CONNECTIONS ON NATURALLY REDUCTIVE SPACES, THEIR DIRAC OPERATOR AND HOMOGENEOUS MODELS IN STRING THEORY

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ABSTRACT. Given a reductive homogeneous space $M = G/H$ endowed with a naturally reductive metric, we study the one-parameter family of connections ∇^t joining the canonical and the Levi-Civita connection ($t = 0, 1/2$). We show that the Dirac operator D^t corresponding to $t = 1/3$ is the so-called "cubic" Dirac operator recently introduced by B. Kostant, and derive the formula for its square for any t, thus generalizing the classical Parthasarathy formula on symmetric spaces. Applications include the existence of a new G-invariant first order differential operator $\mathcal{D}$ on spinors and an eigenvalue estimate for the first eigenvalue of $D^{1/3}$. This geometric situation can be used for constructing Riemannian manifolds which are Ricci flat and admit a parallel spinor with respect to some metric connection ∇ whose torsion $T \neq 0$ is a 3-form, the geometric model for the common sector of string theories. We present some results about solutions to the string equations and give a detailed discussion of some 5-dimensional example.

Contents

1. Introduction

This paper proposes a differential geometric approach to some recent results from B. Kostant on an algebraic object called "cubic Dirac operator" ([Kos99]). The key observation is that one can introduce a metric connection on certain homogeneous spaces whose torsion (viewed as a $(0, 3)$-tensor) is 3-form such that the associated Dirac operator has Kostant's algebraic object as its symbol. At the same time, there has been recently a growing interest in connections with totally skew symmetric

torsion for constructing models in string theory and supergravity. We show that the mentioned class of homogeneous spaces yields interesting candidates for such solutions and use Dirac operator techniques to prove some vanishing theorems.

In a first part of this paper, we consider a reductive homogeneous space $M = G/H$ endowed with a Riemannian metric that induces a naturally reductive metric $\langle\ ,\ \rangle$ on $\mathfrak{m}$, where we set $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The one-parameter family of G-invariant connections defined by

$$\nabla^t_X Y \;=\; \nabla^0_X Y + t\,[X,Y]_{\mathfrak{m}}$$

joins the canonical ($t = 0$) and the Levi-Civita ($t = 1/2$) connection. Its torsion $T(X,Y,Z) = (2t-1) \cdot \langle [X,Y]_{\mathfrak{m}}, Z \rangle$ is a 3-form. For an orthonormal basis $Z_1, \ldots, Z_n$ of $\mathfrak{m}$, it induces the third degree element

$$H \;:=\; \frac{3}{2} \sum_{i<j<k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \, Z_i \cdot Z_j \cdot Z_k$$

inside the Clifford algebra $\mathcal{C}(\mathfrak{m})$ of $\mathfrak{m}$. The fact that the Dirac operator associated with the connection ∇^t may then be written as

$$D^t \psi \;=\; \sum_i Z_i \cdot Z_i(\psi) + t \cdot H \cdot \psi$$

suggested the name "cubic Dirac operator" to B. Kostant. We will show that the main achievement in [Kos99] was to realize that, for the parameter value $t = 1/3$, the square of D^t may be expressed in a very simple way in terms of Casimir operators and scalars only ([Kos99, Thm 2.13], [Ste99, 10.18]). It is a remarkable generalization of the well-known Parthasarathy formula for D^2 on symmetric spaces (Theorem 3.1 in this article, see [Par72]). In fact, S. Slebarski has already noticed independently that the parameter value $t = 1/3$ has distinguished properties (see Theorem 1 and the introduction in [Sle87a]). He uses it to prove a "vanishing theorem" for the kernel of the twisted Dirac operator, which can be easily recovered from Kostant's formula (see [Lan00, Thm 4]). His articles [Sle87a] and [Sle87b] contain several formulas of Weitzenböck type for D^2, but none of them is of Parthasarathy type. We shall compute the general expression for $(D^t)^2$ in Theorem 3.2 and show how it can be simplified for this particular parameter value in Theorem 3.3. We emphasize one difference between our work and [Kos99]. While Kostant studies the algebraic action of $D^{1/3}$ as an element of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{C}(\mathfrak{m})$ on L^2-functions $G \to \Delta_{\mathfrak{m}}$ (the spinor representation), we restrict our attention to spinors, i. e., L^2-sections of the spinor bundle $S = G \times_{\widetilde{\mathrm{Ad}}} \Delta_{\mathfrak{m}}$. In particular, this implies that one of the terms in the formula for $(D^t)^2$ (the "diagonally" embedded Casimir operator of $\mathfrak{h}$) vanishes independently of t. An immediate consequence of Theorem 3.2 is the existence of a new G-invariant first order differential operator

$$\mathcal{D}\psi \;:=\; \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \, Z_i \cdot Z_j \cdot Z_k(\psi)$$

on spinors (Remark 3.5) that has no analogue on symmetric spaces. Furthermore, under some additional hypotheses (the lifted Casimir operator $\Omega_{\mathfrak{g}}$ has to be non negative) Theorem 3.3 yields an eigenvalue estimate, which is discussed in Corollary 3.1.

In the second part of this paper, we use the preceding approach for studying the string equations on naturally reductive spaces. Stated in a differential geometric way, one wants to construct a Riemannian manifold (M, g) with a metric connection ∇ such that its torsion $T \neq 0$ is a 3-form and such that there exists at least one spinor field ψ satisfying the coupled system

$$\mathrm{Ric}^{\nabla} \;=\; 0, \quad \delta(T) \;=\; 0, \quad \nabla\Psi \;=\; 0, \quad T \cdot \Psi \;=\; 0.$$

The number of preserved supersymmetries depends essentially on the number of ∇-parallel spinors. For a general background on these equations, we refer to the article by A. Strominger [Str86], where they appeared for the first time. Thus, if one looks for homogeneous solutions, the family of connections ∇^t yields canonical candidates for the desired connection ∇, and the results on the associated Dirac operator can be used to discuss the solution space to these equations. We discuss the significance of constant spinors (which do not always exist) in Theorem 4.2 and show that the last two string equations cannot have any solutions at all if the lifted Casimir operator $\Omega_{\mathfrak{g}}$ is non

negative (Theorem 4.3). In order to discuss the first equation, we present a representation theoretical expression for the Ricci tensor of the connection ∇^t, which generalizes previous results by Wang and Ziller (Theorem 4.4). The article ends with a thourough discussion of an example, namely, the naturally reductive metrics on the 5-dimensional Stiefel manifold.

Although we rarely refer to it, this paper is in spirit very close (and in some sense complementary) to a recent article by Friedrich and Ivanov ([FI01]). There, the authors study metric connections with totally skew symmetric torsion preserving a given geometry.

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2. A family of connections on naturally reductive spaces

Consider a Riemannian homogeneous space $M = G/H$. We suppose that M is *reductive*, i.e., the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an $\mathrm{Ad}\,(H)$-invariant subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}\,(H)\mathfrak{m} \subset \mathfrak{m}$. We identify $\mathfrak{m}$ with $T_0 M$ by the map $X \mapsto X_0^*$, where X^* is the Killing vector field on M generated by the one parameter group $\exp(tX)$ acting on M. We pull back the Riemannian metric $\langle\ ,\ \rangle_0$ on $T_0 M$ to an inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$. Let $\mathrm{Ad} : H \to \mathrm{SO}(\mathfrak{m})$ be the isotropy representation of M. By a theorem of Wang ([KN96, Ch. X, Thm 2.1]), there is a one-to-one correspondence between the set of G-invariant metric affine connections and the set of linear mappings $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ such that

$$\Lambda_{\mathfrak{m}}(hXh^{-1}) = \mathrm{Ad}\,(h)\Lambda_{\mathfrak{m}}(X)\mathrm{Ad}\,(h)^{-1} \ \text{ for } X \in \mathfrak{m} \text{ and } h \in H\,.$$

Its torsion and curvature are then given for $X, Y \in \mathfrak{m}$ by ([KN96, Ch. X, Prop. 2.3])

$$
\begin{aligned}
T(X,Y) &= \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}}, \\
R(X,Y) &= [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \mathrm{Ad}\,([X,Y]_{\mathfrak{h}})\,,
\end{aligned}
$$

where the Lie bracket is split into its $\mathfrak{m}$ and $\mathfrak{h}$ part, $[X,Y] = [X,Y]_{\mathfrak{m}} + [X,Y]_{\mathfrak{h}}$.

Lemma 2.1. *The $(0,3)$-tensor corresponding to the torsion ($X, Y, Z \in \mathfrak{m}$)*

$$T(X,Y,Z) := \langle T(X,Y), Z \rangle$$

is totally skew symmetric if and only if the map $\Lambda_{\mathfrak{m}}$ satisfies for all $X, Y, Z \in \mathfrak{m}$ the invariance condition

$$\langle \Lambda_{\mathfrak{m}}(X)Y, Z \rangle + \langle \Lambda_{\mathfrak{m}}(Z)Y, X \rangle = \langle [X,Y]_{\mathfrak{m}}, Z \rangle + \langle [Z,Y]_{\mathfrak{m}}, X \rangle\,.$$

Proof. The antisymmetry of $T(X,Y,Z)$ in X and Z is equivalent to

$$\langle \Lambda_{\mathfrak{m}}(X)Y, Z \rangle + \langle \Lambda_{\mathfrak{m}}(Z)Y, X \rangle - \langle \Lambda_{\mathfrak{m}}(Y)X, Z \rangle - \langle \Lambda_{\mathfrak{m}}(Y)Z, X \rangle - \langle [X,Y]_{\mathfrak{m}}, Z \rangle - \langle [Z,Y]_{\mathfrak{m}}, X \rangle = 0\,.$$

The third and fourth term cancel out each other by the assumption that $\Lambda_{\mathfrak{m}}(Y)$ lies in $\mathfrak{so}(\mathfrak{m})$, since this means that the endomorphism $\Lambda_{\mathfrak{m}}(Y)$ is skew symmetric with respect to the inner product of $\mathfrak{m}$. $\qquad\square$

For a general map $\Lambda_{\mathfrak{m}}$, this is all one can say. We are interested in the one parameter family of connections defined by

$$\Lambda_{\mathfrak{m}}^t(X)Y := t \cdot [X,Y]_{\mathfrak{m}}\,.$$

It is well known that $t = 0$ corresponds to the *canonical connection* ∇^0, which, by the Ambrose-Singer theorem, is the unique metric connection on M such that its torsion and curvature are parallel, $\nabla^0 T^0 = \nabla^0 R^0 = 0$. By Lemma 2.1, the torsion of ∇^0 is a 3-form if and only if M is *naturally reductive*.

Definition 2.1. A homogeneous Riemannian metric on M is said to be *naturally reductive* (with respect to G) if the map $[X, -]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew symmetric,

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle \ = \ 0 \text{ for all } X, Y, Z \in \mathfrak{m} \,.$$

Note that if $G_1 \subset G_2$ are two transitive groups of isometries of M, then the properties of being naturally reductive with respect to G_1 and G_2 are independent of each other.

Remark 2.1. Under the assumption that M is naturally reductive, the right-hand side in the criterion of Lemma 2.1 vanishes, and the remaining condition may be restated – using the skew symmetry of $\Lambda_{\mathfrak{m}}(X)$ and $\Lambda_{\mathfrak{m}}(Z)$ – as $\langle Y, \Lambda_{\mathfrak{m}}(X)Z + \Lambda_{\mathfrak{m}}(Z)X \rangle = 0$. Since this equation has to hold for all X, Y and Z in $\mathfrak{m}$, we obtain that the torsion is a 3-form if and only if $\Lambda_{\mathfrak{m}}(X)X = 0$ for all $X \in \mathfrak{m}$.

If M is naturally reductive, then the torsion of the family ∇^t of connections is given by the simple expression

$$T^t(X, Y) \ = \ (2t - 1) \, [X, Y]_{\mathfrak{m}} \,.$$

One sees that the Levi-Civita connection is attained for $t = 1/2$. The general formula for the connection ∇^t is

$$(1) \hspace{4cm} \nabla^t_X Y \ = \ \nabla^0_X Y + t \, [X, Y]_{\mathfrak{m}} \,.$$

Notice that for a symmetric space, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, so all connections of this one-parameter family coincide and are equal to the Levi-Civita connection.

Assumption 2.1. We will assume that $M = G/H$ is naturally reductive with respect to G.

We begin by computing a few characteristic entities for this family of connections, which will be needed in the subsequent sections. We start by recalling a theorem of B. Kostant.

Theorem 2.1 ([Kos56])**.** *Suppose G acts effectively on $M = G/H$. If the inner product $\langle \ , \ \rangle$ is naturally reductive with respect to G, then $\tilde{\mathfrak{g}} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ is an ideal in $\mathfrak{g}$ whose corresponding subgroup $\tilde{G} \subset G$ is transitive on M, and there exists a unique $\mathrm{Ad}\,(\tilde{G})$ invariant, symmetric, non degenerate, bilinear form Q on $\tilde{\mathfrak{g}}$ (not necessarily positive definite) such that*

$$Q(\mathfrak{h} \cap \tilde{\mathfrak{g}}, \mathfrak{m}) \ = \ 0 \quad and \quad Q|_{\mathfrak{m}} \ = \ \langle \ , \ \rangle \,,$$

where $\mathfrak{h} \cap \tilde{\mathfrak{g}}$ will be the isotropy algebra in $\tilde{\mathfrak{g}}$. Conversely, if G is connected, then, for any $\mathrm{Ad}\,(G)$ invariant, symmetric, non degenerate, bilinear form Q on $\mathfrak{g}$, which is non degenerate on $\mathfrak{h}$ and positive definite on $\mathfrak{m} := \mathfrak{h}^{\perp}$, the metric on M defined by $Q|_{\mathfrak{m}}$ is naturally reductive. In this case, $\mathfrak{g} = \tilde{\mathfrak{g}}$. $\hfill \square$

Assumption 2.2. We shall assume from now on that G acts transitively on M (thus, $\mathfrak{g} = \tilde{\mathfrak{g}}$) and use the $\mathrm{Ad}\,(G)$ invariant extension Q of the inner product $\langle \ , \ \rangle$ as well as its restriction $Q|_{\mathfrak{h}} =: Q_{\mathfrak{h}}$ to $\mathfrak{h}$ where needed without further comment.

Lemma 2.2. *The curvature of the connection ∇^t is given by*

$$R^t(X, Y)Z \ = \ t^2 \, [X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} + t^2 \, [Y, [Z, X]_{\mathfrak{m}}]_{\mathfrak{m}} + t \, [Z, [X, Y]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z, [X, Y]_{\mathfrak{h}}] \,.$$

If $Z_i, \dots, Z_n$ is an orthonormal basis of $\mathfrak{m}$, the Ricci tensor and the scalar curvature are

$$\mathrm{Ric}^t(X, Y) \ = \ \sum_i (t - t^2) \, \langle [X, Z_i]_{\mathfrak{m}}, [Y, Z_i]_{\mathfrak{m}} \rangle + Q_{\mathfrak{h}}([X, Z_i], [Y, Z_i])$$

$$\mathrm{Scal}^t \ = \ \sum_{i,j} (t - t^2) \, \langle [Z_i, Z_j]_{\mathfrak{m}}, [Z_i, Z_j]_{\mathfrak{m}} \rangle + Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) \,.$$

Proof. The formula for the curvature follows immediately from the general formula given before. In particular, it implies

$$\left\langle R^t(X, Z)Z, Y \right\rangle \ = \ (t - t^2) \, \langle [X, Z]_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}} \rangle + \langle [Z, [X, Z]_{\mathfrak{h}}], Y \rangle \,.$$

Using the $\mathrm{Ad}(G)$ invariant extension Q of the inner product $\langle\,,\,\rangle$ and the fact that $\mathfrak{m}$ is then perpendicular to $\mathfrak{h}$, we may rewrite the latter term as

$$\langle[Z,[X,Z]_{\mathfrak{h}}],Y\rangle \;=\; Q([Z,[X,Z]_{\mathfrak{h}}],Y) \;=\; Q([X,Z]_{\mathfrak{h}},[Y,Z]) \;=\; Q_{\mathfrak{h}}([X,Z],[Y,Z])\,.$$

Thus, we obtain

$$\langle R^t(X,Z)Z,Y\rangle \;=\; (t-t^2)\,Q_{\mathfrak{m}}([X,Z],[Y,Z]) + Q_{\mathfrak{h}}([X,Z],[Y,Z])$$

and the formula for the Ricci tensor by $\mathrm{Ric}^t(X,Y) = \sum\langle R^t(X,Z_i)Z_i,Y\rangle$. The expression for the scalar curvature is obtained by contraction relative to X and Y. $\qquad\square$

At a later stage, we will give a further expression for the Ricci tensor due to Wang and Ziller ([WZ85]). For the time being, we observe that the connection with $t=1$ has also special properties, for example, it has the same Ricci tensor than the canonical connection. This is why we propose to call it the *anticanonical* connection. We compute the covariant derivative of the torsion tensor.

Lemma 2.3. *As a map* $\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$*, the covariant derivative of* T *is*

$$(\nabla_Z^t T^t)(X,Y) \;=\; t(2t-1)\big([X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[Z,X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}\big)\,.$$

Proof. By definition, the covariant derivative is given by

$$(\nabla_Z^t T^t)(X,Y) \;=\; \nabla_Z^t(T^t(X,Y)) - T^t(\nabla_Z^t X,Y) - T^t(X,\nabla_Z^t Y)\,.$$

We insert the expression for ∇^t from equation (1)

$$
\begin{aligned}
(\nabla_Z^t T^t)(X,Y) \;=\;& \nabla_Z^0(T^t(X,Y)) + t[Z,T^t(X,Y)]_{\mathfrak{m}} - T^t(\nabla_Z^0 X + t[Z,X]_{\mathfrak{m}},Y) \\
& -T^t(X,\nabla_Z^0 Y + t[Z,Y]_{\mathfrak{m}}) \\
\;=\;& \nabla_Z^0(T^t(X,Y)) - T^t(\nabla_Z^0 X,Y) - T^t(X,\nabla_Z^0 Y) \\
& +t(2t-1)\big([X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[Z,X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}\big)\,.
\end{aligned}
$$

But the third line may be rewritten as $-(2t-1)(\nabla_Z^0 T^0)(X,Y)$, which vanishes by the Ambrose-Singer theorem. $\qquad\square$

For the first time we encounter here an expression that will play an important role at different places. Let us define

$$
\begin{aligned}
\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) \;:=\;& [X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[Z,X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}\,, \\
\mathrm{Jac}_{\mathfrak{h}}(X,Y,Z) \;:=\;& [X,[Y,Z]_{\mathfrak{h}}] + [Y,[Z,X]_{\mathfrak{h}}] + [Z,[X,Y]_{\mathfrak{h}}]\,.
\end{aligned}
$$

Notice that the summands of $\mathrm{Jac}_{\mathfrak{h}}(X,Y,Z)$ automatically lie in $\mathfrak{m}$ by the assumption that M is reductive. The Jacobi identity for $\mathfrak{g}$ implies $\langle\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) + \mathrm{Jac}_{\mathfrak{h}}(X,Y,Z),\mathfrak{m}\rangle = 0$. As the connection ∇^t is metric, the covariant derivatives of T viewed as a $(0,3)$- resp. $(1,2)$-tensor are related by

$$(2)\qquad (\nabla_Z^t T^t)(X,Y,V) \;=\; \big\langle(\nabla_Z^t T^t)(X,Y),V\big\rangle \;=\; t(2t-1)\,\langle\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z),V\rangle\,.$$

For completeness, we recall the formula for the exterior derivative of a differential form in terms of a connection with torsion.

Lemma 2.4. *If* ω *is an* r*-form, then*

$$
\begin{aligned}
(d\omega)(X_0,\ldots,X_r) \;=\;& \sum_{i=0}^{r}(-1)^i(\nabla_{X_i}\omega)(X_0,\ldots,\hat{X}_i,\ldots,X_r) \\
& - \sum_{0\le i<j\le r}(-1)^{i+j}\omega(T(X_i,X_j),X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r)\,.
\end{aligned}
$$

Proof. We start with the general formula for the derivative of an r-form ω (see, for example, [KN91, Prop. 3.11]),

$$
\begin{aligned}
(d\omega)(X_0, \ldots, X_r) \quad = \quad & \sum_{i=0}^{r} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_r)) \\
+ \quad & \sum_{0 \le i < j \le r} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r) \,.
\end{aligned}
$$

In the first line, we express every summand in terms of the covariant derivative of ω, i.e.,

$$
\begin{aligned}
X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, , X_r)) \quad = \quad & (\nabla_{X_i}\omega)(X_1, \ldots, X_r) + \omega(\nabla_{X_i}X_0, \ldots, \hat{X}_i, \ldots, X_r) + \ldots + \\
+ \quad & \omega(X_0, X_1, \ldots, \nabla_{X_i}X_r) \,.
\end{aligned}
$$

A simple rearrangement of terms together with the definition $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ of the torsion yields the result. $\qquad\square$

Lemma 2.5. *The codifferential of the 3-form T^t vanishes, $\delta T^t = 0$, while its outer derivative is given by $dT^t(X, Y, Z, V) = 2(2t - 1) \cdot \langle \mathrm{Jac}_{\mathfrak{m}}(X, Y, Z), V \rangle$.*

Proof. For the first claim, one deduces from equation (2) that $X \lrcorner \nabla^t_X T^t = 0$. Then it follows for the orthonormal basis $Z_i, \ldots, Z_n$ of $\mathfrak{m}$ that

$$
\delta^t T^t \quad = \quad \sum_{i=1}^{n} Z_i \lrcorner \nabla^t_{Z_i} T^t \quad = \quad 0 \,.
$$

In particular, the divergence of T with respect to ∇^t coincides with its Riemannian divergence (a more general fact, see [FI01]), $\delta^t T^t = \delta^{1/2} T^t = 0$. Hence we shall drop the superscript, as we did in the statement of the lemma. The second claim follows from Lemma 2.4 by a simple algebraic computation. $\qquad\square$

Remark 2.2. We finish this section with a remark about the connection between the torsion and the Lie algebra structure. If some torsion 3-form T is given as a fundamental datum and is to be the torsion of the canonical connection of some space with naturally reductive metric, then the $\mathfrak{m}$-part of the commutators $[\mathfrak{m}, \mathfrak{m}]$ may be reconstructed by

$$
[X, Y]_{\mathfrak{m}} \quad = \quad - \sum_i T(X, Y, Z_i) Z_i \,.
$$

This formula is fundamental for the point of view taken in the article [Kos99] (formula 1.23). The full Lie algebra structure of $\mathfrak{g}$ can now be viewed as consisting of the torsion 3-form, the isotropy representation and the subalgebra structure of $\mathfrak{h}$, with some compatibility condition resulting from the Jacobi identity. This point of view will be useful in the last section, where we will study examples.

3. The Dirac operator of the family of connections ∇^t

3.1. **General remarks and formal self adjointness.** Assume that there exists a homogeneous spin structure on M, i. e., a lift $\widetilde{\mathrm{Ad}} : H \to \mathrm{Spin}(\mathfrak{m})$ of the isotropy representation such that the diagram

commutes, where λ denotes the spin covering. Moreover, we denote by $\widetilde{\mathrm{ad}}$ the corresponding lift into $\mathfrak{spin}(\mathfrak{m})$ of the differential $\mathrm{ad} : \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ of Ad. Let $\kappa : \mathrm{Spin}(\mathfrak{m}) \to \mathrm{GL}(\Delta_{\mathfrak{m}})$ be the spin

representation, and identify sections of the spinor bundle $S = G \times_{\widetilde{\mathrm{Ad}}} \Delta_{\mathfrak{m}}$ with functions $\psi : G \to \Delta_{\mathfrak{m}}$ satisfying

$$\psi(gh) = \kappa(\widetilde{\mathrm{Ad}}\,(h^{-1}))\psi(g)\,.$$

For any G invariant connection defined by a map $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$, we consider its lift $\tilde{\Lambda}_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{spin}(\mathfrak{m})$, which is given by $\tilde{\Lambda}_{\mathfrak{m}} := d\lambda^{-1} \circ \Lambda_{\mathfrak{m}}$. Then the the covariant derivative on spinors may be expressed as ([Ike75, Lemma 2])

$$(3) \qquad \nabla_Z \psi = Z(\psi) + \tilde{\Lambda}_{\mathfrak{m}}(Z)\psi$$

and thus the Dirac operator associated with this connection has the form

$$(4) \qquad D\psi = \sum_i Z_i \cdot Z_i(\psi) + Z_i \cdot \tilde{\Lambda}_{\mathfrak{m}}(Z_i)\psi\,,$$

where $Z_1, \ldots, Z_n$ denotes any orthonormal basis of $\mathfrak{m}$. In the same article, Ikeda states a criterion for the formal self adjointness of this operator. We restate the result here, since there is some confusion about the assumptions on the scalar product in the original version.

Proposition 3.1. *Let $M = G/H$ be a homogeneous reductive manifold with a homogeneous spin structure, $\langle\, , \,\rangle$ the scalar product on $\mathfrak{m}$ induced by the Riemannian metric on M, and ∇ the G invariant metric connection defined by some map $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$. Then the Dirac operator D associated with the connection ∇ is formally self adjoint if and only if for any vector $X \in \mathfrak{m}$ and any orthonormal basis $Z_1, \ldots, Z_n$ of $\mathfrak{m}$, one has*

$$(*) \qquad \sum_i \langle \Lambda_{\mathfrak{m}}(Z_i)X, Z_i \rangle = \sum_i \langle [Z_i, X]_{\mathfrak{m}}, Z_i \rangle\,.$$

In particular, this condition is always satisfied if the torsion $T(X, Y, Z)$ is totally skew symmetric. If the metric $\langle\, , \,\rangle$ is in addition naturally reductive, condition $(*)$ is equivalent to $\sum \Lambda_{\mathfrak{m}}(Z_i)Z_i = 0$.

Proof. By a result of Friedrich and Sulanke ([FS79]), the Dirac operator D^{∇} associated with any metric connection ∇ is formally self adjoint if and only if the ∇-divergence of any vector X coincides with its Riemannian divergence,

$$\mathrm{div}^{\nabla}(X) := \sum_i \langle Z_i, \nabla_{Z_i} X \rangle = \sum_i \langle Z_i, \nabla^{\mathrm{LC}}_{Z_i} X \rangle =: \mathrm{div}(X)\,,$$

where ∇^{LC} denotes the Levi-Civita connection. But for any vector X, the covariant derivatives are related by

$$\nabla_{Z_i} X = \nabla^{\mathrm{LC}}_{Z_i} X + \frac{1}{2}T(Z_i, X)\,,$$

thus equality of divergences holds if and only if

$$\sum_i \langle T(Z_i, X), Z_i \rangle = 0\,.$$

Inserting the general formula for the torsion and using the fact that $\langle \Lambda_{\mathfrak{m}}(X)Z_i, Z_i \rangle = 0$, one checks that this is equivalent to condition $(*)$. Since $\langle T(Z_i, X), Z_i \rangle = T(Z_i, X, Z_i)$, condition $(*)$ is always fulfilled if the $(0,3)$-tensor T is totally skew symmetric. Alternatively, one easily deduces equation $(*)$ from the antisymmetry condition in Lemma 2.1 by a contraction. Finally, if the metric is naturally reductive, the right-hand side of $(*)$ vanishes, and by the antisymmetry of $\Lambda_{\mathfrak{m}}(Z_i)$ one obtains $\langle X, \sum \Lambda_{\mathfrak{m}}(Z_i)Z_i \rangle = 0$. This finishes the proof. $\square$

Returning to the family ∇^t, our aim is to rewrite the connection term of the Dirac operator in equation (4) as an element of the Clifford algebra $\mathcal{C}(\mathfrak{m})$. Basically this amounts to the identification of $\mathfrak{spin}(\mathfrak{m})$ with the elements of second degree in $\mathcal{C}(\mathfrak{m})$. We implement the Clifford relations via $Z_i \cdot Z_j + Z_j \cdot Z_i = -\delta_{ij}$, in contrast to [Kos99] (see [Fri00] for notational details). The following lemma due to Parthasarathy expresses the lift of the isotropy representation as an element of the Clifford algebra.

Lemma 3.1 ([Par72, 2.1]). *For any element Y in $\mathfrak{h}$, one has*

$$\widetilde{\mathrm{ad}}\,(Y) \;=\; \frac{1}{4}\sum_{i,j=1}^{n} \langle [Y,Z_i], Z_j\rangle\, Z_i\cdot Z_j\,.$$

$\square$

Similarly, any skew symmetric map $\Lambda_{\mathfrak{m}}(X) : \mathfrak{m}\to\mathfrak{m}$ may be expanded in the standard basis E_{ij} of $\mathfrak{so}(\mathfrak{m})$ as

$$\Lambda_{\mathfrak{m}}(X) \;=\; \sum_{i<j} \langle \Lambda_{\mathfrak{m}}(X)Z_i, Z_j\rangle\, E_{ij}.$$

Since E_{ij} lifts to $Z_i\cdot Z_j/2$ in the Clifford algebra, we obtain in complete analogy to the Parthasarathy Lemma:

Lemma 3.2. *For any map $\Lambda_{\mathfrak{m}} : \mathfrak{m}\to\mathfrak{so}(\mathfrak{m})$, one has*

$$\tilde{\Lambda}_{\mathfrak{m}}(X) \;=\; \frac{1}{2}\sum_{i<j} \langle \Lambda_{\mathfrak{m}}(X)Z_i, Z_j\rangle\, Z_i\cdot Z_j \;=\; \frac{1}{4}\sum_{i,j} \langle \Lambda_{\mathfrak{m}}(X)Z_i, Z_j\rangle\, Z_i\cdot Z_j\,.$$

$\square$

In particular, the image of $\Lambda_{\mathfrak{m}}^1(Z_i) = [Z_i, -]_{\mathfrak{m}}$ in $\mathcal{C}(\mathfrak{m})$ may be written

$$\tilde{\Lambda}_{\mathfrak{m}}^1(Z_i) \;=\; \frac{1}{4}\sum_{j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_j\cdot Z_k\,.$$

Thus, by defining the element

$$H \;:=\; \sum_{i=1}^{n} Z_i\cdot \tilde{\Lambda}_{\mathfrak{m}}^1(Z_i) \;=\; \frac{1}{4}\sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_i\cdot Z_j\cdot Z_k \;=\; \frac{3}{2}\sum_{i<j<k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_i\cdot Z_j\cdot Z_k\,,$$

we can rewrite the Dirac operator corresponding to the connection ∇^t from equation (4) as

$$(5) \qquad\qquad D^t\psi \;=\; \sum_i Z_i\cdot Z_i(\psi) + t\cdot H\cdot\psi\,.$$

Remark 3.1. We identify differential forms with elements of the Clifford algebra by

$$\sum_{i_1<\ldots<i_r} \omega_{1\ldots r}\, Z_{i_1}\wedge\ldots\wedge Z_{i_r} \;\longmapsto\; \sum_{i_1<\ldots<i_r} \omega_{1\ldots r}\, Z_{i_1}\cdot\ldots\cdot Z_{i_r}\,.$$

Thus, the torsion form $T^t(X,Y,Z) = (2t-1)\,\langle [X,Y]_{\mathfrak{m}}, Z\rangle$ induces the element

$$T^t \;=\; (2t-1)\sum_{i<j<k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_i\cdot Z_j\cdot Z_k$$

of the Clifford algebra, which differs from H only by a numerical factor,

$$T^t \;=\; \frac{2\,(2t-1)}{3}H\,.$$

The simplicity of equation (5) is the main reason why we prefer to work with the element H instead of T^t.

3.2. **The cubic element H, its square and the Casimir operator.** It is the cubic element H inside the Clifford algebra $\mathcal{C}(\mathfrak{m})$ which suggested the name "cubic Dirac operator" to B. Kostant. We see that the fact that H is of degree 3 inside $\mathcal{C}(\mathfrak{m})$ does not depend on the particular choice for $\Lambda_{\mathfrak{m}}$. The square of H will play an eminent role in our considerations, both for a Kostant-Parthasarathy type formula and for general vanishing theorems. Notice that the square of *any* element of degree 3 inside $\mathcal{C}(\mathfrak{m})$ has only terms of degree zero and 4.

Proposition 3.2. *The terms of degree zero and 4 of H^2 are given by*

$$(H^2)_0 = \frac{3}{8} \sum_{i,j} \langle [Z_i, Z_j]_{\mathfrak{m}}, [Z_i, Z_j]_{\mathfrak{m}} \rangle \,,$$

$$(H^2)_4 = -\frac{9}{2} \sum_{i<j<k<l} \langle Z_i, \mathrm{Jac}_{\mathfrak{m}}(Z_j, Z_k, Z_l) \rangle \, Z_i \cdot Z_j \cdot Z_k \cdot Z_l \,.$$

The first formula is valid for all $n \geq 3$, while the second holds only for $n \geq 5$. For $n = 3, 4$, one has $(H^2)_4 = 0$.

Proof. The contributions of degree zero in H^2 are exactly the squares of the summands of H. Because of $(Z_i \cdot Z_j \cdot Z_k)^2 = 1$, we have

$$(H^2)_0 = \frac{9}{4} \sum_{i<j<k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle^2 = \frac{9}{24} \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \,.$$

For fixed i, j, the sum over k is the coordinate expansion of the scalar product $\langle [Z_i, Z_j]_{\mathfrak{m}}, [Z_i, Z_j]_{\mathfrak{m}} \rangle$, thus

$$(H^2)_0 = \frac{3}{8} \sum_{i,j} \langle [Z_i, Z_j]_{\mathfrak{m}}, [Z_i, Z_j]_{\mathfrak{m}} \rangle \,,$$

as claimed. Contributions of degree 4 occur if $Z_i \cdot Z_j \cdot Z_k$ is multiplied by $Z_{i'} \cdot Z_{j'} \cdot Z_{k'}$ with exactly one common index. Since this requires at least 5 different indices, it follows that there are no terms of fourth degree for $n \leq 4$. For the moment, put aside the overall factor $9/4$ of H^2. We explain the occurrence of the term proportional to $Z_{1234} := Z_1 \cdot Z_2 \cdot Z_3 \cdot Z_4$ in detail, the others are obtained in a similar way. Since H consists of ordered tuples proportional to $Z_{ijk} := Z_i \cdot Z_j \cdot Z_k$, $i < j < k$, the only way to obtain a term in Z_{1234} is to multiply Z_{12k} by Z_{34k}, Z_{13k} by Z_{24k} and Z_{14k} by Z_{23k} for any index $k \geq 5$. First we notice that the order of multiplication is irrelevant, since

$$Z_{12k} \cdot Z_{34k} = Z_{34k} \cdot Z_{12k}, \quad Z_{13k} \cdot Z_{24k} = Z_{24k} \cdot Z_{13k}, \quad \text{and } Z_{14k} \cdot Z_{23k} = Z_{23k} \cdot Z_{14k} \,.$$

Every term will thus have multiplicity two. In the next step, these products have to be rearranged in order to be proportional to Z_{1234}:

$$Z_{12k} \cdot Z_{34k} = -Z_{1234}, \quad Z_{13k} \cdot Z_{24k} = +Z_{1234}, \quad Z_{14k} \cdot Z_{23k} = -Z_{1234} \,.$$

The total contribution coming from the products Z_{12k} by Z_{34k} is thus

$$(*) := -2 \, Z_{1234} \sum_{k \geq 5} \langle [Z_1, Z_2]_{\mathfrak{m}}, Z_k \rangle \langle [Z_3, Z_4]_{\mathfrak{m}}, Z_k \rangle \,.$$

This is equal to the sum over all k, since the additional terms are zero. However, it shows that the sum is precisely the expansion of the scalar product $\langle [Z_1, Z_2]_{\mathfrak{m}}, [Z_3, Z_4]_{\mathfrak{m}} \rangle$:

$$(*) = -2 \, Z_{1234} \sum_{k=1}^{n} \langle [Z_1, Z_2]_{\mathfrak{m}}, Z_k \rangle \langle [Z_3, Z_4]_{\mathfrak{m}}, Z_k \rangle = -2 \, Z_{1234} \langle [Z_1, Z_2]_{\mathfrak{m}}, [Z_3, Z_4]_{\mathfrak{m}} \rangle \,.$$

After a similar simplification of the other two contributions, the fourth degree term in H^2 proportional to Z_{1234} is finally equal to

$$(**) := 2 \left[- \langle [Z_1, Z_2]_{\mathfrak{m}}, [Z_3, Z_4]_{\mathfrak{m}} \rangle + \langle [Z_1, Z_3]_{\mathfrak{m}}, [Z_2, Z_4]_{\mathfrak{m}} \rangle - \langle [Z_1, Z_4]_{\mathfrak{m}}, [Z_2, Z_3]_{\mathfrak{m}} \rangle \right] \cdot Z_{1234} \,.$$

This, in turn, may be rewritten as

$$(**) = -2 \langle Z_1, \mathrm{Jac}_{\mathfrak{m}}(Z_2, Z_3, Z_4) \rangle \cdot Z_{1234} \,.$$

Putting back in the factor $9/4$, we get the factor $-9/2$ as stated in the lemma. $\qquad \square$

For later reference, we compute the anticommutator of H with an element Z_l for arbitrary l.

Lemma 3.3. *For any l, one has* $H \cdot Z_l + Z_l \cdot H = -\frac{3}{2} \sum_{i,j} \langle Z_l, [Z_i, Z_j]_{\mathfrak{m}} \rangle \, Z_i \cdot Z_j$.

Proof. By definition,

$$H \cdot Z_l + Z_l \cdot H = \frac{1}{4} \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \left(Z_i \cdot Z_j \cdot Z_k \cdot Z_l + Z_l \cdot Z_i \cdot Z_j \cdot Z_k \right).$$

If all four indices i, j, k, l are pairwise different,

$$Z_i \cdot Z_j \cdot Z_k \cdot Z_l = -Z_l \cdot Z_i \cdot Z_j \cdot Z_k,$$

and the corresponding summand vanishes. Thus, the sum may be split into those parts where l is one of the indices i, j and k, respectively:

$$\begin{aligned}
H \cdot Z_l + Z_l \cdot H &= \frac{1}{4} \sum_{j,k} \langle [Z_l, Z_j]_{\mathfrak{m}}, Z_k \rangle \left(Z_l \cdot Z_j \cdot Z_k \cdot Z_l + Z_l \cdot Z_l \cdot Z_j \cdot Z_k \right) \\
&\quad + \frac{1}{4} \sum_{i,k} \langle [Z_i, Z_l]_{\mathfrak{m}}, Z_k \rangle \left(Z_i \cdot Z_l \cdot Z_k \cdot Z_l + Z_l \cdot Z_i \cdot Z_l \cdot Z_k \right) \\
&\quad + \frac{1}{4} \sum_{i,j} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_l \rangle \left(Z_i \cdot Z_j \cdot Z_l \cdot Z_l + Z_l \cdot Z_i \cdot Z_j \cdot Z_l \right).
\end{aligned}$$

We simplify the mixed products to get

$$\begin{aligned}
H \cdot Z_l + Z_l \cdot H &= -\frac{1}{2} \sum_{j,k} \langle [Z_l, Z_j]_{\mathfrak{m}}, Z_k \rangle Z_j \cdot Z_k + \frac{1}{2} \sum_{i,k} \langle [Z_i, Z_l]_{\mathfrak{m}}, Z_k \rangle Z_i \cdot Z_k \\
&\quad - \frac{1}{2} \sum_{i,j} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_l \rangle Z_i \cdot Z_j.
\end{aligned}$$

Using the invariance property of the scalar product and renaming the summation indices, this is easily seen to be the desired expression. $\square$

Finally, we compute the image of the quadratic Casimir operator of $\mathfrak{h}$ inside the Clifford algebra. Since the $\mathrm{Ad}\,(G)$ invariant extension Q of $\langle\,,\,\rangle$ is not necessarily positive definite when restricted to $\mathfrak{h}$, it is more appropriate to work with dual rather than with orthonormal bases. So pick bases X_i, Y_i of $\mathfrak{h}$ wich are dual with respect to $Q_{\mathfrak{h}}$, i. e., $Q_{\mathfrak{h}}(X_i, Y_j) = \delta_{ij}$. The lift of the Casimir operator of $\mathfrak{h}$ is defined as

$$\widetilde{C}_{\mathfrak{h}} = -\sum_i \widetilde{\mathrm{ad}}\,(X_i) \circ \widetilde{\mathrm{ad}}\,(Y_i).$$

By the Parthasarathy Lemma (Lemma 3.1),

$$\widetilde{\mathrm{ad}}\,(X_i) = \frac{1}{4} \sum_{j,k} \langle [X_i, Z_j], Z_k \rangle Z_j \cdot Z_k$$

and similarly for $\widetilde{\mathrm{ad}}\,(Y_i)$. Thus,

$$\widetilde{C}_{\mathfrak{h}} = -\frac{1}{16} \sum_i \sum_{j,k,l,p} \langle [X_i, Z_j], Z_k \rangle \langle [Y_i, Z_l], Z_p \rangle Z_j \cdot Z_k \cdot Z_l \cdot Z_p.$$

We may get rid of the sum over i immediately. Since $\mathfrak{m}$ is orthogonal to $\mathfrak{h}$, we can rewrite $\widetilde{C}_{\mathfrak{h}}$ as

$$\begin{aligned}
\widetilde{C}_{\mathfrak{h}} &= -\frac{1}{16} \sum_i \sum_{j,k,l,p} Q([X_i, Z_j], Z_k) Q([Y_i, Z_l], Z_p) Z_j \cdot Z_k \cdot Z_l \cdot Z_p \\
&= -\frac{1}{16} \sum_i \sum_{j,k,l,p} Q(X_i, [Z_j, Z_k]) Q(Y_i, [Z_l, Z_p]) Z_j \cdot Z_k \cdot Z_l \cdot Z_p.
\end{aligned}$$

For fixed j, k, l and p, the sum over i is again the expansion of the $\mathfrak{h}$ part of $Q([Z_j, Z_k], [Z_l, Z_p])$, yielding

$$(6) \qquad \widetilde{C}_{\mathfrak{h}} = -\frac{1}{16} \sum_{j,k,l,p} Q_{\mathfrak{h}}([Z_j, Z_k], [Z_l, Z_p]) Z_j \cdot Z_k \cdot Z_l \cdot Z_p.$$

This expression has the advantage that it does not contain the dual bases X_i, Y_i any more. It turns out that $\widetilde{C}_{\mathfrak{h}}$ has no second degree term, for such a term would occur if the two index pairs (j, k) and (l, p) had exactly one common index, for example, $j = l$. But such a term would appear twice, namely, as $Z_j \cdot Z_k \cdot Z_j \cdot Z_p$ and as $Z_j \cdot Z_p \cdot Z_j \cdot Z_k$, and these cancel out each other.

Proposition 3.3. *The terms of degree zero and 4 of $\widetilde{C}_{\mathfrak{h}}$ are given for all $n \geq 3$ by*

$$(\widetilde{C}_{\mathfrak{h}})_0 = \frac{1}{8} \sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]),$$

$$(\widetilde{C}_{\mathfrak{h}})_4 = -\frac{1}{2} \sum_{i<j<k<l} \langle Z_i, \mathrm{Jac}_{\mathfrak{h}}(Z_j, Z_k, Z_l) \rangle \, Z_i \cdot Z_j \cdot Z_k \cdot Z_l.$$

In particular, $(\widetilde{C}_{\mathfrak{h}})_4$ vanishes identically for $n \leq 3$, but not for $n = 4$.

Proof. As the form of the result suggests, the proof is similar to the computation of H^2 (Proposition 3.2). This is why we shall be brief. For the zero degree term, $(j, k) = (l, p)$, and each term of this kind appears twice, thus

$$(\widetilde{C}_{\mathfrak{h}})_0 = -\frac{1}{8} \sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) Z_i \cdot Z_j \cdot Z_i \cdot Z_j.$$

Since $Z_i \cdot Z_j \cdot Z_i \cdot Z_j = -1$, we obtain the first part of the proposition. For the fourth degree contribution, rewrite the Casimir operator as

$$(7) \qquad \widetilde{C}_{\mathfrak{h}} = -\frac{1}{4} \sum_{j<k, l<p} Q_{\mathfrak{h}}([Z_j, Z_k], [Z_l, Z_p]) Z_j \cdot Z_k \cdot Z_l \cdot Z_p.$$

Then the index pairs (j, k) and (l, p) have to be completely disjoint. Agein we look only at the term that is proportional to $Z_{1234} := Z_1 \cdot Z_2 \cdot Z_3 \cdot Z_4$. It may be obtained by multiplying Z_{12} by Z_{34}, Z_{13} by Z_{24} and Z_{14} by Z_{23}. Again, these elements commute, so we only need to consider each product in the order of multiplication just given and count it twice. Restoring the order of indices in these products, one sees that the term in $(\widetilde{C}_{\mathfrak{h}})_4$ proportional to Z_{1234} looks like

$$(*) := -\frac{2}{4} [Q_{\mathfrak{h}}([Z_1, Z_2], [Z_3, Z_4]) - Q_{\mathfrak{h}}([Z_1, Z_3], [Z_2, Z_4]) + Q_{\mathfrak{h}}([Z_1, Z_4], [Z_2, Z_3])] \cdot Z_{1234}.$$

By the properties of Q, the first scalar product may be formulated differently:

$$Q_{\mathfrak{h}}([Z_1, Z_2], [Z_3, Z_4]) = Q([Z_1, Z_2], [Z_3, Z_4]_{\mathfrak{h}}) = Q(Z_1, [Z_2, [Z_3, Z_4]_{\mathfrak{h}}]).$$

Rewriting the other two products in a similar way, we see that

$$(*) = -\frac{1}{2} Q(Z_1, \mathrm{Jac}_{\mathfrak{h}}(Z_2, Z_3, Z_4)) \cdot Z_{1234}.$$

$\square$

3.3. A Kostant-Parthasarathy type formula for $(D^t)^2$.

If $M = G/H$ is a symmetric space, it is well known that besides the general Schrödinger-Lichnerowicz formula for D^2, which is valid on any Riemannian manifold, there exists a formula expressing D^2 in terms of Casimir operators due to Parthasarathy (see also [Kos99, Remark 1.63]). The Dirac operator D is defined relative to the Levi-Civita connection, which coincides with our one-parameter family ∇^t, and $\langle \, , \, \rangle$ denotes an Ad (G) invariant scalar product on $\mathfrak{g}$ whose restriction to $\mathfrak{m}$ is positive definite. Let Scal be the scalar curvature of the symmetric space M and Ω_G the Casimir operator of G, viewed as a second order differential operator.

Theorem 3.1 ([Par72, Prop.3.1], [Fri00, Ch. 3]). *On a symmetric space $M = G/H$, one has*

$$D^2 = \Omega_G + \frac{1}{8}\mathrm{Scal},$$

and the scalar curvature may be rewritten as $\mathrm{Scal} = 8 \cdot (\langle \varrho_{\mathfrak{g}}, \varrho_{\mathfrak{g}} \rangle - \langle \varrho_{\mathfrak{h}}, \varrho_{\mathfrak{h}} \rangle)$.

This formula is the starting point for vanishing theorems, the realization of discrete series representations in the kernel of D, and it allows the computation of the full spectrum of D on M. If we now go back to the situation studied in this article, i. e., a reductive homogeneous space G/H endowed with a naturally reductive metric $\langle \, , \, \rangle$ on $\mathfrak{m}$, then, a priori, the steps in the proof of Theorem 3.1 cannot be performed any longer. To prove a Kostant-Parthasarathy type formula in this situation, we recall the general expression for the Dirac operator associated with the connection ∇^t from equation (5) and split it into the terms coming from the canonical connection and the 3-form H, respectively:

$$(8) \qquad D^t\psi \;=\; \sum_i Z_i \cdot Z_i(\psi) + Z_i \cdot \tilde{\Lambda}^t_{\mathfrak{m}}(Z_i)\psi \;=:\; D^0\psi + D^t_H\psi \,.$$

First, notice that the equivariance property of spinors implies that the action on spinors of vector fields coming from $\mathfrak{m}$ is by "true" differential operators, while the action of vector fields in $\mathfrak{h}$ is in fact purely algebraic.

Lemma 3.4. *Let ψ be a spinor, i. e., a section in $S = G \times_{\tilde{\mathrm{Ad}}} \Delta_{\mathfrak{m}}$ and X an element of $\mathfrak{h}$, identified with the left invariant vector field it induces. Then*

$$X(\psi) \;=\; -\tilde{\mathrm{ad}}\,(X) \cdot \psi \,,$$

where $\tilde{\mathrm{ad}}\,(X) \cdot \psi$ denotes the Clifford product of the spinor ψ with the element $\tilde{\mathrm{ad}}\,(X) \subset \mathfrak{spin}(\mathfrak{m}) \subset \mathcal{C}(\mathfrak{m})$.

Proof. We identify ψ with a map $\psi : G \to \Delta_{\mathfrak{m}}$ such that $\psi(gh) = \kappa(\tilde{\mathrm{Ad}}\,(h^{-1}))\psi(g)$ for all $g \in G$ and $h \in H$. Then one has

$$X\psi(g) \;=\; \frac{d}{ds}\psi(ge^{sX})\big|_{s=0} \;=\; \frac{d}{ds}\kappa(\tilde{\mathrm{Ad}}\,(e^{-sX}))\psi(g)\big|_{s=0} \;=\; -\kappa(\tilde{\mathrm{ad}}\,(X))\psi(g) \,.$$

Thus, $X(\psi) = -\kappa(\tilde{\mathrm{ad}}\,(X))\psi = -\tilde{\mathrm{ad}}\,(X) \cdot \psi$, as claimed. $\qquad\square$

Remark 3.2. In [Kos99, Section 2] and [Ste99, Chapter 10.5], the map assigning to $X \in \mathfrak{h}$ the sum

$$X(-) + \tilde{\mathrm{ad}}\,(X) \cdot -$$

is called the "diagonal" map from $\mathfrak{h}$ to $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{C}(\mathfrak{m})$. The assumption that the action is on spinors thus implies that this diagonal map is equal to zero. In particular, the diagonal Casimir operator of $\mathfrak{h}$ vanishes in the formula for $(D^t)^2$.

Proposition 3.4. *The square of D^0, the Dirac operator corresponding to the canonical connection, is given by*

$$(D^0)^2\psi \;=\; -\sum_i Z_i^2(\psi) + 2\,\tilde{C}_{\mathfrak{h}} + \frac{1}{2}\sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle\, Z_i \cdot Z_j \cdot Z_k(\psi) \,.$$

Before proceeding to the proof, let us make a short remark on how this formula is to be understood. In the first term, one has to take the derivative of ψ along all vector fields Z_i twice, thus yielding a second order differential operator. By $\tilde{C}_{\mathfrak{h}}$, we mean the image of the Casimir operator of $\mathfrak{h}$ inside $\mathcal{C}(\mathfrak{m})$ as described in Section 3.2. Finally, $Z_i \cdot Z_j\cdot$ denotes the Clifford product of Z_i and Z_j, whereas Z_k acts again as a derivative. Thus the last term is a first order differential operator. Notice that Clifford multiplication by any constant element in $\mathcal{C}(\mathfrak{m})$ commutes with derivation along $\mathfrak{m}$.

Proof. We compute $(D^0)^2$ as follows:

$$(D^0)^2\psi \;=\; \sum_i Z_i \cdot Z_i(\sum_j Z_j \cdot Z_j(\psi)) \;=\; \sum_{i,j} Z_i \cdot Z_j \cdot (Z_iZ_j(\psi)) \,.$$

We divide the sum into the diagonal ($i = j$) and off-diagonal ($i \neq j$) terms and see that this separates the second and the first order differential operator contribution,

$$(D^0)^2\psi \;=\; -\sum_i Z_i^2(\psi) + \frac{1}{2}\sum_{i,j} Z_i \cdot Z_j \cdot [Z_i, Z_j](\psi) \,.$$

We concentrate our attention on the second term. Split the commutator into its $\mathfrak{m}$ and $\mathfrak{h}$ part, then write the $\mathfrak{m}$ part again in the orthonormal basis $Z_1, \ldots, Z_n$ to obtain

$$
\begin{aligned}
\frac{1}{2} \sum_{i,j} Z_i \cdot Z_j \cdot [Z_i, Z_j](\psi) &= \frac{1}{2} \sum_{i,j} Z_i \cdot Z_j \cdot ([Z_i, Z_j]_{\mathfrak{m}}(\psi) + [Z_i, Z_j]_{\mathfrak{h}}(\psi)) \\
&= \frac{1}{2} \sum_{i,j,k} \langle Z_k, [Z_i, Z_j]_{\mathfrak{m}} \rangle Z_i \cdot Z_j \cdot Z_k(\psi) + \frac{1}{2} \sum_{i,j} Z_i \cdot Z_j \cdot [Z_i, Z_j]_{\mathfrak{h}}(\psi).
\end{aligned}
$$

This takes care of the last term in the formula of Proposition 3.4. Thus it remains to show that

$$
\frac{1}{2} \sum_{i,j} Z_i \cdot Z_j \cdot [Z_i, Z_j]_{\mathfrak{h}}(\psi) = 2\,\widetilde{C}_{\mathfrak{h}}.
$$

The action of the commutators $[Z_i, Z_j]_{\mathfrak{h}}$ on the spinor ψ is first transformed into Clifford multiplication by the adjoint representation as explained in Lemma 3.4, then rewritten in terms of an orthonormal basis according to the Parthasarathy Lemma (Lemma 3.1),

$$
\begin{aligned}
\frac{1}{2} \sum_{i,j} Z_i \cdot Z_j \cdot [Z_i, Z_j]_{\mathfrak{h}}(\psi) &= -\frac{1}{2} \sum_{i,j} Z_i \cdot Z_j \cdot \widetilde{\mathrm{ad}}\,([Z_i, Z_j]_{\mathfrak{h}}) \cdot \psi \\
&= -\frac{1}{8} \sum_{i,j} Z_i \cdot Z_j \sum_{p,q} \langle [[Z_i, Z_j]_{\mathfrak{h}}, Z_p], Z_q \rangle Z_p \cdot Z_q \cdot \psi.
\end{aligned}
$$

But since $\langle [[Z_i, Z_j]_{\mathfrak{h}}, Z_p], Z_q \rangle = Q_{\mathfrak{h}}([Z_i, Z_j], [Z_p, Z_q])$, this is $2\,\widetilde{C}_{\mathfrak{h}}$ by equation (6). $\qquad\square$

With the preparations of Section 3.2, the other two terms in the expression for $(D^t)^2$ are relatively easy to compute. We denote the Casimir operator of the full Lie algebra $\mathfrak{g}$ by $\Omega_{\mathfrak{g}}$,

$$
\Omega_{\mathfrak{g}}\psi = -\sum_i Z_i^2(\psi) + \widetilde{C}_{\mathfrak{h}} \cdot \psi.
$$

We decided to use a symbol different from C in order to emphasize that $\Omega_{\mathfrak{g}}$ is a real second order differential operator, as opposed to $\widetilde{C}_{\mathfrak{h}}$, which is a constant element of the Clifford algebra. In particular, the result of Lemma 3.4 may be restated as

$$
(9) \qquad (D^0)^2\psi = \Omega_{\mathfrak{g}} + \widetilde{C}_{\mathfrak{h}} + \frac{1}{2} \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle Z_i \cdot Z_j \cdot Z_k(\psi).
$$

First we state the formula in its most general form.

Theorem 3.2 (General Kostant-Parthasarathy formula). *For $n \geq 5$, the square of D^t is given by*

$$
\begin{aligned}
(D^t)^2\psi = {}& \Omega_{\mathfrak{g}}(\psi) + \frac{1}{2}(1 - 3t) \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle Z_i \cdot Z_j \cdot Z_k(\psi) \\
& - \frac{1}{2} \sum_{i<j<k<l} \langle Z_i, \mathrm{Jac}_{\mathfrak{h}}(Z_j, Z_k, Z_l) + 9t^2 \mathrm{Jac}_{\mathfrak{m}}(Z_j, Z_k, Z_l) \rangle \cdot Z_i \cdot Z_j \cdot Z_k \cdot Z_l \cdot \psi \\
& + \frac{1}{8} \sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j])\psi + \frac{3}{8}t^2 \sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j])\psi.
\end{aligned}
$$

For $n \leq 4$, one has

$$
\begin{aligned}
(D^t)^2\psi = {}& \Omega_{\mathfrak{g}}(\psi) + \frac{1}{2}(1 - 3t) \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle Z_i \cdot Z_j \cdot Z_k(\psi) \\
& - \frac{1}{2} \sum_{i<j<k<l} \langle Z_i, \mathrm{Jac}_{\mathfrak{h}}(Z_j, Z_k, Z_l) \rangle \cdot Z_i \cdot Z_j \cdot Z_k \cdot Z_l \cdot \psi \\
& + \frac{1}{8} \sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j])\psi + \frac{3}{8}t^2 \sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j])\psi.
\end{aligned}
$$

Proof. The mixed term is the first order differential operator

$$(D^0 D_H^t + D_H^t D^0)\psi \;=\; t \sum_p \left[Z_p \cdot Z_p (H \cdot \psi) + H \cdot Z_p \cdot Z_p(\psi) \right]$$

$$=\; t \sum_p \left[Z_p \cdot H + H \cdot Z_p \right] \cdot Z_p(\psi) \,.$$

In Lemma 3.3, we computed the anticommutator of H with the vector Z_p, which leads us to

$$(D^0 D_H^t + D_H^t D^0)\psi \;=\; -\frac{3}{2} t \sum_p \left[\sum_{i,j} \langle Z_p, [Z_i, Z_j]_{\mathfrak{m}} \rangle \, Z_i \cdot Z_j \right] Z_p(\psi) \,.$$

By Lemma 3.2, we have

$$(D_H^t)^2 \psi \;=\; -\frac{9}{2} t^2 \sum_{i<j<k<l} \langle Z_i, \mathrm{Jac}_{\mathfrak{m}}(Z_j, Z_k, Z_l) \rangle \, Z_i \cdot Z_j \cdot Z_k \cdot Z_l \cdot \psi + \frac{3}{8} t^2 \sum_{i,j} \langle [Z_i, Z_j]_{\mathfrak{m}}, [Z_i, Z_j]_{\mathfrak{m}} \rangle \, \psi$$

for $n \geq 5$ and

$$(D_H^t)^2 \psi \;=\; \frac{3}{8} t^2 \sum_{i,j} \langle [Z_i, Z_j]_{\mathfrak{m}}, [Z_i, Z_j]_{\mathfrak{m}} \rangle \, \psi$$

otherwise. Together with equation (9) and the formula for $\widetilde{C}_{\mathfrak{h}}$ from Proposition 3.3, one obtains the desired formulas. □

Now it becomes clear that the particular choice $t = 1/3$ leads to substantial simplifications in case of $n = 3$ or $n \geq 5$. In fact, the second part of the first line vanishes identically, the second line is zero by the Jacobi identity in $\mathfrak{g}$ ($n \geq 5$) or for dimensional reason ($n = 3$), and the scalar contributions in the last line appear in a very precise ratio, which will allow some further simplification. It is a strange effect that no simplification is possible for $n = 4$.

Theorem 3.3 (The Kostant-Parthasarathy formula for $t = 1/3$). *For $n = 3$ or $n \geq 5$ and $t = 1/3$, the general formula for $(D^t)^2$ reduces to*

$$(D^{1/3})^2 \psi \;=\; \Omega_{\mathfrak{g}}(\psi) + \frac{1}{8} \left[\sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) + \frac{1}{3} \sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j]) \right] \psi$$

$$=\; \Omega_{\mathfrak{g}}(\psi) + \frac{1}{8} \left[\mathrm{Scal}^{1/3} + \frac{1}{9} \sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j]) \right] \psi \,.$$

□

Remark 3.3. In particular, one immediately recovers the classical Parthasarathy formula for a symmetric space (Theorem 3.1), since then all scalar curvatures coincide and $[Z_i, Z_j] \in \mathfrak{h}$.

As in the classical Parthasarathy formula, the scalar term as well as the eigenvalues of $\Omega_{\mathfrak{g}}(\psi)$ may be expressed in representation theoretical terms if G (and hence M) is compact. Consider the unique $\mathrm{Ad}\,(G)$ invariant extension Q of the scalar product $\langle\,,\,\rangle$ on $\mathfrak{m}$ to the full Lie algebra $\mathfrak{g}$, which exists by Kostant's Theorem. Thus, Q is a multiple of the Killing form on any simple factor of $\mathfrak{g}$; however, Q is not necessarily positive definite, hence the scaling factors may be of different sign. If they are such that Q is positive definite, the metric $\langle\,,\,\rangle$ is said to be *normal homogeneous*.

We begin with a more careful analysis of the Casimir operator $\Omega_{\mathfrak{g}}(\psi)$. By the same arguments as in the symmetric space case, $\Omega_{\mathfrak{g}}(\psi)$ is a G invariant differential operator, and this property does not depend on the signs of Q. We sketch the argument for completeness: On every simple summand $\mathfrak{g}_i$ of $\mathfrak{g}$, $Q_i := Q|_{\mathfrak{g}_i}$ is either a positive or a negative multiple of the Killing form, and $\mathrm{Ad}\,(g)$ maps $\mathfrak{g}_i$ into itself. Hence, in either case, the adjoint action of G transforms an orthonormal base $\tilde{Z}_1, \ldots, \tilde{Z}_m$ of $\mathfrak{g}_i$ into an orthonormal base, and dual bases $\tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{X}_m, \tilde{Y}_m$ of $\mathfrak{g}_i$ are mapped to dual bases:

$$Q_i(\mathrm{Ad}\,(g)\tilde{Z}_k, \mathrm{Ad}\,(g)\tilde{Z}_l) \;=\; Q_i(\tilde{Z}_k, \tilde{Z}_l), \quad Q_i(\mathrm{Ad}\,(g)\tilde{X}_k, \mathrm{Ad}\,(g)\tilde{Y}_l) \;=\; Q_i(\tilde{X}_k, \tilde{Y}_l) \,.$$

Now consider the Frobenius decomposition of the square integrable spinors into irreducible finite-dimensional representations V_λ of G,

$$L^2(S) \;=\; \sum_{\lambda \in \hat{G}} M_\lambda \otimes V_\lambda,$$

where M_λ denotes the multiplicity space of V_λ. Let $\varrho_\lambda : G \to \mathrm{GL}(V_\lambda)$ be the representation with highest weight λ, and $d\varrho_\lambda$ its differential. Then $\Omega_{\mathfrak{g}_i}$ acts on V_λ by

$$d\varrho_\lambda(\Omega_{\mathfrak{g}_i}) \;=\; -\sum_{k=1}^m d\varrho_\lambda(\tilde{Z}_k)^2 \quad \text{or} \quad d\varrho_\lambda(\Omega_{\mathfrak{g}_i}) \;=\; -\sum_{k=1}^m d\varrho_\lambda(\tilde{X}_k)d\varrho_\lambda(\tilde{Y}_k)\,.$$

However, for any element $X \in \mathfrak{g}_i$, one checks immediately

$$\varrho_\lambda(g)d\varrho_\lambda(X)\varrho_\lambda(g^{-1}) \;=\; d\varrho_\lambda(\mathrm{Ad}\,(g)X),$$

hence $\Omega_{\mathfrak{g}_i}$ commutes with the action of $g \in G$ on V_λ, as claimed. Furthermore, it acts by multiplication by the well-known eigenvalue

$$Q_i(\lambda + \varrho_i, \lambda + \varrho_i) - Q_i(\varrho_i, \varrho_i),$$

whose sign, however, depends on whether Q_i is a positive or a negative multiple of the Killing form on $\mathfrak{g}_i$. Here, ϱ_i denotes the half sum of positive roots of $\mathfrak{g}_i$, as usually. Since the center of G does not contribute to the total eigenvalue of $\Omega_{\mathfrak{g}}$, we conclude:

Lemma 3.5. *For a compact group G, the operator $\Omega_{\mathfrak{g}}$ is non negative if the metric $\langle\,,\,\rangle$ is normal homogeneous or if the negative definite contribution to Q comes from an abelian summand in $\mathfrak{g}$.* $\square$

In a forthcoming paper, we will discuss examples where Q has also a simple summand on which Q is negative definite and show that $\Omega_{\mathfrak{g}}$ has negative eigenvalues. We use these remarks to express the scalar term in Theorem 3.3 in a different way.

Lemma 3.6. *Let G be compact, $n = 3$ or $n \geq 5$, and denote by $\varrho_{\mathfrak{g}}$ and $\varrho_{\mathfrak{h}}$ the half sum of the positive roots of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then the Kostant-Parthasarathy formula for $(D^{1/3})^2$ may be restated as*

$$(D^{1/3})^2\psi \;=\; \Omega_{\mathfrak{g}}(\psi) + [Q(\varrho_{\mathfrak{g}}, \varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}}, \varrho_{\mathfrak{h}})]\,\psi \;=\; \Omega_{\mathfrak{g}}(\psi) + \langle \varrho_{\mathfrak{g}} - \varrho_{\mathfrak{h}}, \varrho_{\mathfrak{g}} - \varrho_{\mathfrak{h}} \rangle\,\psi\,.$$

In particular, the scalar term is positive independently of the properties of Q.

Proof. Consider the eightfold multiple of the term under consideration and regroup it as

$$\begin{aligned}
8((D^{1/3})^2 - \Omega_{\mathfrak{g}}) &= \sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) + \frac{1}{3}\sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j]) \\
&= \frac{1}{3}\left[\sum_{i,j} Q([Z_i, Z_j], [Z_i, Z_j]) + 2\sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) \right].
\end{aligned}$$

The first summand can easily be seen to be a trace over $\mathfrak{m}$,

$$\sum_{i,j} Q([Z_i, Z_j], [Z_i, Z_j]) = -\sum_{i,j} Q([Z_i, [Z_i, Z_j]], Z_j) = -\sum_j Q(\sum_i (\mathrm{ad}\,Z_i)^2, Z_j) = -\mathrm{tr}_{\mathfrak{m}} \sum_i (\mathrm{ad}\,Z_i)^2.$$

For the second term, we first notice that it may be rewritten by expanding and contracting in two different ways as

$$\begin{aligned}
\sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) &= \sum_{i,j,k} Q(X_k, [Z_i, Z_j])Q(Y_k, [Z_i, Z_j]) = \sum_{i,j,k} Q([X_k, Z_i], Z_j)Q([Y_k, Z_i], Z_j) \\
&= \sum_{i,k} Q([X_k, Z_i], [Y_k, Z_i]).
\end{aligned}$$

This, in turn, can be identified with two different kinds of traces: On the one hand, this is

$$-\sum_{i,k} Q([Z_i, [Z_i, X_k]], Y_k) \;=\; -\mathrm{tr}_{\mathfrak{h}} \sum_i (\mathrm{ad}\,Z_i)^2,$$

on the other hand, this reads

$$-\sum_{i,k} Q([X_k,[Y_k,Z_i]],Z_i) \;=\; -\mathrm{tr}_{\mathfrak{m}}\sum_k (\mathrm{ad}\,X_k)(\mathrm{ad}\,Y_k) \;=\; \mathrm{tr}_{\mathfrak{m}}C_{\mathfrak{h}}\,,$$

were $C_{\mathfrak{h}}$ denotes the "unlifted" Casimir operator of $\mathfrak{h}$, i.e., its usual action on $\mathfrak{g}$ via the adjoint representation. Now, since the sum we have just treated appears twice, we use each way of writing it once to obtain

$$
\begin{aligned}
8((D^{1/3})^2 - \Omega_{\mathfrak{g}}) \;&=\; \frac{1}{3}\Bigg[-\mathrm{tr}_{\mathfrak{m}}\sum_i (\mathrm{ad}\,Z_i)^2 - \mathrm{tr}_{\mathfrak{h}}\sum_i (\mathrm{ad}\,Z_i)^2 + \mathrm{tr}_{\mathfrak{m}}C_{\mathfrak{h}} \Bigg] \\
&=\; \frac{1}{3}\Bigg[-\mathrm{tr}_{\mathfrak{g}}\sum_i (\mathrm{ad}\,Z_i)^2 + \mathrm{tr}_{\mathfrak{g}}C_{\mathfrak{h}} - \mathrm{tr}_{\mathfrak{h}}C_{\mathfrak{h}} \Bigg] \\
&=\; \frac{1}{3}\Big[\mathrm{tr}_{\mathfrak{g}}C_{\mathfrak{g}} - \mathrm{tr}_{\mathfrak{h}}C_{\mathfrak{h}} \Big].
\end{aligned}
$$

Again, $C_{\mathfrak{g}}$ is not to be confused with the action of the Casimir operator of $\mathfrak{g}$ on spinors. By looking separately on every simple summand where Q is just a multiple of the Killing form, one easily sees that these traces are the rescaled lengths of the half sum of positive roots,

$$\mathrm{tr}_{\mathfrak{g}}C_{\mathfrak{g}} \;=\; 24\,Q(\varrho_{\mathfrak{g}},\varrho_{\mathfrak{g}}),$$

and similarly for $\mathfrak{h}$ (Proposition 1.84 in [Kos99]). This proves the formula. To see that the scalar is positive even for non normal homogeneous metrics, decompose $\varrho_{\mathfrak{g}} = \varrho_{\mathfrak{h}} + R$, where $R \in \mathfrak{m}$. Since $\mathfrak{m}$ and $\mathfrak{h}$ are orthogonal with respect to Q, one obtains

$$Q(\varrho_{\mathfrak{g}},\varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}},\varrho_{\mathfrak{h}}) \;=\; Q(\varrho_{\mathfrak{h}}+R,\varrho_{\mathfrak{h}}+R) - Q(\varrho_{\mathfrak{h}},\varrho_{\mathfrak{h}}) \;=\; Q(R,R) \;=\; \langle R,R\rangle > 0,$$

since by dimensional reasons $R \neq 0$ and the scalar product on $\mathfrak{m}$ is positive definite. $\qquad\square$

We can formulate our first conclusion from Theorem 3.3:

Corollary 3.1. *Let G be compact. If the operator $\Omega_{\mathfrak{g}}$ is non negative, the first eigenvalue $\lambda_1^{1/3}$ of the Dirac operator $D^{1/3}$ satisfies the inequality*

$$\left(\lambda_1^{1/3}\right)^2 \;\geq\; Q(\varrho_{\mathfrak{g}},\varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}},\varrho_{\mathfrak{h}}).$$

Equality occurs if and only if there exists an algebraic spinor in $\Delta_{\mathfrak{m}}$ which is fixed under the lift $\kappa(\widetilde{\mathrm{Ad}}\,H)$ of the isotropy representation.

Proof. By our assumption on $\Omega_{\mathfrak{g}}$, its eigenvalue on a spinor ψ can be zero if and only if the Casimir eigenvalue of every simple summand $\mathfrak{g}_i$ of $\mathfrak{g}$ vanishes, hence ψ has to lie in the trivial G-representation and is thus constant. $\qquad\square$

Remark 3.4. This eigenvalue estimate is remarkable for several reasons. Firstly, for homogeneous non symmetric spaces, it is sharper than the classical Parthasarathy formula. For a symmetric space, one classically obtains $\lambda_1^2 \geq \mathrm{Scal}/8$. But since the Schrödinger-Lichnerowicz formula yields immediately $\lambda_1^2 \geq \mathrm{Scal}/4$, the lower bound in the classical Parthasarathy formula is never attained, and hence of small interest. In contrast, there exist many examples of homogeneous non symmetric spaces with constant spinors. Secondly, it uses a lower bound wich is *always* strictly positive; for naturally reductive metrics where the non positive definite part of the metric comes from an abelian factor, the scalar curvature can become negative and hence a pure curvature bound would again be of small interest. Finally, of the known formulas of Weitzenböck type which generalize the Schrödinger-Lichnerowicz formula, one does not yield an eigenvalue estimate ([FI01, Thm. 3.1] for any metric connection with skew symmetric torsion), another one yields an eigenvalue estimate which is not sharp and applies only to the normal homogeneous case (see [Goe99, Lemma 1.17]). The example of Section 5 illustrates the situation described here.

Remark 3.5. Since D^t is a G-invariant differential operator on M by construction, Theorem 3.2 implies that the linear combination of the first order differential operator and the multiplication by the element of degree four in the Clifford algebra appearing in the formula for $(D^t)^2$ is again G invariant for all t. Hence, the first order differential operator

$$\mathcal{D}\psi := \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle Z_i \cdot Z_j \cdot Z_k(\psi)$$

has to be a G invariant differential operator, a fact that cannot be seen directly by any simple arguments. It has no analogue on symmetric spaces and certainly deserves further separate investigations.

4. The equations of type II string theory on naturally reductive spaces

4.1. The field equations. The common sector of type II string theories may be geometrically described as a tuple $(M^n, \langle\,,\,\rangle, H, \Phi, \Psi)$ consisting of a manifold M^n with a Riemannian metric $\langle\,,\,\rangle$, a 3-form H, a so-called dilaton function Φ and a spinor field Ψ satisfying the coupled system of field equations

$$\text{Ric}^{\text{LC}}_{ij} - \frac{1}{4} H_{imn} H_{jmn} + 2\nabla^{\text{LC}}_i \partial_j \Phi = 0, \; \delta(e^{-2\Phi} H) = 0, \; (\nabla^{\text{LC}}_X + \frac{1}{4} X \lrcorner H)\Psi = 0, \; (d\Phi - \frac{1}{2} H)\Psi = 0.$$

The first equation generalizes the Einstein equation, the second is a conservation law, while the first of the spinorial field equations suggests that the 3-form H should be the torsion of some metric connection ∇ with totally skew-symmetric torsion tensor $T = H$. Then the equations may be rewritten in terms of ∇:

$$\text{Ric}^\nabla + \frac{1}{2}\delta(T) + 2\nabla^{\text{LC}} d\Phi = 0, \quad \delta(T) = 2 \cdot d\Phi^\# \lrcorner T, \quad \nabla\Psi = 0, \quad (d\Phi - \frac{1}{2}T) \cdot \Psi = 0.$$

If the dilaton Φ is constant, the equations may be simplified even further,

$$\text{Ric}^\nabla = 0, \quad \delta(T) = 0, \quad \nabla\Psi = 0, \quad T \cdot \Psi = 0.$$

In particular, the last equation becomes a purely algebraic condition. The number of preserved supersymmetries depends essentially on the number of ∇-parallel spinors. For a general background on these equations, we refer to the article by A. Strominger where they appeared first [Str86]. A routine calculation shows that $\delta(T)$ is the skew symmetric part of the Ricci curvature, hence the first equation implies the second (see [FI01]). In any event, for the family of connections ∇^t, the second equation is always satisfied by Lemma 2.5.

Before proceeding further, we add a general observation which follows easily from the formulas in [FI01] and which was pointed out to us by Bogdan Alexandrov.

Theorem 4.1. *Let M^n be a compact Riemannian manifold with metric $\langle\,,\,\rangle$ and a metric connection ∇ with totally skew symmetric torsion T. Suppose that there exists a spinor field ψ such that all the equations*

$$\text{Ric}^\nabla = 0, \quad \nabla\Psi = 0, \quad T \cdot \Psi = 0$$

hold. Then $T = 0$ and ∇ is the Levi-Civita connection.

Proof. If ψ is ∇-parallel, the Riemannian Dirac operator D^{LC} acts on ψ by $D^{\text{LC}}\psi = -3T \cdot \psi/4$. The last equation thus implies $D^{\text{LC}}\psi = 0$. By the classical Schrödinger-Lichnerowicz formula,

$$0 = \int_{M^n} ||\nabla^{\text{LC}}\psi||^2 dM^n + \frac{1}{4} \int_{M^n} \text{Scal}^{\text{LC}} ||\psi||^2 dM^n.$$

On the other hand, the two Ricci tensors are related by the equation

$$\text{Ric}^{\text{LC}}(X,Y) = \text{Ric}^\nabla(X,Y) + \frac{1}{2}(\delta T)(X,Y) + \frac{1}{4} \sum_{i=1}^n \langle T(X, e_i), T(Y, e_i) \rangle,$$

where $e_1, \ldots, e_n$ denotes an orthonormal basis. If $\mathrm{Ric}^\nabla = 0$, then $\delta T = 0$ (see above), and this implies that the Riemannian scalar curvature is non negative and given by

$$4\,\mathrm{Scal}^{\mathrm{LC}} = \sum_{i,j=1}^n \langle T(e_i, e_j), T(e_i, e_j)\rangle\,.$$

Consequently, the scalar curvature $\mathrm{Scal}^{\mathrm{LC}}$ has to vanish identically, and the torsion form T is zero, too. $\qquad\square$

Hence, compact solutions to all equations have to be Calabi-Yau manifolds in dimensions 4 and 6, Joyce manifolds in dimensions 7 and 8 etc.

4.2. **Some particular spinor fields.** Consider the situation that the lift of the isotropy representation $\kappa(\widetilde{\mathrm{Ad}}\, H)$ contains the trivial representation, i.e., an algebraic spinor ψ that is fixed under the action of H. Any such spinor induces a section of the spinor bundle $S = G \times_{\kappa(\widetilde{\mathrm{Ad}})} \Delta_{\mathfrak{m}}$ if viewed as a constant map $G \to \Delta_{\mathfrak{m}}$ and is thus of particular interest.

Theorem 4.2.

(1) *Any constant spinor field ψ satisfies the equation*

$$\nabla_Z^t \psi = \frac{t}{3}(Z \lrcorner H)\psi\,.$$

   In particular, it is parallel with respect to the canonical connection ($t = 0$). Conversely, any spinor field ψ satisfying $\nabla^0\psi = 0$ is necessarily constant.

(2) *Any constant spinor field ψ is an eigenspinor of the square of the Dirac operator $(D^t)^2$, and its eigenvalue does not depend of the special choice of ψ:*

$$(D^t)^2\psi = 9t^2\big[Q(\varrho_{\mathfrak{g}}, \varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}}, \varrho_{\mathfrak{h}})\big]\,\psi\,.$$

   In particular, $H \cdot \psi \neq 0$ and hence the last string equation can never hold for a constant spinor.

Proof. For a constant spinor field, the formula for the covariant derivative of a spinor field (equation 3) reduces to $\nabla_Z^t\psi = 0 + \tilde{\Lambda}_{\mathfrak{m}}^t(Z)\psi$. By Lemma 3.2, $\tilde{\Lambda}^t(Z)$ may be expressed in terms of an orthonormal basis as

$$\nabla_Z^t\psi = \frac{t}{2}\sum_{j<k}\langle[Z, Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_j \cdot Z_k \cdot \psi\,.$$

By the definition of H, this is easily seen to be $t(Z \lrcorner H)/3$. Conversely, assume that ψ is parallel with respect to the canonical connection, i.e. $Z_i(\psi) = 0$ for all i. Then $[Z_i, Z_j](\psi) = 0$, and the commutator $[Z_i, Z_j]$ may be split into its $\mathfrak{m}$ and $\mathfrak{h}$ part. But the $\mathfrak{m}$ part acts again trivially on ψ, hence we obtain

$$[Z_i, Z_j]_{\mathfrak{h}}(\psi) = 0\,.$$

By Assumption 2.2, $[\mathfrak{m}, \mathfrak{m}]$ spans all of $\mathfrak{h}$, hence $\mathfrak{h}$ also acts trivially on ψ, which finishes the argument. For the second part of the Theorem, we use that the Dirac operator on a constant spinor is given by $D^t\psi = t\,H \cdot \psi$ for any t. Since any constant spinor lies in the trivial G-representation in the Frobenius decomposition of $\Gamma(S)$, the eigenvalue of $\Omega_{\mathfrak{g}}$ on ψ is zero. For $t = 1/3$, the Kostant-Parthasarathy formula (Theorem 3.3) thus yields

$$(D^{1/3})^2\psi = \big[Q(\varrho_{\mathfrak{g}}, \varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}}, \varrho_{\mathfrak{h}})\big]\psi = \frac{1}{9}H^2\psi\,.$$

This may be understood as a formula for $H^2\psi$, from which we immediately derive the general formula through $(D^t)^2\psi = t^2\,H^2\psi$. In particular, $H \cdot \psi$ cannot vanish. $\qquad\square$

Remark 4.1. Easy examples show that ψ might not be an eigenspinor of D^t itself, since not all constant spinors are eigenspinors of H. For the canonical connection, $\nabla^0 T^0 = 0$ implies that the space of parallel spinors is invariant under T^0, hence there exists a basis of the space of parallel spinors consisting of eigenspinors.

4.3. **Vanishing theorems.** This section is devoted to non-existence theorems for solutions in certain geometric configurations. It allows us to draw quite a precise picture of what a promising naturally reductive metric should look like. First, the Kostant-Parthasarathy formula yields that we should be interested in precisely those metrics where $\Omega_{\mathfrak{g}}$ is *not* non negative.

Theorem 4.3. *If the operator $\Omega_{\mathfrak{g}}$ is non negative and ∇^t is not the Levi-Civita connection, there do not exist any non trivial solutions to the system of equations*

$$\nabla^t \psi \; = \; 0, \quad T^t \cdot \psi = 0 \,.$$

Proof. If the spinor ψ is ∇^t-parallel, then it lies in the kernel of $D^t = D^0 + tH$. Since ∇^t is assumed not to be the Levi-Civita connection, T^t does not vanish and hence $T^t \cdot \psi = 0$ implies $H \cdot \psi = 0$. Thus ψ is also in the kernel of D^0. For the Dirac operator to the parameter $t = 1/3$, we obtain

$$D^{1/3} \psi \; = \; D^0 \psi + \frac{1}{3} H \cdot \psi \; = \; 0 \,,$$

which contradicts Corollary 3.1. $\qquad\square$

For the Levi-Civita connection, it is well known that the existence of a parallel spinor implies vanishing Ricci curvature. By repetition of the same argument, one sees that this conclusion does no longer hold for a metric connection with torsion. Rather, we get restrictions on the algebraic type of the derivatives of the torsion.

Proposition 4.1. *If the canonical connection ∇^0 is Ricci flat and admits a parallel spinor, then the exterior derivative of its torsion T^0 satisfies $(X \lrcorner dT^0) \cdot \psi = 0$ for all vectors X in $\mathfrak{m}$.*

Proof. In [FI01, Cor. 3.2], Friedrich and Ivanov showed that a spin manifold with some connection ∇ whose torsion T is totally skew symmetric and a ∇-parallel spinor ψ satisfies

$$\left[\frac{1}{2} X \lrcorner dT + \nabla_X T \right] \cdot \psi \; = \; \mathrm{Ric}^\nabla(X) \cdot \psi \,.$$

Since the canonical connection satisfies $\nabla^0 T^0 = 0$, the claim follows. $\qquad\square$

These conditions are independent of the equation $T^0 \cdot \psi = 0$. If $dT^0 \neq 0$ and the dimension is sufficiently small, it can happen that the intersection of all kernels of $X \lrcorner dT^0$ is already empty, thus showing the non-existence of solutions. Models with $dT^0 = 0$ are of particular interest and are called *closed* in string theory.

For further investigations of the Ricci tensor

$$\mathrm{Ric}^t(X, Y) \; = \; \sum_i (t - t^2) \left\langle [X, Z_i]_{\mathfrak{m}}, [Y, Z_i]_{\mathfrak{m}} \right\rangle + Q_{\mathfrak{h}}([X, Z_i], [Y, Z_i]) \,,$$

it is useful to describe it from a more representation theoretical point of view. Wang and Ziller derived the general formula we shall present for $t = 1/2$ in [WZ85]. Their proof may easily be generalized to the case of arbitrary t, hence we omit it here. The main idea is to use a more elaborate version of the core computation in the proof of Lemma 3.6. Recall that $C_{\mathfrak{h}}$ denotes the (unlifted) Casimir operator of $\mathfrak{h}$, i. e.,

$$C_{\mathfrak{h}} \; = \; -\sum_i \mathrm{ad}\, X_i \mathrm{ad}\, Y_i \,.$$

It defines a symmetric endomorphism $A : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by $A(X, Y) := \langle C_{\mathfrak{h}} X, Y \rangle$. Similarly, we denote by $\beta(X, Y) = -\mathrm{tr}_{\mathfrak{g}}\, \mathrm{ad}\, X \mathrm{ad}\, Y$ the Killing form of the full Lie algebra $\mathfrak{g}$. We make no notational difference between β itself and its restriction to $\mathfrak{m}$.

Theorem 4.4. *The endomorphisms A and β satisfy the identities*

$$A(X, Y) \; = \; \sum_i Q_{\mathfrak{h}}([X, Z_i], [Y, Z_i]), \quad \beta(X, Y) \; = \; \sum_i \langle [X, Z_i]_{\mathfrak{m}}, [Y, Z_i]_{\mathfrak{m}} \rangle + 2A(X, Y) \,.$$

$\dim G/H$	5	6	7	8
$H_{\max}$	$\mathrm{SU}(2)$	$\mathrm{SU}(3)$	G_2	$\mathrm{Spin}(7)$

Table 1. Maximal holonomy groups for the existence of a parallel spinor

Thus, the Ricci tensor is given by

$$\mathrm{Ric}^t(X,Y) \;=\; (t-t^2)\beta(X,Y) + (2t^2 - 2t + 1)A(X,Y)\,.$$

$\square$

Remark 4.2. We observe that the coefficient of β vanishes for $t = 0$ and $t = 1$, and is positive between these parameter values, whereas the coefficient of A is always positive and attains its minimum for the Levi-Civita connection ($t = 1/2$).

The endomorphism A has block diagonal structure, with every block corresponding to an irreducible summand of the isotropy representation. In particular, the block of the trivial representation vanishes, since its Casimir eigenvalue is zero. Since β is positive definite for G compact, we can deduce:

Proposition 4.2. *Assume that G is compact. If the isotropy representation* $\mathrm{Ad} : H \to \mathrm{SO}(\mathfrak{m})$ *has fixed vectors, only the connections $t = 0$ and $t = 1$ can be Ricci flat.* $\square$

Typically, the eigenvalues of $C_{\mathfrak{h}}$ are linear functions of some deformation parameters, hence, they can vanish for some particular parameter choices *without* belonging to a trivial $\mathfrak{h}$-summand of $\mathfrak{m}$. This makes it difficult to make more precise predictions for the vanishing of the Ricci tensor.

Proposition 4.3. *If the canonical connection has vanishing scalar curvature, H cannot be simple and the metric cannot be normal homogeneous.*

Proof. The scalar curvature for the canonical connection is

$$\sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j])\,.$$

By Assumption 2.2, not all vectors $[Z_i, Z_j]_{\mathfrak{h}}$ can be zero. Since $Q_{\mathfrak{h}}$ is non degenerate, we conclude that $Q_{\mathfrak{h}}$ can be neither positive nor negative definite. However, on every simple factor of $\mathfrak{h}$, $Q_{\mathfrak{h}}$ has to be a multiple of the Killing form; hence $\mathfrak{h}$ cannot be simple. $\square$

This fact, as elementary as its proof might be, has far reaching consequences for the geometry of homogeneous models of string theory. The existence of a parallel spinor severely restricts the holonomy group of ∇. In fact, it needs to be a subgroup of the isotropy subgroup of a spinor inside $\mathrm{SO}(n)$, and these subgroups are well-known. By a theorem of Wang ([KN96, Ch.X, Cor. 4.2]), the Lie algebra of the holonomy group is spanned by

$$\mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] + \dots\,,$$

where the subspace $\mathfrak{m}_0$ is defined as

$$\mathfrak{m}_0 \;=\; \{[\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \mathrm{ad}\,([X,Y]_{\mathfrak{h}}) : \; X, Y \in \mathfrak{m}\,\}\,.$$

For the canonical connection and using our assumption that $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}$ spans all of $\mathfrak{h}$, we conclude that its holonomy Lie algebra is precisely $\mathfrak{h}$. For $t \neq 0$, the holonomy can only increase, hence we obtain Table 1 for the maximally possible subgroups $H_{\max}$. If we restrict our attention to the canonical connection, Proposition 4.3 implies that H cannot be equal to $H_{\max}$ itself, but rather has to be a non simple subgroup of it. This excludes many homogeneous spaces that would naturally come to one's mind. Of course, they might yield models for other connections than the canonical one, but such an analysis can only be performed on a case by case basis.

5. Examples

5.1. The Jensen metric on $V_{4,2}$. The 5-dimensional Stiefel manifold $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2)$ carries a one-parameter family of metrics constructed by G. Jensen [Jen75] with many remarkable properties. Embed $H = \mathrm{SO}(2)$ into $G = \mathrm{SO}(4)$ as the lower diagonal 2×2 block. Then the Lie algebra $\mathfrak{so}(4)$ splits into $\mathfrak{so}(2) \oplus \mathfrak{m}$, where $\mathfrak{m}$ is given by

$$\mathfrak{m} \;=\; \left\{ \left[\begin{array}{cc|cc} 0 & -a & \multicolumn{2}{c}{} \\ a & 0 & \multicolumn{2}{c}{-X^t} \\ \hline \multicolumn{2}{c|}{X} & 0 & 0 \\ \multicolumn{2}{c|}{} & 0 & 0 \end{array} \right] =: (a, X) \;:\; a \in \mathbb{R}, \; X \in \mathcal{M}_{2,2}(\mathbb{R}) \right\} .$$

Denote by $\beta(X, Y) := \mathrm{tr}(X^t Y)$ the Killing form of $\mathfrak{so}(4)$. Then the Jensen metric on $\mathfrak{m}$ to the parameter $s \in \mathbb{R}$ is defined by

$$\langle (a, X), (b, Y) \rangle \;=\; \frac{1}{2} \beta(X, Y) + s\beta(a, b) \;=\; \frac{1}{2}\beta(X, Y) + 2s \cdot ab .$$

For $s = 2/3$, G. Jensen proved that this metric is Einstein, and Th. Friedrich showed that it admits two Riemannian Killing spinors [Fri80] and thus realizes the equality case in his estimate for the first eigenvalue of the Dirac operator. A more careful analysis shows that $V_{4,2}$ carries three different contact structures, one of which is Sasakian, one quasi-Sasakian but not Sasakian, and the third one has no special name, although special properties. It will become clear in the discussion that this metric is only naturally reductive with respect to $G = \mathrm{SO}(4)$ for $s = 1/2$. In the following sections, we shall describe the Jensen metrics on $V_{4,2}$ first from the point of view of contact geometry and then from the point of view of naturally reductive spaces.

5.2. The contact geometry approach. Denote by E_{ij} the standard basis of $\mathfrak{so}(4)$. Then the elements

$$Z_1 := E_{13}, \; Z_2 := E_{14}, \; Z_3 = E_{23}, \; Z_4 = E_{24}, \; Z_5 = \frac{1}{\sqrt{2s}} E_{12}$$

form an orthonormal base of $\mathfrak{m}$. To start with, we compute all nonvanishing commutators in $\mathfrak{m}$. These are

$$\begin{aligned}
[Z_1, Z_3]_{\mathfrak{m}} &= \sqrt{2s}\, Z_5, & [Z_1, Z_5]_{\mathfrak{m}} &= -\frac{1}{\sqrt{2s}} Z_3, & [Z_2, Z_4]_{\mathfrak{m}} &= \sqrt{2s}\, Z_5, \\
[Z_2, Z_5]_{\mathfrak{m}} &= -\frac{1}{\sqrt{2s}} Z_4, & [Z_3, Z_5]_{\mathfrak{m}} &= \frac{1}{\sqrt{2s}} Z_1, & [Z_4, Z_5]_{\mathfrak{m}} &= \frac{1}{\sqrt{2s}} Z_2.
\end{aligned}$$

$(*)$

Notice that all these commutators have no $\mathfrak{h}$-contribution. Identifying $\mathfrak{m}$ with $\mathbb{R}^5$ via the chosen basis, the isotropy representation of an element $g(\theta) = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right] \in H = \mathrm{SO}(2)$ may be written as follows:

$$\mathrm{Ad}\, g(\theta) \;=\; \left[\begin{array}{ccccc} \cos\theta & -\sin\theta & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] .$$

In particular, Z_5 is invariant under the isotropy action. As in [Fri80], we use a suitable basis $\psi_1, \ldots, \psi_4$ for the 4-dimensional spinor representation $\kappa : \mathrm{Spin}(\mathbb{R}^5) \to \mathrm{GL}(\Delta_5)$. One derives the expression for the lift of the isotropy representation,

$$\kappa\big(\widetilde{\mathrm{Ad}}\, g(\theta)\big) \;=\; \left[\begin{array}{cccc} \cos\theta + i\sin\theta & 0 & 0 & 0 \\ 0 & \cos\theta - i\sin\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] .$$

Thus, the elements ψ_3 and ψ_4 define sections of the spinor bundle $S = G \times_{\kappa(\widetilde{\mathrm{Ad}})} \Delta_5$ if viewed as constant maps $G \to \Delta_5$. In fact, for $s = 2/3$, $\psi^{\pm} := \psi_3 \mp i\psi_3$ are exactly the Riemannian Killing

spinors from [Fri80] as we will see below. The sections induced by ψ_1 and ψ_2 are not constant and thus more difficult to treat. We will not consider them in our discussion. In [Jen75, Prop. 3], the author computed the map $\Lambda_{\mathfrak{m}}^{\mathrm{LC}} : \mathfrak{m} \cong \mathbb{R}^5 \to \mathfrak{so}(5)$ (see Wang's Theorem in Section 2) defining the Levi-Civita connection:

$$\Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_\alpha)Z_\beta = \frac{1}{2}[Z_\alpha, Z_\beta], \ \Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_5)Z_\alpha = (1-s)[Z_5, Z_\alpha], \ \Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_\alpha)Z_5 = s[Z_\alpha, Z_5] \quad \text{for } \alpha, \beta = 1, \ldots, 4.$$

Indeed, one easily verifies that this is the unique map $\Lambda_{\mathfrak{m}}$ verifying the conditions

$$\langle \Lambda_{\mathfrak{m}}(X)Y, Z \rangle + \langle Y, \Lambda_{\mathfrak{m}}(X)Z \rangle = 0 \ \text{ and } \ \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X = [X, Y]_{\mathfrak{m}}.$$

Thus, one sees that for $s \neq 1/2$, $\Lambda_{\mathfrak{m}}(X)Y$ is not globally proportional to the commutator $[X, Y]_{\mathfrak{m}}$, and both $\langle \Lambda_{\mathfrak{m}}(X)Y, Z \rangle$ and $-\langle [X, Y]_{\mathfrak{m}}, Z \rangle$ (the torsion of the canonical connection) fail to define a 3-form: The first is not skew symmetric in X and Y, the second is not skew symmetric in X and Z. In any case, by using the commutator relations $(*)$, the Levi-Civita connection can be identified with an endomorphism of $\mathbb{R}^5$ as follows:

$$\Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_1) = \sqrt{\frac{s}{2}}E_{35}, \quad \Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_2) = \sqrt{\frac{s}{2}}E_{45}, \quad \Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_3) = -\sqrt{\frac{s}{2}}E_{15}, \quad \Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_4) = -\sqrt{\frac{s}{2}}E_{25},$$

$$\Lambda_{\mathfrak{m}}^{\mathrm{LC}}(Z_5) = \frac{1-s}{\sqrt{2s}}(E_{13} + E_{24}).$$

The lift into the spin representation yields a global factor $1/2$ and replaces E_{ij} by $Z_i \wedge Z_j$. By setting

$$\tilde{T} := (Z_1 \wedge Z_3 + Z_2 \wedge Z_4) \wedge Z_5,$$

the Levi-Civita connection may be rewritten in a unified way as

$$(10) \qquad \tilde{\Lambda}_{\mathfrak{m}}^{\mathrm{LC}}(Z_5) = \frac{1}{4}\frac{2(1-s)}{\sqrt{2s}}(Z_5 \lrcorner \tilde{T}), \quad \tilde{\Lambda}_{\mathfrak{m}}^{\mathrm{LC}}(Z_\alpha) = \frac{1}{4}\sqrt{2s}(Z_\alpha \lrcorner \tilde{T}) \ \text{ for } \alpha = 1, \ldots, 4.$$

Now we discuss the three different metric almost contact structures existing on $V_{4,2}$. The space $\mathfrak{m}$ has a preferred direction, namely $\xi = Z_5$, which is fixed under the isotropy representation. Denote its dual 1-form, $\eta(X) = \langle Z_5, X \rangle$ by η. The following operators

$$\varphi_S = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \varphi_{qS} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \varphi_* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

intertwine the isotropy representation, and thus define compatible complex structures on the linear span of $Z_1, \ldots, Z_4$. Then one checks for all three choices for φ that the compatibility conditions defining a metric almost contact structure hold:

$$\varphi^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \langle \varphi(X), \varphi(Y) \rangle = \langle X, Y \rangle - \eta(X) \cdot \eta(Y), \quad \varphi(\xi) = 0.$$

The fundamental form of the structure is defined by $F(X, Y) = \langle X, \varphi(Y) \rangle$, thus yielding

$$F_S = Z_1 \wedge Z_3 + Z_2 \wedge Z_4, \quad F_{qS} = Z_1 \wedge Z_2 + Z_3 \wedge Z_4, \quad F_* = Z_1 \wedge Z_2 - Z_3 \wedge Z_4,$$

respectively. Since Z_5 is constant under the isotropy action, its exterior derivative may be computed using the general formula as stated at the beginning of the proof of Lemma 2.4,

$$d\omega^1(X_0, X_1) = X_0(\omega^1(X_1)) - X_1(\omega^1(X_0)) - \omega^1([X_0, X_1]).$$

For the constant vector field Z_5, we thus obtain $dZ_5(Z_i, Z_j) = -\langle Z_5, [Z_i, Z_j] \rangle$. Applying again the commutator relations implies

$$dZ_5 = -\sqrt{2s}(Z_1 \wedge Z_3 + Z_2 \wedge Z_4).$$

In particular, dZ_5 is proportional to F_S, turning it into a Sasaki structure (up to rescaling) and implying immediately $dF_S = 0$. For the other two structures, remark that $Z_1 \wedge Z_2$ and $Z_3 \wedge Z_4$ are also invariant forms under the isotropy action, thus their exterior differential may be computed in a

similar way. One gets that $dF_{qS} = 0$, turning it into a non Sasakian quasi-Sasakian structure, and dF_* is proportional to $Z_2 \wedge Z_3 \wedge Z_5$, which implies $dF_*^{\varphi^*} = 0$. We can then compute the Nijenhuis tensor

$$N(X,Y) := [\varphi(X), \varphi(Y)] + \varphi^2([X,Y]) - \varphi([\varphi(X), Y]) - \varphi([X, \varphi(Y)]) + d\eta(X,Y) \cdot \xi$$

and see that it vanishes for all three metric almost contact structures. By [FI01, Thm. 8.2], the Stiefel manifold $V_{4,2}$ admits a unique almost contact connection ∇ with torsion

$$T = \eta \wedge d\eta = -\sqrt{2s}\,(Z_1 \wedge Z_3 + Z_2 \wedge Z_4) \wedge Z_5\,.$$

Next we discuss the existence of spinors that are parallel with respect to the connection ∇ as well as the existence of Killing spinors, since we consider the analogy and differences to the previous case to be instructive.

Theorem 5.1.
  (1) *The constant spinors are parallel with respect to the contact connection ∇ if and only if $s = 1/2$;*
  (2) *The constant spinors $\psi^\pm$ are Riemannian Killing spinors if and only if $s = 2/3$.*

Proof. In equation (10), we gave the general formula for the Levi-Civita connection in direction Z_i as the inner product of Z_i and the 3-form $\tilde{T}$. If a constant spinor ψ is to be parallel with respect to ∇,

$$0 = \nabla_X \psi = (\tilde{\Lambda}_{\mathfrak{m}}^{\mathrm{LC}}(X) + \frac{1}{4} X \lrcorner T)\psi\,,$$

then the coefficients of $\tilde{\Lambda}_{\mathfrak{m}}^{\mathrm{LC}}$ as in equation (10) have to be equal for all Z_i, hence, $2(1-s)/\sqrt{2s} = \sqrt{2s}$, which means that $s = 1/2$. For this value, the combination $\tilde{\Lambda}^{\mathrm{LC}}(X) + \frac{1}{4} X \lrcorner T$ vanishes, so both constant spinors are parallel indeed. For the discussion of Riemannian Killing spinors, we use the following realization of the spin representation:

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},\quad e_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\quad e_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\quad e_5 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

Then one checks that

$$(Z_5 \lrcorner \tilde{T}) \cdot \psi^\pm = \pm 2 Z_5 \cdot \psi^\pm, \quad (Z_\alpha \lrcorner \tilde{T}) \cdot \psi^\pm = \pm Z_\alpha \cdot \psi^\pm \text{ for } \alpha = 1, \dots, 4.$$

Looking at Z_5, we conclude that the Killing equation $\nabla_X^{\mathrm{LC}} \psi = \mu X \cdot \psi$ implies that the coefficients in equation (10) have to satisfy $2(1-s)/\sqrt{2s} = \sqrt{2s}/2$. The solution is now $s = 2/3$, and one checks that $\psi^\pm$ are Killing spinors indeed. $\square$

In [FI01a], Friedrich and Ivanov studied the 5-dimensional contact case in more detail.

5.3. The naturally reductive space approach. We would like to interpret the metric $\langle\,,\,\rangle$ as a naturally reductive metric with respect to some other group $\bar{G}$, and the connection with the torsion

$$T = -\sqrt{2s}\,(Z_1 \wedge Z_3 + Z_2 \wedge Z_4) \wedge Z_5$$

as its canonical connection. So write $M = \bar{G}/\bar{H}$ with the Lie algebra decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \bar{\mathfrak{m}}$, and assume that the original isotropy representation is a subrepresentation of the new isotropy representation, i.e., the action of $\mathfrak{h} \subset \bar{\mathfrak{h}}$ on $\mathfrak{m} \cong \bar{\mathfrak{m}}$ remains unchanged. This point of view necessarily enlarges the holonomy group H already for dimensional reasons. In fact, we can deduce a lot of information about the new isotropy representation from the formula for T. In Remark 2.2, we

explained the relation between $\mathfrak{m}$-commutators and the torsion. For example, the formula above implies

$$[Z_1, Z_3]_{\bar{\mathfrak{m}}} = \sqrt{2s}\, Z_5, \quad [Z_4, Z_5]_{\bar{\mathfrak{m}}} = \sqrt{2s}\, Z_2, \quad [Z_1, Z_4]_{\bar{\mathfrak{m}}} = [Z_3, Z_4]_{\bar{\mathfrak{m}}} = 0.$$

Then we can compute

$$\mathrm{Jac}_{\bar{\mathfrak{m}}}(Z_1, Z_3, Z_4) = 2s\, Z_2.$$

On the other hand,

$$\mathrm{Jac}_{\bar{\mathfrak{h}}}(Z_1, Z_3, Z_4) = -Z_2 + [Z_4, [Z_1, Z_3]_{\bar{\mathfrak{h}}}] + [Z_3, [Z_4, Z_1]_{\bar{\mathfrak{h}}}] \overset{!}{=} -\mathrm{Jac}_{\bar{\mathfrak{m}}}(Z_1, Z_3, Z_4).$$

Thus, there must be two elements $H_1 := [Z_1, Z_3]_{\bar{\mathfrak{h}}}$ and $H_2 := [Z_4, Z_1]_{\bar{\mathfrak{h}}}$ in $\bar{\mathfrak{h}}$, not both zero, such that

$$[H_1, Z_4] + [H_2, Z_3] = (2s-1)Z_2.$$

By some more careful analysis, one obtains $H_2 = 0$, $H_1 = [Z_2, Z_4]_{\bar{\mathfrak{h}}}$ and the action of H_1 on the other vectors Z_i. The systematic description of $\langle\, ,\, \rangle$ as a naturally reductive metric can be given using a deformation construction due to Chavel and Ziller ([Cha70], [Zil77]). It is based on the remark that for $s = 1/2$, $\mathfrak{m}$ splits into an orthogonal direct sum of $\mathfrak{m}_1 := \{(0, X)\}$ and $\mathfrak{m}_2 := \{(a, 0)\}$ such that

$$[\mathfrak{h}, \mathfrak{m}_2] = 0 \text{ and } [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2.$$

Let $M_2 \subset G$ be the subgroup of G with Lie algebra $\mathfrak{m}_2$, and set $\bar{G} = G \times M_2$, $\bar{H} = H \times M_2$. An element (k, m) of $\bar{G}$ acts on $M = G/H$ by $(k, m)gH = kgHm^{-1}$, and then $\bar{H}$ can indeed be identified with the isotropy group of this action. We endow $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{m}_2$ with the direct sum Lie algebra structure. The trick is now to choose a realization of $\bar{\mathfrak{m}}$ that depends on the deformation parameter s of the metric. Writing all elements of $\bar{\mathfrak{g}}$ as 4-tuples (H, U, X, Y) with $H \in \mathfrak{h}$, $U \in \mathfrak{m}_1$ and $X, Y \in \mathfrak{m}_2$, we can realize the Lie algebra of $\bar{H}$ as

$$\bar{\mathfrak{h}} = \{(H, 0, X, X) \subset \bar{\mathfrak{g}} : H \in \mathfrak{h}, X \in \mathfrak{m}_2\}$$

and choose

$$\mathfrak{m} = \{(0, X, 2s\, Y, (2s-1)Y) : X \in \mathfrak{m}_1, Y \in \mathfrak{m}_2\}$$

as an orthogonal complement. Here, $(0, 0, 2s\, Y, (2s-1)Y)$ will be identified with $Y \in \mathfrak{m}_2$. Since $\mathfrak{m}_2$ is abelian in this example, the Lie algebra structure of $\bar{\mathfrak{g}}$ is particularly simple. $\bar{\mathfrak{h}}$ is a Lie algebra with commutator

$$[(H, 0, X, X), (H', 0, X', X')] = ([H, H'], 0, 0, 0),$$

the full isotropy representation is

$$[(H, 0, X, X), (0, U, 2sY, (2s-1)Y)] = (0, [H + X, U], 0, 0)$$

and the commutator of two elements in $\bar{\mathfrak{m}}$ splits into its $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{m}}$ part as follows:

$$[(0, U, 2s\, X, (2s-1)X), (0, V, 2s\, Y, (2s-1)Y)] = ([U, V]_{\mathfrak{h}}, 0, -(2s-1)[U, V]_{\mathfrak{m}_2}, -(2s-1)[U, V]_{\mathfrak{m}_2})$$
$$+ (0, [U, V]_{\mathfrak{m}_1} + 2s([U, Y] + [X, V]), 2s[U, V]_{\mathfrak{m}_2}, (2s-1)[U, V]_{\mathfrak{m}_2}).$$

With these choices for $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{m}}$, the metric $\langle\, ,\, \rangle$ is naturally reductive with respect to $\bar{G}$, the torsion of its canonical connection is precisely T and the Ricci tensor is given by

$$\mathrm{Ric}^0 = 2(1-s)\mathrm{diag}(1, 1, 1, 1, 0).$$

For $s = 1$, the canonical connection is thus Ricci flat, and by Proposition 4.2, we know that no other connection can have this property. However, the holonomy $\bar{H} \cong \mathrm{SO}(2) \times \mathrm{SO}(2)$ is too large to admit parallel spinors. For $s = 1/2$, we have two parallel spinors for the canonical connection as seen in the preceding section, but the Ricci curvature does not vanish. In this case, one can ask the question whether some other connection of the family ∇^t admits parallel spinors. But using Wang's Theorem ([KN96, Ch.X, Cor. 4.2]) for computing the holonomy, one sees that ∇^t has full holonomy $\mathrm{SO}(\mathfrak{m})$ for $t \neq 0$, excluding again the existence of parallel spinors.

We close this section with a look at the eigenvalue estimate for $(D^{1/3})^2$. Since the extension of H is by the abelian group $\mathrm{SO}(2)$, the Casimir operator $\Omega_{\mathfrak{g}}$ is non negative by Lemma 3.5 and

Corollary 3.1 can be applied. We compute the scalar in the general Kostant-Parthasarathy formula (Theorem 3.2)

$$\frac{1}{8}\sum_{i,j} Q_{\mathfrak{h}}([Z_i, Z_j], [Z_i, Z_j]) + \frac{3}{8}t^2 \sum_{i,j} Q_{\mathfrak{m}}([Z_i, Z_j], [Z_i, Z_j]) \;=\; \frac{1}{8} \cdot 8(1-s) + \frac{3}{8}t^2 \cdot 24s \;=\; 1 + (9t^2 - 1)s$$

and see that it is independent of the deformation parameter s precisely for the Kostant connection $t = 1/3$. If $s \neq 1/2$, there exist no constant spinors and hence Corollary 3.1 is a strict inequality,

$$(\lambda^{1/3})^2 \;>\; 1 \,.$$

For $s = 1/2$, there exists a constant spinor ψ and it satisfies by Theorem 4.2

$$(D^t)^2 \psi \;=\; 9t^2 \cdot 1 \cdot \psi \;=\; 9t^2 \psi \,.$$

Unfortunately, we have been unable to relate this bound with the infimum of the spectrum of $(D^t)^2$ for other values of t. In particular, it seems to be difficult to deduce from Corollary 3.1 any information about the Riemannian Dirac spectrum.

References

[Cha70] I. Chavel, *A class of Riemannian homogeneous spaces*, J. Differ. Geom. **4** (1970), 13–20.

[FI01a] Th. Friedrich and S. Ivanov, *Almost contact manifolds and type II string equations*, Tech. report, SFB 288 preprint Nr. 523, 2001, math.DG/0111131.

[FI01] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Tech. report, SFB 288 preprint Nr. 492, 2001, math.DG/0102142.

[Fri80] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. **97** (1980), 117–146.

[Fri00] Th. Friedrich, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics, vol. 25, American Mathematical Society, Prov., 2000.

[FS79] Th. Friedrich and S. Sulanke, *Ein Kriterium für die formale Selbstadjungiertheit des Dirac-Operators*, Coll. Math. **XL** (1979), 239–247.

[Goe99] S. Goette, *Equivariant η-invariants on homogeneous spaces*, Math. Z. **232** (1999), 1–42.

[Ike75] A. Ikeda, *Formally self adjointness for the Dirac operator on homogeneous spaces*, Osaka J. Math. **12** (1975), 173–185.

[Jen75] G. Jensen, *Imbeddings of Stiefel manifolds into Grassmannians*, Duke Math. J. **42** (1975), no. 3, 397–407.

[KN91] S. Kobayashi and K Nomizu, *Foundations of differential geometry I*, Wiley Classics Library, Wiley Inc., Princeton, 1963, 1991.

[KN96] S. Kobayashi and K Nomizu, *Foundations of differential geometry II*, Wiley Classics Library, Wiley Inc., Princeton, 1969, 1996.

[Kos56] B. Kostant, *On differential geometry and homogeneous spaces II*, Proc. N. A. S. **42** (1956), 354–357.

[Kos99] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), no. 3, 447–501.

[Lan00] G. Landweber, *Harmonic spinors on homogeneous spaces*, Repr. Theory **4** (2000), 466–473.

[Par72] R. Parthasarathy, *Dirac operator and the discrete series*, Ann. of Math. **96** (1972), no. 1, 1–30.

[Sle87a] S. Slebarski, *The Dirac operator on homogeneous spaces and representations of reductive Lie groups I*, Amer. J. Math. **109** (1987), 283–301.

[Sle87b] S. Slebarski, *The Dirac operator on homogeneous spaces and representations of reductive Lie groups II*, Amer. J. Math. **109** (1987), 499–520.

[Ste99] S. Sternberg, *Lie algebras*, Lecture notes, 1999.

[Str86] A. Strominger, *Superstrings with torsion*, Nucl. Phys. B **274** (1986), 253–284.

[WZ85] M. Y. Wang and W. Ziller, *On normal homogeneous Einstein manifolds*, Ann. Sci. Éc. Norm. Sup., 4e série **18** (1985), 563–633.

[Zil77] W. Ziller, *The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces*, Comment. Math. Helv. **52** (1977), 573–590.

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