $T/3$ (see [4], [5]). In fact, $D^{1/3}$ coincides with the so-called "cubic Dirac operator" introduced by B. Kostant ([25], [1]) on naturally reductive spaces and with the Dolbeault operator on Hermitian manifolds ([14], [22]).

The aim of the present paper is to estimate the eigenvalues of this new Dirac operator in case the torsion form $T$ is $\nabla$-parallel. In this situation $(D^{1/3})^2$ and the torsion form $T$ commute on spinors (see [5]) and we can estimate the eigenvalues separately in any eigensubbundle of the symmetric endomorphism defined by $T$. The classically well-known classes of manifolds with parallel characteristic torsion—nearly Kähler manifolds, Sasakian manifolds, nearly parallel $G_2$-manifolds, naturally reductive spaces—have been considerably enlarged in more recent investigations (see [27], [23], [19], [8], [9], [6], [18], [26]), leading eventually to an abundant supply of manifolds to which our results can be

applied. It was known that the general formula of Schrödinger-Lichnerowicz type (S-L-formula for short) for the operator $D^{1/3}$ derived in [4], [5] does not yield optimal lower bounds for the spectrum. In this article, we deform the connection $\nabla$ by polynomials of the torsion form. The resulting connections are not affine connections anymore, they exist only on the spinor bundle. In 1980, Friedrich had used a similar spinorial modification of the lift of the Levi-Civita connection to derive his estimate for the eigenvalue of $D^g$; the main difference is, however, that there was no torsion form to use there.

The article is organized as follows. In section 2, we prove the necessary integral formulas for perturbations of $D^{1/3}$ by some parallel symmetric endomorphism $S$ of the spinor bundle and we describe the general strategy for proving bounds of the spectrum of $D^{1/3}$. In order to obtain an explicit estimate through this method, one needs to know the algebraic type of $T$ and the splitting of the spinor bundle, hence, every special geometry requires a separate investigation. Section 3 is devoted to the determination of the lower bound for the first eigenvalue $\lambda$ of $(D^{1/3})^2$ by this method on 4-dimensional compact spin manifolds with positive scalar curvature $\mathrm{Scal}^g_{\min} > 0$ and parallel torsion $T \neq 0$. This applies for example to generalized Hopf manifolds, i.e. Hermitian 4-manifolds with Lee form parallel with respect to the Levi-Civita connection (see [27], [23], [13]). We prove the following estimate depending on the ratio $c := \mathrm{Scal}^g_{\min}/||T||^2 > 0$:

$$\lambda \geq \begin{cases} \frac{1}{16}\left[\sqrt{6\,\mathrm{Scal}^g_{\min}} - ||T||\right]^2 = \frac{||T||^2}{16}\left[\sqrt{6\,c} - 1\right]^2 & \text{for } 1/6 \leq c \leq 3/2 \\ \frac{1}{4}\left[\mathrm{Scal}^g_{\min} - \frac{1}{2}||T||^2\right] = \frac{||T||^2}{4}\left[c - \frac{1}{2}\right] & \text{for } c \geq 3/2 \end{cases}$$

We refer to Theorem 3.1 for details. In section 4 we apply our method to 5-dimensional compact Sasakian manifolds with $\mathrm{Scal}^g_{\min} > -4$. Their characteristic torsion is always parallel of length 8 and we get the optimal estimate

$$\lambda \geq \begin{cases} \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2 & \text{for } -4 < \mathrm{Scal}^g_{\min} \leq 4(9 + 4\sqrt{5}) \\ \frac{5}{16}\,\mathrm{Scal}^g_{\min} & \text{for } \mathrm{Scal}^g_{\min} \geq 4(9 + 4\sqrt{5}) \simeq 71,78. \end{cases}$$

Full statements are to be found in Theorem 4.1. The lower bound is attained, for example, on every $\eta$-Einstein-Sasakian manifold. It is a curious fact that this is, to our knowledge, the first eigenvalue estimate for a Dirac operator with a *quadratic* dependence on the scalar curvature. In both dimensions, we have the effect that the main improvement is on an intervall corresponding to 'small' scalar curvatures and eigenvalues (upper line in the estimates above). Beyond this bound, the estimate in dimension 4 can be obtained relatively easy by universal arguments. In dimension 5, this bound is exactly the Riemannian estimate $n/4(n-1)\mathrm{Scal}^g_{\min}$, a non-trivial fact that can be traced back to the particular property that 0 is an eigenvalue of the characteristic torsion $T$ of a Sasaki manifold (this can never happen in dimension 4).

## 2. Schrödinger-Lichnerowicz formulas for the deformation of the connection

Consider a compact Riemannian spin manifold $(M^n, g, T)$ with Levi-Civita connection $\nabla^g$ as well as a metric connection with skew-symmetric torsion $T \in \Lambda^3(M)$,

$$\nabla_X Y := \nabla^g_X Y + \frac{1}{2} \cdot T(X, Y, -).$$

This connection can be lifted to the spinor bundle $\Sigma M$ of $M$, where it takes the expression

$$\nabla_X \psi := \nabla^g_X \psi + \frac{1}{4}(X \lrcorner T) \cdot \psi.$$

Its Dirac operator is given by $D = D^g + (3/4)\, T$, where $D^g$ denotes the Riemannian Dirac operator. Besides this, the connection with torsion $T/3$—henceforth denoted $\nabla^{1/3}$—will play a crucial role in our considerations. Its associated Dirac operator is accordingly given by $D^{1/3} = D^g + T/4 = D - T/2$. Similarly, the spinor Laplacian of $\nabla$ will be written $\Delta$. The Laplacian of $\nabla^{1/3}$ will never be used. We define an algebraic 4-form derived from $T$ by

$$\sigma_T \;:=\; \frac{1}{2} \sum_k (e_k \lrcorner T) \wedge (e_k \lrcorner T),$$

where $e_1, \ldots, e_n$ is an orthonormal basis. Recall that any $k$-form acts on spinors by the extension of Clifford multiplication; henceforth, we shall make no notational difference between a $k$-form and the endomorphism on the spinor bundle that it induces.

**Proposition 2.1** ([4]). *Let $T$ be a 3-form in dimension $n \geq 5$, and denote by the same symbol its associated $(2,1)$-tensor. Then its square inside the Clifford algebra has no contributions of degree $2$ and $6$, and its scalar and fourth degree part are given by*

$$T_0^2 \;=\; \frac{1}{6} \sum_{i,j=1}^n \|T(e_i, e_j)\|^2 \;=:\; \|T\|^2, \quad T_4^2 \;=\; -2\,\sigma_T.$$

*For $n = 3, 4$, one has $T^2 = \|T\|^2$.*

Let's now state the main S-L-formula for $(D^{1/3})^2$. It links the Dirac operator for the torsion $T/3$ with the Laplacian for the torsion $T$. Here, $\mathrm{Scal}^g$ and $\mathrm{Scal}$ denote the scalar curvatures of the Levi-Civita connection and the new connection $\nabla$, respectively. They are related by $\mathrm{Scal} = \mathrm{Scal}^g - (3/2)\|T\|^2$.

**Theorem 2.1** ([4]). *The spinor Laplacian $\Delta$ and the square of the Dirac operator $D^{1/3}$ are related by*

$$(D^{1/3})^2 \;=\; \Delta + \frac{1}{4}\, dT + \frac{1}{4}\, \mathrm{Scal}^g - \frac{1}{8}\, \|T\|^2.$$

Eigenvalue estimates for $D^{1/3}$ footing on this relation will be called *universal*, to distinguish them from the new estimates obtained by deforming $\nabla$ to be discussed later.

In this article, we are only interested in connections $\nabla$ for which $T$ is parallel, $\nabla T = 0$. In this case, $T$ has constant length and it is well-known that $dT = 2\,\sigma_T$, hence Proposition 2.1 implies

$$(1) \qquad\qquad\qquad dT \;=\; -T^2 + \|T\|^2.$$

Combined with the main result of Theorem 2.1, we obtain in the case of parallel torsion the "universal" S-L-formula

$$(2) \qquad\qquad (D^{1/3})^2 \;=\; \Delta - \frac{1}{4}\, T^2 + \frac{1}{4}\, \mathrm{Scal}^g + \frac{1}{8}\|T\|^2.$$

Now let $S : \Sigma M \to \Sigma M$ be a symmetric endomorphism that is parallel itself, $\nabla S = 0$. The main case we have in mind in our applications are polynomials $S = P(T)$ in $T$. Then we can define a new, *S-deformed spin connection* $\nabla^S$ on $\Sigma M$ by

$$\nabla_X^S \psi \;:=\; \nabla_X \psi - \frac{1}{2}(X \cdot S + S \cdot X) \cdot \psi$$

which is metric again. Indeed, the symmetry of $S$ implies for any two spinors $\varphi, \psi \in \Sigma M$

$$(3) \qquad\qquad \langle (X \cdot S + S \cdot X)\,\varphi, \psi \rangle + \langle \varphi, (X \cdot S + S \cdot X)\,\psi \rangle \;=\; 0.$$

In the following technical proposition, we gather all necessary computations involving $S$. Here, all lengths and inner products refer to the $L^2$-scalar product on spinors.

**Proposition 2.2.**

$$\text{a) } D^{1/3}S + SD^{1/3} = \sum_{i=1}^{n}(e_i \cdot S + S \cdot e_i)\nabla_{e_i} - \frac{1}{2}(TS + ST)$$

$$\text{b) } ||\nabla^S \psi||^2 = ||\nabla \psi||^2 + \sum_{i=1}^{n}\langle(e_i \cdot S + S \cdot e_i)\nabla_{e_i}\psi, \psi\rangle + \frac{1}{4}\sum_{i=1}^{n}||(e_i \cdot S + S \cdot e_i)\psi||^2$$

$$\text{c) } \langle(D^{1/3} + S)^2\psi, \psi\rangle = ||\nabla^S \psi||^2 - \frac{1}{4}\sum_{i=1}^{n}||(e_i \cdot S + S \cdot e_i)\psi||^2 - \frac{1}{4}||T\psi||^2 + \frac{1}{8}||T||^2 \cdot$$

$$||\psi||^2 + \frac{1}{4}\int_{M^n}\text{Scal}^g\,||\psi||^2 + ||S\psi||^2 - \langle TS\,\psi, \psi\rangle$$

*Proof.* Identity a) is easy:

$$\left[D^{1/3}S + SD^{1/3}\right]\psi = [(D - T/2)S + S(D - T/2)]\psi = DS\psi + SD\psi - \frac{1}{2}(TS + ST)\psi$$

$$= \sum_{i=1}^{n}e_i \cdot \nabla_{e_i}(S\psi) + \sum_{i=1}^{n}Se_i\nabla_{e_i}\psi - \frac{1}{2}(TS + ST)\psi.$$

Since $S$ is assumed to be parallel, the claim follows. The second relation is a direct consequence of the definition of $\nabla^S$ and the antisymmetry property stated in equation (3).

For the last statement, observe that $D^{1/3} + S$ is again a symmetric first order differential operator. First, we have

$$(D^{1/3} + S)^2 = (D^{1/3})^2 + (D^{1/3}S + SD^{1/3}) + S^2.$$

Hence, we can insert identity (2) and relation a)

$$(D^{1/3} + S)^2 = \Delta - \frac{1}{4}T^2 + \frac{1}{4}\text{Scal}^g + \frac{1}{8}||T||^2 + \sum_{i=1}^{n}(e_i \cdot S + S \cdot e_i)\nabla_{e_i} - \frac{1}{2}(TS + ST) + S^2,$$

which implies for the scalar product

$$\langle(D^{1/3} + S)^2\psi, \psi\rangle = \sum_{i=1}^{n}\langle(e_i \cdot S + S \cdot e_i)\nabla_{e_i}\psi, \psi\rangle - \frac{1}{4}||T\psi||^2 + \frac{1}{4}\int_{M^n}\text{Scal}^g\,||\psi||^2$$

$$+ \frac{1}{8}||T||^2 \cdot ||\psi||^2 + ||S\psi||^2 - \frac{1}{2}\langle(TS + ST)\psi, \psi\rangle + ||\nabla\psi||^2.$$

Now the result follows from relation b). For the very last term, observe that

$$\langle TS\psi, \psi\rangle = \langle S\psi, T\psi\rangle = \langle\psi, ST\psi\rangle$$

and that the imaginary part is irrelevant when integrating.                    □

**Remark 2.1.** We will refer to formula c) as the *S-deformed Schrödinger-Lichnerowicz formula.*

The general strategy for deducing eigenvalue estimates for any eigenvalue $\lambda$ of $(D^{1/3})^2$ from the deformed connection $\nabla^S$ is as follows. The torsion $T$ is a $\nabla$-parallel symmetric endomorphism of the spinor bundle. We split the spinor bundle under $T$,

$$\Sigma M = \bigoplus_{\mu}\Sigma_\mu.$$

The connection $\nabla$ preserves this splitting. Moreover, the fact that $T$ and $(D^{1/3})^2$ commute (see [5, Prop. 3.4]) implies that $(D^{1/3})^2$ preserves this splitting, too. Remark that, in general, the first order operator $D^{1/3}$ does not commute with $T$. Next we make a suitable Ansatz for the endomorphism $S$. A natural restriction on $S$ is that the new connection $\nabla^S$ should preserve the decomposition of $\Sigma M$ again. If $S$ is a polynomial in $T$, we obtain a simple characterization of such endomorphisms. Indeed, let us fix one subbundle $\Sigma_{\mu_0}$ and consider the minimal number of indices $\mu_1, \ldots, \mu_k$ such that

$$TM^n \cdot \Sigma_{\mu_0} \; = \; \big\{ X \cdot \psi \, : \, X \in TM^n \, , \, \psi \in \Sigma_{\mu_0} \big\} \; \subset \; \Sigma_{\mu_1} \oplus \ldots \oplus \Sigma_{\mu_k} \, .$$

Then an easy computation yields the following result.

**Proposition 2.3.** *Let* $S = P(T)$ *be a polynomial in the torsion form* $T$. *Then* $\nabla^S$ *is a connection in the subbundle* $\Sigma_{\mu_0}$ *if and only if, for all* $1 \leq i \leq k$:

$$\big(P(\mu_0) \, + \, P(\mu_i)\big)\, \mu_i \; = \; \big(P(\mu_0) \, + \, P(\mu_i)\big)\, \mu_0 .$$

Maximizing over all admissible endomorphisms $P(T) = S$ in a fixed subbundle $\Sigma_{\mu_0}$, the $S$-deformed S-L-formula yields a lower bound for the first eigenvalue $\lambda$ of $(D^{1/3})^2$ on $\Sigma_{\mu_0}$. However, this estimate typically still involves the eigenspinor $\psi$ of $(D^{1/3})^2$, in particular the term $\langle D^{1/3}\psi, \psi\rangle$. In general, $\psi \in \Gamma(\Sigma_{\mu_0})$ will not be an eigenspinor of $D^{1/3}$, hence this term cannot be expressed through $\lambda$ in any simple way. Nevertheless, it will be possible to overcome this difficulty in special geometric situations by some additional arguments. Lower estimates for $\lambda$ obtained by this strategy will, for better reference, be called $S$-*deformed eigenvalue bounds*. In the next sections, we will show that this strategy yields non-trivial new eigenvalue estimates in concrete special geometries.

## 3. 4-dimensional manifolds with parallel torsion

A particular property of the dimension 4 is that $\sigma_T = 0$ for purely algebraic reasons, hence $\nabla T = 0$ implies $dT = 0$ and $T^2$ acts by scalar multiplication with $||T||^2$. Thus equation (2) reduces to

$$(D^{1/3})^2 \; = \; \Delta + \frac{1}{4}\operatorname{Scal}^g - \frac{1}{8}||T||^2,$$

and we obtain for any eigenvalue $\lambda$ of $(D^{1/3})^2$ the following universal lower bound

$$(4) \qquad\qquad \lambda \; \geq \; \frac{1}{4}\left[\operatorname{Scal}^g_{\min} - \frac{1}{2}||T||^2\right].$$

Remark that even for $T = 0$ a better estimate for the first eigenvalue of the Riemannian Dirac operator is known, $\lambda \geq \operatorname{Scal}^g_{\min}/3$, see [15]. Equality is obtained if and only if $\nabla$ admits at least one parallel spinor $\psi \neq 0$. A look at the usual integrability condition $\nabla\nabla\psi = 0$ and $\sigma_T = \nabla T = 0$ yields the $\operatorname{Ric}^\nabla$-flatness of the connection $\nabla$. In particular, $0 = \operatorname{Scal} = \operatorname{Scal}^g - 3||T||^2/2$ holds in this situation, while in general, $\operatorname{Scal}^g_{\min}$ and $||T||$ are independent geometric quantities. Obviously, the universal bound becomes useless for $2\operatorname{Scal}^g_{\min} \leq ||T||^2$. For complex Hermitian spin surfaces of non-negative conformal scalar curvature, a discussion of the universal bound may be found in [11, Thm 1.2].

Consider the symmetric endomorphism $S := a_0 \operatorname{Id} + a_1 T : \Sigma M \to \Sigma M$ for two real constants $a_0, a_1$, which is symmetric and again parallel. Higher order terms in $T$ are not
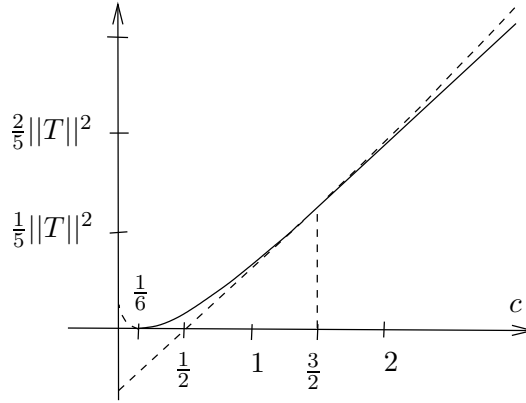
Figure 1. Lower bound for $\lambda$ according to Theorem 3.1.

needed, as $T^2$ is already acting by a scalar. In particular, the spinor bundle splits into two subbundles of equal dimension,

$$\Sigma M \; = \; \Sigma_- \oplus \Sigma_+, \quad \Sigma_\pm \; := \; \{\psi \in \Sigma M \, : \, T \cdot \psi = \pm ||T|| \, \psi\}.$$

We shall use the strategy outlined in Section 4.1 to prove the following result.

**Theorem 3.1.** *Let $(M^4, g)$ be a compact, 4-dimensional spin manifold with $\mathrm{Scal}^g_{\min} > 0$ and $0 \neq T \in \Lambda^3(M)$ a 3-form such that the connection $\nabla$ it defines satisfies $\nabla T = 0$. The first eigenvalue $\lambda$ of $(D^{1/3})^2$ satisfies the following estimate depending on the ratio $c := \mathrm{Scal}^g_{\min}/||T||^2 > 0$:*

$$\lambda \; \geq \; \begin{cases} \frac{1}{4}\left[\mathrm{Scal}^g_{\min} - \frac{1}{2}||T||^2\right] \; = \; \frac{||T||^2}{4}\left[c - \frac{1}{2}\right] & \text{for } c \geq 3/2 \\[2ex] \frac{1}{16}\left[\sqrt{6\,\mathrm{Scal}^g_{\min}} - ||T||\right]^2 \; = \; \frac{||T||^2}{16}\left[\sqrt{6\,c} - 1\right]^2 & \text{for } 1/6 \leq c \leq 3/2 \end{cases}$$

*For $c = 3/2$, both estimates coincide. For $c < 1/6$, no lower bound can be given.*

**Remark 3.1.** Hence, for $c \geq 3/2$, the universal bound is still the best one, while we can improve it for $c \in [1/2, 3/2]$ and obtain a new estimate in the range $c \in [1/6, 1/2]$, where the universal bound is negative. A graph of these estimates is given in Figure 1 (solid line). The dashed line shows the values of the other estimate in the interval where it is not applicable. In particular, we obtain no estimate for $c \leq 1/6$, though the curve corresponding to the $S$-deformed estimate would be positive. Recall that $c = 3/2$ corresponds to vanishing $\nabla$-scalar curvature, and that examples with $\nabla$-parallel spinors and hence $\lambda = ||T||^2/4$ do exist.

*Proof.* Let $\psi \in \Sigma_\pm$ be an eigenspinor of $(D^{1/3})^2$ with eigenvalue $\lambda$. Our starting point is the $S$-deformed S-L-formula c) from Proposition 2.2. First, an easy computation shows that

$$\langle (D^{1/3} + S)^2 \psi, \psi \rangle \; = \; \lambda \, ||\psi||^2 + 2(a_0 \pm a_1 ||T||)\langle D^{1/3}\psi, \psi \rangle + ||S\psi||^2.$$

For the right hand side, choose an orthonormal basis such that $T$ is proportional to $e_{123}$. Then $e_4 \cdot T + T \cdot e_4 = 0$, while $e_i \cdot T = T \cdot e_i$ for $i = 1, 2, 3$. Thus, we can compute

$$\sum_{i=1}^4 ||(e_i \cdot S + S \cdot e_i)\psi||^2 = \sum_{i=1}^4 ||2a_0\, e_i \cdot \psi + a_1(e_i \cdot T + T \cdot e_i) \cdot \psi||^2$$

$$= ||2\, a_0\, e_4 \cdot \psi||^2 + \sum_{i=1}^3 ||2\, a_0\, e_i \cdot \psi + 2\, a_1\, e_i \cdot T \cdot \psi||^2$$

$$= 4a_0^2 ||e_4 \cdot \psi||^2 + 4\sum_{i=1}^3 ||(a_0 \pm a_1 ||T||)e_i \cdot \psi||^2 = 4\left[3\,(a_0 \pm a_1||T||)^2 + a_0^2\right]||\psi||^2.$$

Thus, evaluation of the $S$-deformed S-L-formula yields

$$\lambda||\psi||^2 + 2(a_0 \pm a_1||T||)\langle D^{1/3}\psi, \psi\rangle = ||\nabla^S \psi||^2 - \left[3\,(a_0 \pm a_1||T||)^2 + a_0^2\right]||\psi||^2$$
$$+ \left[\frac{1}{4}\mathrm{Scal}^g - \frac{1}{8}||T||^2 - (a_0 \pm a_1||T||)||T||\right]||\psi||^2.$$

We divide by $||\psi||^2$, exploit that $||\nabla^S \psi||^2/||\psi||^2 \geq 0$ and introduce the abbreviation $y := \langle D^{1/3}\psi, \psi\rangle/||\psi||^2$,

$$\lambda \geq -(a_0 \pm a_1||T||)(2y + ||T||) - 4a_0^2 - 3a_1^2||T||^2 \mp 6\,a_0 a_1||T|| + \frac{\mathrm{Scal}^g_{\min}}{4} - \frac{1}{8}||T||^2.$$

This is the rough $S$-deformed eigenvalue estimate which we now need to concretize. We search the global maximum with respect to the variables $a_0, a_1$. As partial derivatives, we obtain

$$-(2y + ||T||) - 8\,a_0 \mp 6\,a_1||T|| = 0, \quad \mp(2y + ||T||)||T|| - 6\,a_1||T||^2 \mp 6\,a_0||T|| = 0.$$

The vanishing of both expressions is only possible for $a_0 = 0$ or $T = 0$. If $T = 0$, the solution $a_0 = -y/4$ yields Friedrich's inequality, see [15]. For the case we are interested in, $T \neq 0$, we conclude $a_0 = 0$ and $a_1 = \mp(2y + ||T||)/6||T||$. Thus, the eigenvalue estimate becomes

$$\lambda \geq \frac{1}{12}(2y + ||T||)^2 + \frac{1}{4}\mathrm{Scal}^g_{\min} - \frac{1}{8}||T||^2.$$

At this point, the distinction between $\Sigma_+$ and $\Sigma_-$ disappears. The Cauchy-Schwarz inequality implies $y^2 \leq \lambda$, i.e. we are interested in the minimum of $(2y + ||T||)^2$ on the interval $[-\sqrt{\lambda}, \sqrt{\lambda}]$.

**Case 1:** $\lambda \geq ||T||^2/4$. In this situation, the minimum of $(2y + ||T||)^2$ on $[-\sqrt{\lambda}, \sqrt{\lambda}]$ vanishes, hence the estimate is reduced to the universal bound known before.

**Case 2:** $\lambda < ||T||^2/4$. The minimum of $(2y + ||T||)^2$ is realized at $y = -\sqrt{\lambda}$, where it takes the value $(-2\sqrt{\lambda} + ||T||)^2 = 4\lambda - 4\sqrt{\lambda}||T|| + ||T||^2$. Thus we obtain

$$\lambda \geq \frac{\lambda}{3} - \frac{\sqrt{\lambda}||T||}{3} + \frac{1}{4}\mathrm{Scal}^g - \frac{1}{24}||T||^2, \quad \text{i.e.} \quad 2\lambda + \sqrt{\lambda}||T|| + \frac{||T||^2}{8} - \frac{3}{4}\mathrm{Scal}^g \geq 0.$$

This quadratic inequality yields that $\sqrt{\lambda} \geq (\sqrt{6\,\mathrm{Scal}^g} - ||T||)/4$. For $c \geq 1/6$, the right hand side is positive and we obtain

$$\lambda \geq \frac{1}{16}(\sqrt{6\,\mathrm{Scal}^g} - ||T||)^2.$$

Now let us discuss which estimate is to be taken for different values of $c$. We have $\lambda \geq ||T||^2/4$ in the first case, and $\lambda \geq ||T||^2(\sqrt{6\,c} - 1)^2/16$ in the second case; but there

is no way of knowing which case is realized, hence we have to take the minimum of both values.

If $c \in [1/6, 3/2]$, the inequality $1/4 \geq (\sqrt{6\,c} - 1)^2/16$ implies that we can only conclude $\lambda \geq ||T||^2(\sqrt{6\,c} - 1)^2/16$. Since, on the other side, this estimate is better than the universal one on $[1/6, 3/2]$, the claim for this interval follows.

For $c \geq 3/2$, $1/4 \leq (\sqrt{6\,c} - 1)^2/16$, so we were to conclude that $\lambda \geq ||T||^2/4$, obviously a rather bad estimate. In this situation, the universal bound is better and should hence be taken. $\qquad \square$

**Remark 3.2** (limiting case)**.** If the lower bound of Theorem 3.1 ($T \neq 0$) is an eigenvalue and $\psi$ the eigenspinor, there exists an endomorphism $S = a_1 T$ such that $\nabla^S \psi = 0$, i.e.

$$\nabla^g_X \psi = -(a_1 + 1/4)(X \lrcorner T) \cdot \psi.$$

Observe that from the proof, $a_1$ is not an arbitrary constant, but depends implicitly on $\psi$ and the action of $D^{1/3}$ on it. The $\nabla^g$-parallel vector field $*T$ splits the universal covering $\tilde{M} = N^3 \times \mathbb{R}^1$ and $\psi$ is a real Killing spinor on the 3-dimensional manifold $N^3$, $\nabla^{N^3}_X = a_1 ||T|| X \cdot \psi$. Consequently, $N^3$ is either flat ($a_1 = 0$) or isometric to a sphere ($a_1 \neq 0$) (see [15], [17], [11]).

**Remark 3.3.** By using the generalized Casimir operator, it was shown in Proposition 3.5 of [5] that $\mathrm{Scal}^g_{\min} \geq \frac{3}{16} ||T||^2$, i.e. $c \geq 3/16$ implies that the operator $D^{1/3}$ has trivial kernel. Theorem 3.1 shows that this conclusion holds even for all $c > 1/6$. Notice that the Casimir operator can also be used to derive an eigenvalue estimate for $(D^{1/3})^2$, but it typically stays below the universal bound, hence is rather useless. For example, in the 4-dimensional case considered here it yields $\lambda \geq \frac{1}{8}(\mathrm{Scal}^g_{\min} + ||T||^2/2)$ for $c \geq 3/2$ [5, Prop. 3.2, Prop. 3.3].

**Remark 3.4.** The condition $\nabla T = 0$ implies $\nabla^g *T = 0$, i.e. there exists an LC-parallel 1-form on $(M^4, g)$. For such manifolds, it has been shown by Alexandrov, Grantcharov and Ivanov in [10] that the eigenvalues of the Riemannian Dirac operator are bounded by

$$\lambda\big((D^g)^2\big) \geq \frac{3}{8} \mathrm{Scal}^g_{\min}.$$

However, $(D^{1/3})^2 = (D^g)^2 + (TD^g + D^gT)/4 + T^2/16$ is a perturbation of $(D^g)^2$ by an unbounded operator, i.e., the spectra of $(D^g)^2$ and $(D^{1/3})^2$ are not related in any obvious manner.

In [11], the same authors proved that for the Dolbeault operator on a Hermitian spin surface with strictly positive conformal scalar curvature $k$, the estimate

$$(*) \qquad\qquad\qquad \lambda^2 \geq \frac{1}{2} k_{\min}$$

holds. Under our assumption $\nabla T = 0$, the Lee form $\theta$ is coclosed, hence the relation between Riemannian scalar and $*$-scalar curvature reads $\mathrm{Scal}^g - \mathrm{Scal}^* = ||\theta||^2 = ||T||^2$. By definition, $k = (3\mathrm{Scal}^* - \mathrm{Scal}^g)/2$, so $k > 0$ is equivalent to $c > 3/2$ and the estimate $(*)$ is equivalent to $\lambda^2 \geq ||T||^2(2c-3)/4$. This is a line going through zero at $c = 3/2$ and that becomes better than the universal estimate for $c \geq 5/2$. We think that it cannot be derived in the more general framework described here (where no Hermitian structure is assumed).

**Example 3.1.** Consider the 2-dimensional sphere $X^2$ equipped with a Riemannian metric of positive Gaussian curvature $G$. Denote by $G_{\min}$ its minimum. Moreover, we

fix a positive number $||T||$ such that the following conditions hold:

$$||T||^2 \; < \; G_{\min}, \qquad \frac{||T|| \, \text{vol}(X^2)}{\pi} \; = \; k \;\; \text{is an integer}.$$

The 2-form $F := 2||T|| \, dX^2$ satisfies the condition

$$\frac{1}{2\pi} \int_{X^2} F \; = \; k.$$

Consequently, there exists an $S^1$-principal bundle $N^3 \to X^2$ and a connection form $\eta$ on $N^3$ such that $d\eta = \pi^*(F)$. We split the tangent bundle $TN^3 = T^v \oplus T^h$ into its vertical and horizontal part and we define a metric $g$ on $N^3$ by pulling back the metric of the surface $X^2$. The complex structure of $X^2$ lifts to an endomorphism $\varphi : TN^3 \to TN^3$ such that

$$\varphi^2 \; = \; -\,\text{Id} + \eta \otimes \eta, \quad g(\varphi(X), \varphi(Y)) \; = \; g(X, Y) - \eta(X)\eta(Y), \quad d\eta \; = \; 2||T|| \, \pi^*(dX^2)$$

holds. The tuple $(N^3, g, \eta, \varphi)$ is a 3-dimensional compact $\alpha$-Sasakian manifold with fundamental form $\pi^*(dX^2)$. The scalar curvature $\text{Scal}^g$ of $N^3$ is given by the formula

$$\text{Scal}^g \; = \; 2\,G \; - \; 2\,||T||^2 \; > \; 0.$$

The Riemannian product $M^4 := N^3 \times S^1$ admits a canonical complex structure $J$ such that the Kähler form is given by the formula $\Omega = dt \wedge \eta + \pi^*(dX^2)$. Consider the 1-form $\Gamma := ||T|| dt$. Then we obtain

$$d\Omega \; = \; -\, dt \wedge d\eta \; = \; -\, 2\,||T|| \, dt \wedge \pi^*(dX^2) \; = \; -\, 2\,\Gamma \wedge \Omega.$$

Consequently, $(M^4, g \oplus dt^2, J)$ is a compact Hermitian 4-manifold with positive scalar curvature and parallel characteristic torsion (see [6] and [9]),

$$T^c \; = \; -\,||T|| \, dN^3, \quad \nabla^c \; = \; \nabla^g + \frac{1}{2}T^c, \quad \nabla^c\Omega \; = \; 0, \quad \nabla^c T^c \; = \; 0.$$

By the way, any Hermitian 4-manifold with parallel characteristic torsion and positive scalar curvature is locally isometric to a manifold of our family. Now, the universal lower bound $\beta_{\text{univ}}$, the $S$-deformed lower bound $\beta_S$ for $(D^{1/3})^2$ on $M^4$ as well as the relevant ratio $c$ are given by

$$\beta_{\text{univ}} \; = \; \frac{1}{2}G_{\min} - \frac{5}{8}||T||^2, \quad \beta_S \; = \; \frac{1}{16}\big(\sqrt{12(G_{\min} - ||T||^2)} - ||T||\big)^2, \quad c \; = \; \frac{2G_{\min}}{||T||^2} - 2.$$

Let us discuss these estimates in case of an ellipsoid with the induced metric (see for example [3] for the relevant formulas)

$$X^2 \; = \; \big\{(x, y, z) \in \mathbb{R}^3 \, : \, x^2 + y^2 + \frac{z^2}{a^2} \; = \; 1\big\}.$$

Computing the minimum of the Gaussian curvature and the volume,

$$G_{\min} = \frac{1}{a^2}, \quad \text{vol}(X^2) \; = \; 2\pi + 2\pi a^2 \frac{\arcsin\big[\frac{1}{a}\sqrt{a^2 - 1}\big]}{\sqrt{a^2 - 1}},$$

we see that the conditions $||T||^2 < G_{\min}$ and $||T|| \, \text{vol}(X^2) = k\,\pi$ imply that there are only three admissible values for $k$, namely $k = 1, 2, 3$. If $a > 1$, the ratio $c = \text{Scal}^g_{\min}/||T||^2$ is always larger than $1/6$ and $c < 3/2$ is true for $1.02 < a < \infty$ (see Figure 2 for $k = 3$), although $\lim_{a \to 1+} c = 14/9 = 1.555\ldots$. Figure 3 shows the universal bound $\beta_{\text{univ}}$ (lower curve) and the new bound $\beta_S$ (upper curve) as a function of $a$ for $a \geq 3/2$. While $\beta_{\text{univ}}(a)$ becomes negative for $a > 2.33$, the new bound $\beta_S(a)$ stays strictly positive for
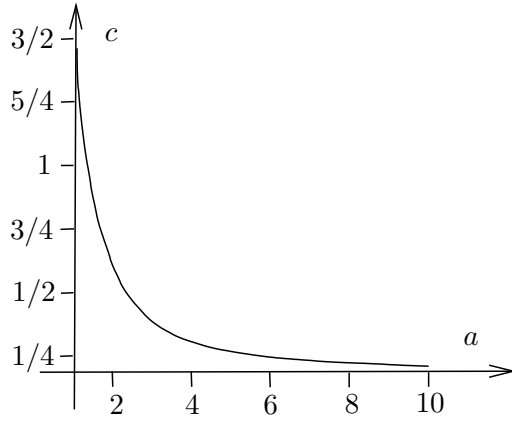
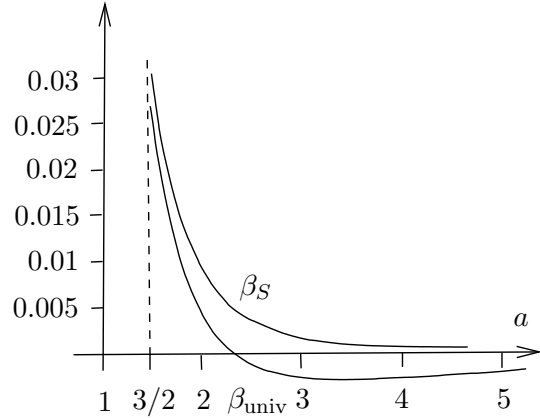Figure 2. The parameter $c = \mathrm{Scal}^g_{\min}/||T||^2$ for $a > 1$.

Figure 3. Lower bounds $\beta_{\mathrm{univ}}$ and $\beta_S$ for $a > 1$.

all $a$ (and has limit $0$ when $a \to \infty$). For $a \to 1^+$, the two curves approach each other more and more, though $\beta_S(a) > \beta_{\mathrm{univ}}(a)$ holds for all $a$. Their limits are

$$\lim_{a \to 1^+} \beta_{\mathrm{univ}} = \frac{19}{128} \cong 0.1484375, \quad \lim_{a \to 1^+} \beta_S = \frac{93}{256} - \frac{3\sqrt{21}}{64} \cong 0.1484730143.$$

Consequently, the $S$-deformed lower bound on $M^4$ is better than the universal bound for ellipsoidal deformations of the sphere with $a > 1.02$. A similar discussion applies in case of $a < 1$.

**Remark 3.5.** The example also shows that the $S$-deformed lower bound $\beta_S$ applies *only* under the condition that $c \leq 3/2$. Indeed, consider the (forbidden) case $||T|| = 0$. Then $M^4 = X^2 \times T^2$ is the Riemannian product of $X^2$ by the flat torus $T^2$. The scalar curvature $\mathrm{Scal}^g$ of $M^4$ equals $2G$ and $D^{1/3}$ coincides with the Riemannian Dirac operator $D^g$ of $M^4$. The $S$-deformed lower bound for $(D^{1/3})^2$ yields $3\mathrm{Scal}^g/2$. However, the sharp estimate of the Dirac operator on a 4-dimensional Kähler manifold is $\mathrm{Scal}^g/2$ and this lower bound is realized on $S^2 \times T^2$ (see [24], [16]).

## 4. 5-dimensional Sasakian manifolds

Let $(M^5, g, \xi, \eta, \varphi)$ be a compact 5-dimensional Sasakian manifold with a fixed spin structure. There exists a unique connection $\nabla$ with totally skew-symmetric torsion and preserving the Sasakian structure (see [19]). The torsion form is given by the formula $T = \eta \wedge d\eta$, $||T|| = 8$. $T$ splits the spinor bundle into two 1-dimensional bundles and one 2-dimensional bundle,

$$\Sigma_{\pm 4} = \left\{ \psi \in \Sigma M^5 : T\psi = \pm 4\,\psi \right\}, \quad \Sigma_0 = \left\{ \psi \in \Sigma M^5 : T\psi = 0 \right\}.$$

In particular, we obtain $T^3 = 16\,T$ and any admissible polynomial in $T$ is quadratic, $S = P(T) = a_0 I + a_1 T + a_2 T^2$. We will estimate $(D^{1/3})^2$ in $\Sigma_{\pm 4}$ and in $\Sigma_0$ separately.

4.1. **The estimate of $(D^{1/3})^2$ in the subbundle $\Sigma_0$.** The operator $D^{1/3} = D^g + T/4$ coincides in the subbundle $\Sigma_0$ with the Riemannian Dirac operator. Consequently, we

can use the estimate of the Riemannian Dirac operator (see [15]),

$$\lambda_{\min}\big((D^{1/3})^2_{|\Sigma_0}\big) \;\geq\; \lambda_{\min}\big((D^g)^2\big) \;\geq\; \frac{5}{16}\mathrm{Scal}^g_{\min} \;=:\; \beta^g.$$

If the scalar curvature is small, there is a better estimate. Indeed, the universal S-L-formula for $(D^{1/3})^2$ on spinors stated in equation (2) reduces on $\Gamma(\Sigma_0)$ to

$$(D^{1/3})^2 \;=\; \Delta + 1 + \frac{1}{4}\mathrm{Scal}^g.$$

Both inequalities together yield the following estimate

$$\lambda_{\min}\big((D^{1/3})^2_{|\Sigma_0}\big) \;\geq\; \max\left\{\frac{5}{16}\mathrm{Scal}^g_{\min},\; 1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right\}.$$

In case of $-4 = \mathrm{Scal}^g_{\min}$, any spinor in the kernel of $(D^{1/3})^2$ is parallel and the Sasakian manifold is isometric to a compact quotient of the 5-dimensional Heisenberg group, see [20]. If $-4 < \mathrm{Scal}^g_{\min}$, the eigenvalues of $(D^{1/3})^2$ on spinors in $\Gamma(\Sigma_0)$ are positive and bounded by the eigenvalues on spinors in $\Gamma(\Sigma_{\pm 4})$. More precisely, we prove the following

**Proposition 4.1.** *Let $(M^5, g, \xi, \eta, \varphi)$ be a compact 5-dimensional Sasakian manifold such that $-4 < \mathrm{Scal}^g_{\min}$ holds. If $\psi \in \Gamma(\Sigma_0)$ is an eigenspinor of the operator $(D^{1/3})^2$ and $\lambda$ is the eigenvalue, then any part of the decomposition*

$$D^{1/3}\psi \;=\; \alpha_4 + \alpha_0 + \alpha_{-4}, \quad \alpha_k \in \Gamma(\Sigma_k), \quad k \in \{-4, 0, 4\}$$

*is an eigenspinor of the operator $(D^{1/3})^2$ with the same eigenvalue $\lambda$. Moreover, at least one of the spinors $\alpha_{\pm 4}$ is nontrivial. In particular, the eigenvalues of $(D^{1/3})^2$ on $\Gamma(\Sigma_0)$ are bounded by the eigenvalues of $(D^{1/3})^2$ on $\Gamma(\Sigma_{\pm 4})$,*

$$\lambda_{\min}\big((D^{1/3})^2_{|\Sigma_0}\big) \;\geq\; \lambda_{\min}\big((D^{1/3})^2_{|\Sigma_{\pm 4}}\big).$$

*Proof.* The operator $(D^{1/3})^2$ preserves the splitting of the spinor bundle. Therefore, the components $\alpha_k$ are eigenspinors of $(D^{1/3})^2$. If $\alpha_{\pm 4}$ vanish for any eigenspinor in $\Sigma_0$, the operator $D^{1/3}$ acts on the corresponding eigenspace as a symmetric operator. Consequently, there exists a spinor field $\psi$ such that

$$D^{1/3}\psi \;=\; \pm\sqrt{\lambda}\,\psi, \quad \psi \in \Gamma(\Sigma_0).$$

By Proposition 2.3, the connection $\nabla^S$ defined by the polynomial $S = a(-8\,\mathrm{Id} + T^2)$ preserves the bundle $\Sigma_0$. We compute that

$$\sum_{i=1}^{5} \|(e_i S + S e_i)\psi\|^2 \;=\; 2 \cdot 16^2 a^2 \|\psi\|^2$$

holds for any spinor $\psi \in \Sigma_0$. The $S$-deformed S-L-formula of Proposition 2.2 yields the following inequality

$$\lambda \;\geq\; 16a\frac{\langle\psi, D^{1/3}\psi\rangle_{L^2}}{\|\psi\|^2} - \frac{1}{4}16^2 a^2 + 1 + \frac{1}{4}\mathrm{Scal}^g_{\min}.$$

The optimal parameter

$$a \;=\; \frac{1}{8}\frac{\langle\psi, D^{1/3}\psi\rangle_{L^2}}{\|\psi\|^2}$$

implies the inequality

$$\lambda \;\geq\; y^2 + 1 + \frac{1}{4}\mathrm{Scal}^g_{\min},$$

where $y := \langle \psi, D^{1/3}\psi \rangle_{L^2}/||\psi||^2 = \pm\sqrt{\lambda}$. Finally we conclude that $-4 \geq \mathrm{Scal}^g_{\min}$, a contradiction. $\square$

4.2. **The estimate of $(D^{1/3})^2$ in the subbundle $\Sigma_{\pm 4}$.** Let us consider the subbundle $\Sigma_4 = \{\psi \in \Sigma; \ T\psi = 4\psi\}$. Then $TM^5 \cdot \Sigma_4$ is a proper subbundle of $\Sigma$. Indeed, we have

$$TM^5 \cdot \Sigma_4 \ \subset \ \Sigma_0 \oplus \Sigma_4.$$

By Proposition 2.3, the connection $\nabla^S$ defined by the polynomial $S = (-2a_1 - 8a_2)\mathrm{Id} + a_1 T + a_2 T^2$ preserves the bundle $\Sigma_4$. In particular, $S$ acts in $\Sigma_4$ by multiplication with $x = 2a_1 + 8a_2$. The operator $(D^{1/3})^2$ on spinors in $\Gamma(\Sigma_4)$ is given by the formula (see eq. (2))

$$(D^{1/3})^2_{|\Sigma_4} \ = \ \Delta - 3 + \frac{1}{4}\mathrm{Scal}^g$$

and we obtain the following universal estimate for all eigenvalues $\lambda$ of the operator $(D^{1/3})^2_{|\Sigma_4}$,

$$\lambda \ \geq \ -3 + \frac{1}{4}\mathrm{Scal}^g_{\min} \ =: \ \beta_{\mathrm{univ}}.$$

Proposition 4.1 implies that this inequality also holds for all eigenvalues of $(D^{1/3})^2$. In order to evaluate the $S$-deformed eigenvalue bound, we first compute for spinors $\psi \in \Sigma_4$

$$\sum_{i=1}^{5} ||(e_i S + S e_i)\psi||^2 \ = \ 16(a_1 + 4a_2)^2 ||\psi||^2.$$

The $S$-deformed S-L-formula of Proposition 2.2 then yields the inequality

$$\lambda \ \geq \ -x^2 - 4x - 2x\frac{\langle \psi, D^{1/3}\psi \rangle}{||\psi||^2} - 3 + \frac{1}{4}\mathrm{Scal}^g_{\min}, \quad x := 2a_1 + 8a_2.$$

The optimal parameter

$$x \ = \ -2 - \frac{\langle \psi, D^{1/3}\psi \rangle_{L^2}}{||\psi||^2_{L^2}}$$

yields the refined estimate

$$\lambda \ \geq \ 1 + 4y + y^2 + \frac{1}{4}\mathrm{Scal}^g_{\min}, \quad y := \left\langle \psi, D^{1/3}\psi \right\rangle_{L^2}/||\psi||^2.$$

Let us discuss the limiting case of $\lambda = 1 + 4y + y^2 + \frac{1}{4}\mathrm{Scal}^g_{\min}$. Then the scalar curvature is constant and the eigenspinor $\psi \in \Gamma(\Sigma_{\pm 4})$ is parallel with respect to a connection given by the formula

$$\nabla^S \psi \ = \ 0, \quad S \ = \ (-2a_1 - 8a_2)\mathrm{Id} + a_1 T + a_2 T^2.$$

Conversely, an easy algebraic computation yields that any $\nabla^S$-parallel spinor in $\Sigma_{\pm 4}$ is an eigenspinor of the operator $D^{1/3}$ with eigenvalue $y = -(2 + 2a_1 \pm 8a_2)$. In the limiting case $y^2 = \lambda = 1 + 4y + y^2 + \frac{1}{4}\mathrm{Scal}^g_{\min}$ we obtain

$$\lambda \ = \ y^2 \ = \ \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2.$$

**Proposition 4.2.** *For any eigenspinor $\psi \in \Gamma(\Sigma_{\pm 4})$ with eigenvalue $\lambda$ holds*

$$\lambda \ \geq \ 1 + 4y + y^2 + \frac{1}{4}\mathrm{Scal}^g_{\min}, \quad y := \left\langle \psi, D^{1/3}\psi \right\rangle_{L^2}/||\psi||^2.$$

*Equality occurs if and only if the scalar curvature is constant and there exist numbers $a_1, a_2$ such that*

$$\nabla^S \psi = 0, \quad S = (-2a_1 - 8a_2)I + a_1 T + a_2 T^2, \quad \mathrm{Scal}^g = 28 + 32a_1 \pm 128a_2.$$

*In this case we have*

$$\lambda = y^2 = \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2.$$

Since $\Sigma_{\pm 4}$ are 1-dimensional bundles, the integrability condition for the existence of a parallel spinor, $\nabla^S \psi = 0$, can be formulated using the Ricci tensor only. A standard argument (see [21]) yields the following characterization.

**Proposition 4.3.** *A 5-dimensional simply connected Sasakian manifold admits a spinor field $\psi \in \Gamma(\Sigma_{\pm 4})$ such that*

$$\nabla^S \psi = 0, \quad S = (-2a_1 - 8a_2)I + a_1 T + a_2 T^2,$$

*if and only if it is $\eta$-Einstein. The Riemannian Ricci tensor has the eigenvalues $6 + 8a_1 \pm 32a_2$ with multiplicity four and $4$ with multiplicity one.*

**Corollary 4.1.** $\frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g\right]^2$ *is an eigenvalue of the operator $(D^{1/3})^2$ on any 5-dimensional $\eta$-Einstein-Sasakian manifold.*

Now we estimate the eigenvalues of $(D^{1/3})^2$ by purely geometric data. The Cauchy-Schwarz inequality $y^2 \le \lambda$ restricts the parameter $y$. If $\lambda_{\min} < 4$, we obtain

$$\min_{y \in [-\sqrt{\lambda}, \sqrt{\lambda}]} \left\{1 + 4y + y^2\right\} = \left\{1 + 4y + y^2\right\}_{|y = -\sqrt{\lambda}} = 1 - 4\sqrt{\lambda} + \lambda.$$

Finally, we estimated the operator $(D^{1/3})^2$ on spinors in $\Gamma(\Sigma_4)$,

$$\lambda_{\min} \ge \min\left\{4, \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2\right\}.$$

However, for large scalar curvature $\mathrm{Scal}^g_{\min} \ge 28$, the universal estimate lies above 4, $\beta_{\mathrm{univ}} = -3 + \mathrm{Scal}^g_{\min}/4 \ge 4$, hence

$$\lambda \ge \begin{cases} \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2 & \text{for } -4 \le \mathrm{Scal}^g_{\min} \le 28, \\ -3 + \frac{1}{4}\mathrm{Scal}^g_{\min} & \text{for } \mathrm{Scal}^g_{\min} \ge 28. \end{cases}$$

We study the case $\mathrm{Scal}^g_{\min} \ge -4$ more carefully. Let $\psi \in \Gamma(\Sigma_4)$ be an eigenspinor of $(D^{1/3})^2$ realizing the minimal eigenvalue. The spinor $D^{1/3}\psi$ is thus a section of the bundle $\Sigma_0 \oplus \Sigma_4$. We decompose $D^{1/3}\psi = \alpha_0 + \alpha_4$ into its components. Then $\alpha_0$ and $\alpha_4$ are eigenspinors of $(D^{1/3})^2$. If $\alpha_0 \ne 0$ we conclude that
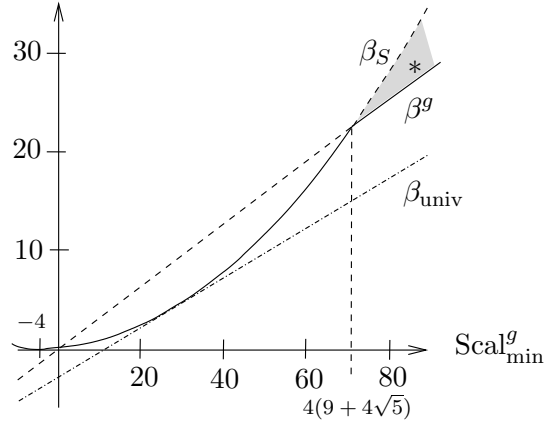
$$\lambda_{\min}\left((D^{1/3})^2_{|\Sigma_{\pm 4}}\right) \ge \lambda_{\min}\left((D^{1/3})^2_{|\Sigma_0}\right).$$

If $\alpha_0 = 0$ for any eigenspinor related to the minimal eigenvalue $\lambda_{\min}$, the operator $D^{1/3}$ is a symmetric operator acting in the eigenspace $\{\psi : (D^{1/3})^2\psi = \lambda_{\min}\psi, \psi \in \Gamma(\Sigma_4)\}$. Consequently, there exists an eigenspinor of $D^{1/3}$ inside the bundle $\Sigma_4$,

$$D^{1/3}\psi = \lambda_{\min}\psi, \quad \psi \in \Gamma(\Sigma_4).$$

Then we obtain $y = \pm\sqrt{\lambda_{\min}}$ and the inequality $\lambda \ge 1 + 4y + y^2 + \frac{1}{4}\mathrm{Scal}^g_{\min}$ yields the estimate

$$\lambda_{\min} \ge \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2 =: \beta_S.$$

Figure 4. Lower bound for $\lambda$ according to Theorem 4.1.

In particular, we proved

**Proposition 4.4.**

$$\lambda_{\min}((D^{1/3})^2_{\Sigma_{\pm 4}}) \ \geq \ \min \left\{ \frac{1}{16} \left[ 1 + \frac{1}{4} \mathrm{Scal}^g_{\min} \right]^2 , \lambda_{\min}\big((D^{1/3})^2_{|\Sigma_0}\big) \right\}.$$

Let us summarize the previous discussion. The inequalities of Proposition 4.1, Proposition 4.4 as well as the inequality $\lambda_{\min}\big((D^{1/3})^2_{|\Sigma_0}\big) \geq \frac{5}{16}\mathrm{Scal}^g_{\min} =: \beta^g$ together yield the following result.

**Theorem 4.1.** *Let $(M^5, g, \xi, \eta, \varphi)$ be a compact Sasakian manifold with $\mathrm{Scal}^g_{\min} > -4$, $T = \eta \wedge d\eta$ its characteristic torsion. The first eigenvalue $\lambda_{\min}$ of $(D^{1/3})^2$ satisfies:*

$$\lambda_{\min} \ \geq \ \begin{cases} \frac{1}{16} \left[ 1 + \frac{1}{4} \mathrm{Scal}^g_{\min} \right]^2 & \text{for} \ -4 < \mathrm{Scal}^g_{\min} \leq 4(9 + 4\sqrt{5}) \\ \frac{5}{16} \mathrm{Scal}^g_{\min} & \text{for} \ \mathrm{Scal}^g_{\min} \geq 4(9 + 4\sqrt{5}) \simeq 71,78. \end{cases}$$

*For $\mathrm{Scal}^g_{\min} = 4(9+4\sqrt{5})$, both estimates coincide. Furthermore, the smallest eigenvalues of the operators $(D^{1/3})^2_{|\Sigma_{\pm 4}}$ and $(D^{1/3})^2_{|\Sigma_0}$ satisfy:*

(1) *$\lambda_{\min}((D^{1/3})^2_{|\Sigma_0}) \geq \lambda_{min}((D^{1/3})^2_{|\Sigma_{\pm 4}})$ for all $\mathrm{Scal}^g_{\min} > -4$, hence $\lambda_{\min}$ can always be realized in $\Sigma_{\pm 4}$;*

(2) *If, for $\mathrm{Scal}^g_{\min} > 4(9 + 4\sqrt{5})$, $\lambda_{\min}$ happens to lie in the intermediate range $\frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2 \geq \lambda_{\min} \geq \frac{5}{16}\mathrm{Scal}^g_{\min}$, then $\lambda_{\min}((D^{1/3})^2_{|\Sigma_0}) = \lambda_{\min}((D^{1/3})^2_{|\Sigma_{\pm 4}})$.*

*Finally, $\frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g\right]^2$ is an eigenvalue of $(D^{1/3})^2$ on any $\eta$-Einstein-Sasakian manifold.*

**Remark 4.1.** A graph of these estimates is given in Figure 4. The bottom dashed line corresponds to the universal lower bound $\beta_{\mathrm{univ}} = -3 + \mathrm{Scal}^g_{\min}/4$ and turns out to be useless for all values of $\mathrm{Scal}^g_{\min}$. The $S$-deformed lower bound $\beta_S = \frac{1}{16}\left[1 + \frac{1}{4}\mathrm{Scal}^g_{\min}\right]^2$ is drawn as a solid line within its range of application $\mathrm{Scal}^g_{\min} \in [-4, 4(9 + 4\sqrt{5})]$, and dashed outside. The Riemannian estimate $\beta^g = 5/16\,\mathrm{Scal}^g_{\min}$ is to be taken for

$\mathrm{Scal}^g_{\min} \geq 4(9 + 4\sqrt{5})$, corresponding to the solid part of the line. A particularly interesting region is the shaded intermediate range $*$ between $\beta^g$ and $\beta^S$; here, $\lambda_{\min}((D^{1/3})^2_{|\Sigma_0}) = \lambda_{\min}((D^{1/3})^2_{|\Sigma_{\pm 4}})$ holds.

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