THE ALGEBRA OF K-INVARIANT VECTOR FIELDS ON A SYMMETRIC SPACE G/K

ILKA AGRICOLA AND ROE GOODMAN

Abstract. When G is a complex reductive algebraic group and G/K is a reductive symmetric space, the decomposition of $\mathbf{C}[G/K]$ as a K-module was obtained (in a non-constructive way) by Richardson, generalizing the celebrated result of Kostant-Rallis for the linearized problem (the harmonic decomposition of the isotropy representation). To obtain a constructive version of Richardson's results, this paper studies the infinite dimensional Lie algebra $\mathfrak{X}(G/K)^K$ of K-invariant regular algebraic vector fields using the geometry of G/K and the K-spherical representations of G. Assume G is semisimple and simply-connected and let $\mathcal{J}$ be the algebra of K biinvariant functions on G. An explicit set of free generators for the localization $\mathfrak{X}(G/K)^K_\psi$ is constructed for a suitable $\psi \in \mathcal{J}$. A commutator formula is obtained for K-invariant vector fields in terms of the corresponding K-covariant maps from G to the isotropy representation of G/K. Vector fields on G/K whose horizontal lifts to G are tangent to the Cartan embedding of G/K into G are called *flat*. When G is simple and simply connected, it is shown that every element of $\mathfrak{X}(G/K)^K$ is flat if and only if K is semisimple. The gradients of the fundamental characters of G are shown to generate all conjugation-invariant vector fields on G. These results are applied in the case of the adjoint representation of $G = \mathrm{SL}(2, \mathbf{C})$ to construct a conjugation invariant differential operator whose kernel furnishes a harmonic decomposition of $\mathbf{C}[G]$.

1. Introduction

In this paper we study the infinite dimensional Lie algebra of K-invariant vector fields on a reductive symmetric space G/K. Our motivation was the investigation of the algebra of invariant differential operators for non transitive group actions on smooth affine varieties, and in particular the abstract Howe duality theorem one has for this situation (see for example [Agr01, Satz 2.2]). Correspondingly, we shall work in the algebraic category, i. e. G is a complex connected reductive linear algebraic group and K is the fixed points of an involutory automorphism θ of G (thus G/K is the complexification of a Riemannian symmetric space).

There is a canonical G-module isomorphism between the space $\mathfrak{X}(G/K)$ of regular algebraic vector fields on G/K and the algebraically induced representation $\mathrm{Ind}_K^G(\sigma)$, where σ is the isotropy representation of K. In particular, the space $\mathfrak{X}(G/K)^K$ of K-invariant vector fields on G/K corresponds to the K-fixed vectors in the induced representation. When G is simple and simply connected, Richardson's results [Ric82] imply that $\mathfrak{X}(G/K)$ is a free module over the algebra $\mathcal{J}$ of K-biinvariant functions on G. In Theorem 2.2 we obtain an explicit set of free generators for a localization $\mathfrak{X}(G/K)^K_\psi$, for some $\psi \in \mathcal{J}$.

We next study $\mathfrak{X}(G/K)^K$ as a Lie algebra in Section 3 and obtain a formula for the commutator of K-invariant vector fields in terms of the associated K-covariant mappings. The Cartan embedding $G/K \longrightarrow P \subset G$ given by $gK \mapsto g\theta(g)^{-1}$ is a fundamental tool in the study of symmetric spaces, and it is natural to use it to study $\mathfrak{X}(G/K)^K$. Invariant vector fields on G/K whose horizontal lifts to G are tangent to P are called *flat* (in fact, the Cartan embedding induces *a priori* two different notions of flatness, which we show to be equivalent). We obtain

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a commutator formula with no curvature term for the action on P of these vector fields. For G simple and simply connected, we prove (Theorem 3.1) that every element of $\mathfrak{X}(G/K)^K$ is flat if and only if K is semisimple (i.e. G/K is not the complexification of a hermitian symmetric space).

In Section 4 we study the conjugation action of a semisimple group G on itself. This is an example of the Cartan embedding of a symmetric space for the group $G \times G$ and involution $\theta(g,h) = (h,g)$. In this case all conjugation-invariant vector fields on G are flat. Assuming G is simply-connected, we show that the gradients of the characters of the fundamental representations of G give a free basis for the conjugation-invariant vector fields on all of G, with no localization needed (Theorem 4.1). In the special case of $\mathrm{SL}(n,\mathbf{C})$ we calculate the commutators of an explicit basis of conjugation-invariant vector fields. When $G = \mathrm{SL}(2,\mathbf{C})$ we construct a $\mathbf{C}$-basis for $\mathfrak{X}_2 = \mathfrak{X}(G)^{\mathrm{Ad}\,G}$ and compute the commutators and the action on invariants of this basis. We show that $\mathfrak{X}_2$ is isomorphic to a subalgebra of the Witt algebra (Theorem 4.5) and we find the highest weight vectors inside $\mathbf{C}[\mathrm{SL}(2,\mathbf{C})]$.

Section 5 establishes a separation of variables theorem for $\mathrm{SL}(2,\mathbf{C})$. More precisely, using the preceding results, we construct explicitly a conjugation-invariant differential operator on $\mathrm{SL}(2,\mathbf{C})$ such that its kernel H realizes the isomorphism

$$\mathbf{C}[\mathrm{SL}(2,\mathbf{C})] \;\cong\; \mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,\mathrm{SL}(2,\mathbf{C})} \otimes H\,.$$

This result (Theorem 5.3) is the global version of the separation of variables in the isotropy representation going back to Kostant and Kostant-Rallis ([Kos63], [KR71]). However, our proof requires extensive representation-theoretic calculations and does not seem to extend to arbitrary conjugation actions or symmetric spaces in any obvious way.

A preliminary version of some of the results in this article (in particular, Theorem 5.3) appeared in the first author's dissertation ([Agr00]). While writing this paper, the first author learned from P. Michor (Vienna) that he and B. Kostant had obtained results concerning the conjugation action of an algebraic reductive group G on itself. In the preprint [KM01], conjugation-equivariant maps and their properties are studied using an approach somewhat similar to our description via spherical representations. They also obtain an explicit algebraic separation of variables theorem for $\mathrm{SL}(n,\mathbf{C})$.

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2. K-Invariant vector fields on G/K

2.1. **Vector fields on G and G/K.** Let G be a connected complex reductive linear algebraic group and let θ be an involutive automorphism of G. Let K be the fixed point set G^θ. We denote the decomposition of the Lie algebra $\mathfrak{g}$ into the ±1-eigenspaces of θ by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Since G/K is an affine variety, we can identify the regular functions on G/K with the right K-invariant regular functions on G.

We denote by $\mathfrak{X}(G)$ (respectively $\mathfrak{X}(G/K)$) the regular (algebraic) vector fields on G (respectively G/K). We fix a trivialization of the tangent bundle $T\,G \cong G \times \mathfrak{g}$ so that a regular vector field X on G corresponds to a regular map $\Phi : G \to \mathfrak{g}$ and a left-invariant vector field corresponds to a constant map. The relation between X and Φ is given by

$$Xf(g) = \frac{d}{dt} f(g(1 + t\Phi(g)))\Big|_{t=0} \tag{1}$$

for all $f \in \mathbf{C}[G]$ (we may assume $G \subset \mathrm{GL}(n,\mathbf{C})$; then f is the restriction to G of a regular function on $\mathrm{GL}(n,\mathbf{C})$ so the right side makes sense, with the sums and products being matrix operations).

Proposition 2.1. *Let σ be the isotropy representation of K on $\mathfrak{p}$, and let $\operatorname{Ind}_K^G(\sigma)$ be the space of regular mappings $\Phi : G \to \mathfrak{p}$ satisfying the right K-covariance condition*

$$\Phi(gk) = \sigma(k)^{-1}\Phi(g) \quad \text{for all } k \in K \text{ and } g \in G. \tag{2}$$

Let G act by left translations on $\operatorname{Ind}_K^G(\sigma)$. Then $\mathfrak{X}(G/K) \cong \operatorname{Ind}_K^G(\sigma)$ as a G-module, where the vector field $X \in \mathfrak{X}(G/K)$ corresponding to $\Phi \in \operatorname{Ind}_K^G(\sigma)$ acts by formula (1) on $f \in \mathbf{C}[G/K]$. In particular, the K-invariant regular vector fields on G/K correspond to the K-fixed elements in $\operatorname{Ind}_K^G(\sigma)$.

Proof. The inclusion $\mathbf{C}[G/K] \subset \mathbf{C}[G]$ and the bundle isomorphism

$$T(G/K) \;\cong\; G \times_K \mathfrak{p}$$

imply that a regular vector field X on G/K can be identified with a map $\Phi \in \operatorname{Ind}_K^G(\sigma)$ by formula (1). The covariance condition (2) on Φ implies that $Xf \in \mathbf{C}[G/K]$ for all $f \in \mathbf{C}[G/K]$; the assumption that Φ has values in $\mathfrak{p}$ then makes the correspondence $\Phi \longmapsto X$ bijective. ☐

2.2. **Some finiteness results.** Let $\mathcal{J} = \mathbf{C}[G/K]^K$ be the algebra of K-biinvariant regular functions on G. Then $\mathfrak{X}(G/K)$ and $\operatorname{Ind}_K^G(\sigma)$ are $\mathcal{J}$-modules under pointwise multiplication. Furthermore, if $X \in \mathfrak{X}(G/K)$ corresponds to the map $\Phi : G \to \mathfrak{p}$, then, for $f \in \mathcal{J}$, the vector field fX corresponds to the map $g \mapsto f(g)\Phi(g)$.

Fix a maximal θ-*anisotropic* algebraic torus $A \subset G$ with Lie algebra $\mathfrak{a}$. Let M be the centralizer of A in K, and M' the normalizer of A in K. Let $W = M'/M$ be the "little Weyl group". Then under the restriction map $\mathcal{J} \cong \mathbf{C}[A]^W$ (see [Ric82, Cor. 11.5]).

Theorem 2.1. *Let G be semisimple. Assume that $\mathbf{C}[A]^W$ is a polynomial algebra (this is always true if G is simply connected). Then the space $\mathfrak{X}(G/K)^K$ of K-invariant vector fields on G/K is a free $\mathcal{J}$-module of rank $\dim(\mathfrak{p}^M)$.*

Proof. We have $\mathfrak{X}(G/K)^K \cong \operatorname{Ind}_K^G(\sigma)^K$ as a $\mathcal{J}$-module. But

$$\operatorname{Ind}_K^G(\sigma)^K = \operatorname{Mor}_K(K\backslash G, \mathfrak{p}),$$

the space of K-equivariant regular maps from the right coset space $K\backslash G$ to $\mathfrak{p}$. By [Ric82, Theorems 12.3 and 14.3], there is a K-stable vector subspace E of $\mathbf{C}[K\backslash G]$ so that the pointwise multiplication map $\mathcal{J} \otimes E \to \mathbf{C}[K\backslash G]$ is a vector space isomorphism. Furthermore, the multiplicity of the isotropy representation σ in E is $q = \dim(\mathfrak{p}^M)$. Since K is reductive, it follows that there are maps $\Phi_1, \ldots, \Phi_q$ from $K\backslash G$ to $\mathfrak{p}$ that are linearly independent over $\mathcal{J}$ and span $\operatorname{Mor}_K(K\backslash G)$ as an $\mathcal{J}$-module. ☐

Remark 2.1. In [Ric82, Section 15] Richardson indicates how to determine all pairs (G, θ) with G semisimple, such that $\mathbf{C}[A]^W$ is a polynomial algebra.

2.3. **K-invariant vector fields and spherical functions.** Given $\psi \in \mathcal{J}$, we use the trivialization of the tangent bundle of G from Section 2.1 to identify the differential of ψ with the map $d\psi : G \to \mathfrak{g}^*$ defined by

$$d\psi(g)(X) = \frac{d}{dt}\psi(g(1+tX))\Big|_{t=0} \qquad \text{for } X \in \mathfrak{g}.$$

Since $d\psi(g)(X) = 0$ for $X \in \mathfrak{k}$, we can view $g \mapsto d\psi(g)$ as a map from G to $\mathfrak{p}^*$. From the K-biinvariance of ψ it is clear that

$$d\psi(kgk')(X) = d\psi(g)(\operatorname{Ad}(k')X) \quad \text{for } k, k' \in K. \tag{3}$$

We fix a bilinear form on $\mathfrak{g}$ invariant under $\operatorname{Ad} G$ and θ. This defines an isomorphism $\mathfrak{p} \cong \mathfrak{p}^*$ as a K-module, and we let $\operatorname{grad}\psi(g) \in \mathfrak{p}$ be the element corresponding to $d\psi(g) \in \mathfrak{p}^*$. From (3) we see that $\operatorname{grad}\psi \in \operatorname{Ind}_K^G(\sigma)^K$. Hence by Proposition 2.1 $\operatorname{grad}\psi$ determines a K-invariant regular vector field X_ψ on G/K.

If (π_λ, V_λ) is a finite-dimensional irreducible K-spherical representation of G with highest weight λ, then the dual representation $(\pi_\lambda^*, V_\lambda^*)$ is also K-spherical. We fix $v_\lambda \in V_\lambda^K$ and $v_{\lambda^*} \in V_\lambda^{*K}$, normalized so that $\langle v_\lambda, v_{\lambda^*}\rangle = 1$, and let

$$\psi_\lambda(g) = \langle \pi_\lambda(g)v_\lambda, v_{\lambda^*}\rangle$$

be the corresponding spherical function on G. Then $\psi_\lambda \in \mathcal{J}$ and hence it determines a K-invariant regular vector field that we denote by X_λ. Recall that when $\mathfrak{g}$ is simple and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, then $\mathfrak{k}$ is either semisimple or else has a one-dimensional center ([Hel78]).

Theorem 2.2. *Assume that G is simply connected, $\mathfrak{g}$ is simple, and G/K has rank r. Let $\varphi_1, \dots, \varphi_r$ be algebraically independent generators for $\mathcal{J}$, and let $X_1, \dots, X_r$ be the corresponding K-invariant vector fields on G/K. Then there is a nonzero function $\psi \in \mathcal{J}$ so that the following holds (where $\mathcal{J}_\psi$ and $\mathfrak{X}(G/K)_\psi^K$ denote localizations at ψ):*

- (i) *If the Lie algebra $\mathfrak{k}$ is semisimple, $X_1, \dots, X_r$ generate the $\mathcal{J}_\psi$-module $\mathfrak{X}(G/K)_\psi^K$.*
- (ii) *If the center of $\mathfrak{k}$ is non-zero and has basis J with $(\mathrm{ad}J)^2 = -1$, let Y_i be the vector field corresponding to the map $g \mapsto (\mathrm{ad}J)\mathrm{grad}\,\varphi_i(g)$. Then $X_1, \dots, X_r, Y_1, \dots, Y_r$ generate the $\mathcal{J}_\psi$-module $\mathfrak{X}(G/K)_\psi^K$.*

Proof. We use a modification of arguments from [Ste65, Theorem 8.1], [Sli-B89] and [Sol63]. We first observe that

$$\mathrm{grad}\,\varphi(A) \subset \mathfrak{a} \quad \text{for all } \varphi \in \mathcal{J}. \tag{4}$$

This is a consequence of the KAK polar coordinate decomposition of G, and holds for any reductive G. For the sake of completeness, we give a proof. Consider the restricted root space decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_\alpha \mathfrak{g}_\alpha\,,$$

where $\mathfrak{m} = \mathrm{Lie}(M)$. We claim that

$$d\varphi(a)(X) = 0 \quad \text{for all } a \in A \text{ and } X \in \mathfrak{g}_\alpha. \tag{5}$$

To prove this, observe that $X + \theta X \in \mathfrak{k}$, so $d\varphi(a)(X + \theta(X)) = 0$. The left K-invariance of φ gives

$$\begin{aligned} 0 &= \frac{d}{dt}\varphi(a + t(X+\theta X)a)|_{t=0} = d\varphi(a)(\mathrm{Ad}\,(a)^{-1}(X+\theta X)) \\ &= d\varphi(a)(a^{-\alpha}X + a^\alpha\theta X). \end{aligned}$$

Since we already know that $d\varphi(a)(a^\alpha X + a^\alpha\theta X) = 0$, we conclude that

$$(a^\alpha - a^{-\alpha})d\varphi(a)(X) = 0.$$

Thus (5) holds on the dense open set in A where $a^\alpha \neq a^{-\alpha}$, and hence it holds on all of A. But (5) implies that $\mathrm{grad}\,\varphi(A) \subset (\mathfrak{m} + \mathfrak{a}) \cap \mathfrak{p} = \mathfrak{a}$, proving assertion (4).

Now assume G is simply connected. Then the set Λ_+ of K-spherical highest weights is a free monoid generated by dominant weights $\mu_1, \dots, \mu_r$ in $\mathfrak{a}^*$. Let $\Lambda = \mathbf{Z}\mu_1 + \cdots + \mathbf{Z}\mu_r$ be the lattice generated by these weights. For $\lambda \in \Lambda_+$, define the *monomial symmetric function* $m_\lambda \in \mathbf{C}[A]^W$ by

$$m_\lambda(a) = \sum_{\mu \in W\cdot\lambda} a^\mu.$$

Put a partial order $\prec$ on Λ by $\mu \prec \lambda$ if $\lambda - \mu$ is a sum of positive restricted roots with nonnegative coefficients. If $\lambda \in \Lambda_+$, the spherical function ψ_λ is given on A by a character sum of the form

$$\psi_\lambda(a) = c_0 m_\lambda(a) + \sum_{\substack{\mu\in\Lambda_+ \\ \mu \prec \lambda}} c_\mu m_\mu(a), \tag{6}$$

where $c_0 \neq 0$ ([Vre76]; see also [Hel94, Prop. 9.4]). When $\lambda = \mu_i$, we write $\psi_\lambda = \psi_i$. Let ω_i be the character $\omega_i(a) = a^{\mu_i}$ of A. Then

$$\Omega = \frac{d\omega_1}{\omega_1} \wedge \cdots \wedge \frac{d\omega_r}{\omega_r}$$

is a nowhere vanishing top-degree differential form on A. By formula (4) we can write

$$d\psi_1 \wedge \cdots \wedge d\psi_r|_A = f\,\Omega, \tag{7}$$

where f is a regular function on A that we can calculate using the differentials of $\psi_i|_A$. Set $\rho = \mu_1 + \cdots + \mu_r$. From formula (6) we see that

$$f(a) = ca^\rho + \sum_{\substack{\mu\in\Lambda \\ \mu\prec\rho}} c_\mu a^\mu,$$

with $c \neq 0$. Hence $f \neq 0$, so we conclude from formulas (3) and (7) that $\{d\psi_1, \ldots, d\psi_r\}$ is linearly independent on a dense open set in G. Now

$$d\varphi_1 \wedge \cdots \wedge d\varphi_r = \frac{\partial(\varphi_1, \ldots, \varphi_r)}{\partial(\psi_1, \ldots, \psi_r)} d\psi_1 \wedge \cdots \wedge d\psi_r\,.$$

Since $\varphi_1, \ldots, \varphi_r$ are assumed to generate $\mathcal{J}$, the Jacobian factor is nonzero. Hence the differentials $d\varphi_1, \ldots, d\varphi_r$ are also linearly independent on a dense open set in G.

When $\mathfrak{k}$ is semisimple, $\mathfrak{p}$ is an irreducible K-module and $\mathfrak{p}^M = \mathfrak{a}$ has dimension r by [Ban-J90, Prop. 5.14]. Let $\{Z_1, \ldots, Z_r\}$ be a set of free generators for $\mathfrak{X}(G/K)^K$ given by Theorem 2.1. Then there are functions $\psi_{ij} \in \mathcal{J}$ such that

$$X_j = \sum_i \psi_{ij}\, Z_i\,.$$

Set $\psi = \det[\psi_{ij}]$. Then $\psi \neq 0$, since the vector fields $X_1, \ldots, X_r$ are linearly independent on a dense open set of G/K. This implies statement (i) of the theorem.

Now assume $\mathfrak{k}$ has center $\mathbf{C}J$ with $\mathrm{ad}(J)^2 = -1$. The vector fields Y_i in statement (ii) of the theorem are K-invariant. Since $\mathfrak{p}^M = \mathfrak{a} \oplus \mathrm{ad}(J)\mathfrak{a}$ by [Ban-J90, Lemma 5.7 and Prop. 5.14], the vector fields $X_1, \ldots, X_r, Y_1, \ldots, Y_r$ are linearly independent on a dense open set of G/K by the argument above. Hence statement (ii) of the theorem follows from Theorem 2.1 and the argument used for statement (i). □

3. Lie algebra structure

3.1. Commutator formula on G/K. The symmetric space G/K is the base of a holomorphic principal K-fibre bundle with total space G. The canonical connection $Z : TG \to \mathfrak{k}$ on G has horizontal space

$$T_g^h G \;=\; \{X \in T_g G :\; Z(X) = 0\} \;=\; \{X \in T_g G :\; dL_{g^{-1}}(X) \in \mathfrak{p}\} \;=\; dL_g(\mathfrak{p})$$

at the point $g \in G$. Since we are working in the context of linear algebraic groups, we can take the differential of left and right translation as usual matrix multiplication; thus we write $dL_g(X) = g \cdot X$ (matrix product) for $g \in G$ and $X \in \mathfrak{g}$.

Let X, Y be vector fields on G/K corresponding to maps Φ, Ψ in $\mathrm{Ind}_K^G(\sigma)$, and let X^*, Y^* be their horizontal lifts to vector fields on G. It is clear from the definition of the canonical connection Z that the horizontal lift X^* of X to a vector field on G is given by formula (1). If f is any regular function on G, we have by definition

$$(X^*Y^*f)(g) = \frac{d^2}{ds\,dt} f\big((g + sg\Phi(g))(1 + t\Psi(g + sg\Phi(g)))\big)\Big|_{s=t=0}\,.$$

Taking a first-order Taylor expansion of Ψ to determine the coefficient of st in the argument of f on the right side of this equation, we find that

$$(X^*Y^*f)(g) = \frac{d}{dr} f\big(g(1 + rH(g))\big)\Big|_{r=0} ,$$

where $H(g) = \Phi(g)\Psi(g) + d\Psi_g(g\Phi(g))$. Using the same formula again with the order of X and Y interchanged, we conclude that for any regular function f on G,

$$[X^*, Y^*]f(g) = \frac{d}{dr} f\big(g(1 + r[\Phi(g), \Psi(g)] + r\Phi \circledast \Psi(g))\big)\Big|_{r=0} , \tag{8}$$

where we have set

$$\Phi \circledast \Psi(g) := d\Psi_g(g\Phi(g)) - d\Phi_g(g\Psi(g)). \tag{9}$$

In formula (8) the term $[\Phi(g), \Psi(g)]$ is in $\mathfrak{k}$ and arises from the curvature of the canonical connection. When f is right K-invariant, however, this term can be omitted and we obtain the commutator of X and Y as vector fields on G/K. Thus we have proved the following.

Proposition 3.1. *Let X and Y be vector fields on G/K corresponding to the maps Φ and Ψ in* $\mathrm{Ind}_K^G(\sigma)$*, respectively. Then the commutator $[X, Y]$ corresponds to the map $\Phi \circledast \Psi$ defined in formula* (9).

Remark 3.1. Each term on the right side of (9) satisfies the right K-covariance condition (2). Indeed, if $x \in \mathfrak{p}$, then

$$\begin{aligned} d\Phi_{gk}(gkx) &= \frac{d}{dt}\Phi\big(gk(1 + tx)\big)\Big|_{t=0} = \frac{d}{dt}\Phi\big(g(1 + t\mathrm{Ad}\,(k)x)k\big)\Big|_{t=0} \\ &= \mathrm{Ad}\,(k^{-1}) d\Phi_g(g\mathrm{Ad}\,(k)x) \end{aligned}$$

by the K-covariance property of Φ. Hence

$$d\Phi_{gk}(gk\Psi(gk)) = \mathrm{Ad}\,(k^{-1})d\Phi_g(g\mathrm{Ad}\,(k)\Psi(gk)) = \mathrm{Ad}\,(k^{-1})d\Phi_g(g\Psi(g))$$

as claimed. Likewise, if Φ and Ψ are left K-invariant, then so is the map $g \longmapsto d\Phi_g(g\Psi(g))$.

Remark 3.2. The commutator formula (9) can also be obtained from Cartan's structural equation for the canonical connection, using the fact that this connection is torsion free.

3.2. **Cartan embedding and flat vector fields.** The Cartan embedding of the symmetric space G/K into G furnishes an alternate description of vector fields on G/K. This will allow us to discuss the properties of Lie algebra $\mathfrak{X}(G/K)^K$ in more detail in some cases. The algebraic group version of this embedding is treated in [Ric82] (see also [GW97, Section 11.2.3]). We summarize the results as follows.

Proposition 3.2 (Cartan embedding)**.** *For $g, y \in G$ the formula $g \star y = gy\theta(g)^{-1}$ defines an action of G on itself. The orbit of the identity $P = G \star e = \{g\,\theta(g)^{-1} : g \in G\}$ is a closed irreducible subset of G isomorphic to G/K as a G-space (relative to this action).*

This embedding will be denoted by $j : G/K \longrightarrow P \subset G, \quad gK \longmapsto g\,\theta(g)^{-1}$. Thus we have a commutative diagram

$$\begin{tikzcd} & G/K \arrow[d, "j"] \\ G \arrow[ur] \arrow[r] & P \end{tikzcd}$$

where the map $G \to P$ is $g \mapsto g\theta(g)^{-1}$ and the map $G \to G/K$ is $g \mapsto gK$. The $\star$-action of K on P is the usual conjugation action. By abuse of notation, we shall often write $\mathrm{Ad}\,g$ both for the conjugation action of G on G as well as the adjoint representation of G on $\mathfrak{g}$. We also denote by θ the involution on $\mathfrak{g}$ as well as on G. At any point y of P, one has the inclusion of tangent spaces $T_yP \subset T_yG$. Set $\theta_y = (\mathrm{Ad}\, y)^{-1}\theta$. This is an involution on $\mathfrak{g}$, and we define $\mathfrak{k}_y$ and $\mathfrak{p}_y$ to be the ± 1-eigenspaces of θ_y:

$$\mathfrak{k}_y \;=\; \{X \in \mathfrak{g} :\; \theta_y X = +X\}, \quad \mathfrak{p}_y \;=\; \{X \in \mathfrak{g} :\; \theta_y X = -X\}\,.$$

Let κ_y and π_y be the projections on these spaces:

$$\kappa_y \;=\; \frac{1}{2}(1+\theta_y), \quad \pi_y \;=\; \frac{1}{2}(1-\theta_y)\,.$$

Then $\mathfrak{p}_y$ is exactly the tangent space T_yP, viewed as a subspace of $\mathfrak{g}$ via left translation by y^{-1} [GW97, Section 11.2.7], and may be realized as

$$T_yP \;=\; \mathfrak{p}_y \;=\; \{\mathrm{Ad}\, y^{-1}X - \theta(X) :\; X \in \mathfrak{g}\}\,.$$

The group K permutes the subspaces $\mathfrak{p}_y$, leaving $\mathfrak{p}_e$ is invariant. More precisely, $\mathrm{Ad}\, k$ maps $\mathfrak{p}_y$ to $\mathfrak{p}_{kyk^{-1}}$ in an equivariant way, as follows.

Lemma 3.1. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathfrak{p}_e \arrow[r, "\pi_y"] \arrow[d, "\mathrm{Ad}\, k"'] & \mathfrak{p}_y \arrow[d, "\mathrm{Ad}\, k"] \\
\mathfrak{p}_e \arrow[r, "\pi_{kyk^{-1}}"'] & \mathfrak{p}_{kyk^{-1}}
\end{tikzcd}$$

Proof. If X is in $\mathfrak{p}_y$ and $Y = \mathrm{Ad}\, kX$, then

$$\begin{aligned}
\theta(Y) &= \theta(\mathrm{Ad}\, kX) = \mathrm{Ad}\, \theta(k)\, \theta(X) = -\mathrm{Ad}\, k\, \mathrm{Ad}\, yX \\
&= -\mathrm{Ad}\,(kyk^{-1})\mathrm{Ad}\, kX = -\mathrm{Ad}\,(kyk^{-1})Y.
\end{aligned}$$

Hence $\mathrm{Ad}\, k$ maps $\mathfrak{p}_y$ to $\mathfrak{p}_{kyk^{-1}}$, as claimed. The commutativity of the diagram is as easily verified. ☐

If $\Phi : P \to \mathfrak{g}$ is any regular map, then we can define a regular vector field $\tilde{\Phi}$ on P by

$$\tilde{\Phi} f(y) = \left.\frac{d}{dt} f(y + ty\pi_y\Phi(y))\right|_{t=0}$$

for $f \in \mathbf{C}[P]$ and $y \in P$. Now assume that $\Phi(y) \in \mathfrak{p}$ for all $y \in P$. Since $\theta(\Phi(y)) = -\Phi(y)$ in this case, we can write

$$y\pi_y\Phi(y) = \frac{1}{2}(y\Phi(y) + y\mathrm{Ad}\,(y^{-1})\Phi(y)) = \{y, \Phi(y)\},$$

where $\{a, b\} = (1/2)(ab + ba)$ is the (normalized) anti-commutator of the matrices a, b. This gives the alternate formula

$$\tilde{\Phi} f(y) = \left.\frac{d}{dt} f(y + t\{y, \Phi(y)\})\right|_{t=0} \tag{10}$$

for maps Φ with values in $\mathfrak{p}$. If we assume that Φ is K-equivariant:

$$\Phi(kyk^{-1}) = \mathrm{Ad}\,(k)\Phi(y) \quad \text{for all } k \in K \text{ and } y \in P,$$

then a brief calculation (using Lemma 3.1) shows that $\tilde{\Phi}$ is a K-invariant vector field on P.

Definition 3.1. The vector field $\tilde{\Phi}$ is said to be *flat* if

(i) $\Phi : P \to \mathfrak{p}$ is K-equivariant
(ii) $\mathrm{Ad}\,(y)\Phi(y) = \Phi(y)$ for all $y \in P$.

Since $\{y, \Phi(y)\} = y\Phi(y)$ for a flat field, formula (10) becomes

$$\tilde{\Phi} f(y) = \left.\frac{d}{dt} f(y + ty\Phi(y))\right|_{t=0} \tag{11}$$

in this case.

Lemma 3.2. *Let $\Phi : P \to \mathfrak{p}$ be a regular, K-equivariant map. The following are equivalent:*

(i) $\tilde{\Phi}$ *is flat;*
(ii) $\tilde{\Phi}_y \in T_y(P)$ *for all* $y \in P$*;*
(iii) $\Phi(A) \subset \mathfrak{a}$.

Proof. (i) $\iff$ (ii): From the identification of T_yP with a subspace of $\mathfrak{g}$, condition (ii) is equivalent to

$$\pi_y\Phi(y) = \Phi(y) \quad \text{for all } y \in P.$$

But $\Phi(y) \in \mathfrak{p}$ when $y \in P$, so $\theta\Phi(y) = -\Phi(y)$, and hence

$$\pi_y\Phi(y) = \frac{1}{2}(1 + \mathrm{Ad}\,(y))\Phi(y)\,.$$

This gives the equivalence of (i) and (ii).

(i) $\Longrightarrow$ (iii): Let $a \in A$ be a regular element. Then $\Phi(a) \in \mathfrak{a}$ if and only if $\mathrm{Ad}\,(a)\Phi(a) = \Phi(a)$. In particular, (i) implies that Φ maps the regular elements of A into $\mathfrak{a}$. Since the regular elements are dense in A, this implies (iii).

(iii) $\Longrightarrow$ (i): Let $a \in A$ and $k \in K$. Set $y = kak^{-1}$. Then $\Phi(y) = \mathrm{Ad}\,(k)\Phi(a)$ by the K-covariance properties of Φ, so $\mathrm{Ad}\,(y)\Phi(y) = \mathrm{Ad}\,(k)\mathrm{Ad}\,(a)\Phi(a)$. Now use (iii) and the K-covariance again to obtain

$$\mathrm{Ad}\,(y)\Phi(y) = \mathrm{Ad}\,(k)\Phi(a) = \Phi(y)\,.$$

Since $\mathrm{Ad}\,(K)A$ is dense in P, this equation holds everywhere on P, and hence $\tilde{\Phi}$ is flat. □

Proposition 3.3. *Let $\mathfrak{X}(P)^K_{\mathrm{flat}}$ be the set of all flat vector fields on P.*

(i) *$\mathfrak{X}(P)^K_{\mathrm{flat}}$ is a $\mathcal{J}$-submodule of $\mathfrak{X}(P)^K$.*
(ii) *If $X, Y \in \mathfrak{X}(P)^K_{\mathrm{flat}}$ correspond to the maps Φ, Ψ respectively, then $[X, Y] = Z$, where Z is the flat vector field corresponding to the map $\Phi \circledast \Psi$. Hence $\mathfrak{X}(P)^K_{\mathrm{flat}}$ is a Lie subalgebra of $\mathfrak{X}(P)^K$.*

Proof. (i): This is obvious from the definition.

(ii): If Φ and Ψ are any regular maps from P to $\mathfrak{p}$, then a straightforward calculation as in the proof of formula (8) shows that $[\tilde{\Phi}, \tilde{\Psi}] = \widetilde{\Phi\#\Psi}$, where $\Phi\#\Psi : P \to \mathfrak{g}$ is defined by

$$\Phi\#\Psi(y) = d\Psi_y(\{y, \Phi(y)\}) - d\Phi_y(\{y, \Psi(y)\}) + \frac{1}{2}[\Phi(y), \Psi(y)]\,. \tag{12}$$

Note however that $\Phi\#\Psi$ has values in $\mathfrak{g}$ rather than $\mathfrak{p}$, in general, so formula (12) does not define a Lie algebra structure on the set of regular maps from P to $\mathfrak{p}$. The projection onto T_yP of the $\mathfrak{k}$ component $\frac{1}{2}[\Phi(y), \Psi(y)]$ in formula (12) is the *curvature term.*

Now assume that Φ and Ψ correspond to flat vector fields X and Y. Let $a \in A$. Then the pointwise commutator $[\Psi(a), \Phi(a)] = 0$ by condition (iii) of Lemma 3.2. Hence

$$[\Psi(kak^{-1}), \Phi(kak^{-1}] = \mathrm{Ad}\,(k)[\Psi(a), \Phi(a)] = 0 \quad \text{for } k \in K$$

by K-covariance. Since $\mathrm{Ad}\,(K)A$ is dense in P, it follows that $[\Psi(y), \Phi(y)] = 0$ for all $y \in P$. Hence the curvature term is zero, $\{y, \Phi(y)\} = y\Phi(y)$, and $\Phi\#\Psi = \Phi \circledast \Psi$. It is clear that $\Phi \circledast \Psi$ satisfies condition (iii) of Lemma 3.2, so $[X, Y]$ is a flat vector field. □

Definition 3.2. Let $X \in \mathfrak{X}(G/K)^K$ and let X^* be the horizontal lift of X to a vector field on G. The vector field X is said to be *horizontally flat* if $X^*_y \in T_y(P)$ for all $y \in P$.

If X is a horizontally flat K-invariant vector field on G/K and f is a regular function on G that vanishes on P, then $X^*f|_P = 0$ also. Hence X^* restricts to a well-defined vector field on P that we denote by $X^\natural$. If X is defined by a map $\Phi \in \mathrm{Ind}^G_K(\sigma)^K$, we see from formulas (1) and (11) that $X^\natural = \tilde{\Phi}$. We note that Φ is uniquely determined by its restriction to P, since KP is dense in G, so X is determined by $X^\natural$ when X is horizontally flat. Also $\Phi(kyk^{-1}) = \Phi(y)$ for $k \in K$ and $y \in P$. Thus, by Lemma 3.2, the flatness of X is equivalent to the condition $\Phi(A) \subset \mathfrak{a}$. In this case, $X^\natural \in \mathfrak{X}(P)^K_{\mathrm{flat}}$.

Proposition 3.4. *Let $\varphi \in \mathcal{J}$. Then X_φ is a horizontally flat vector field.*

Proof. This follows from formula (4), Lemma 3.2, and the remarks just made. □

Let $j^* : \mathbf{C}[P] \to \mathbf{C}[G/K]$ be the algebra isomorphism obtained from the Cartan embedding ($j^* f = f \circ j$ for $f \in \mathbf{C}[P]$). Define the push-forward vector field $j_*(X) = j^{*-1} \circ X \circ j^*$ for $X \in \mathfrak{X}(G/K)$. Then j_* gives an isomorphism between $\mathfrak{X}[G/K]^K$ and $\mathfrak{X}[P]^K$. Suppose $X \in \mathfrak{X}(G/K)^K$ is defined by a map $\Phi : G \to \mathfrak{p}$. The left K-invariance of Φ and the isomorphism $G/K \cong P$ given by the Cartan embedding imply the existence of a regular map $\Psi : G \to \mathfrak{p}$ such that

$$2\Phi(g) = \Psi(\theta(g)^{-1} g) \quad \text{for all } g \in G.$$

Let $f \in \mathbf{C}[P]$ and $y \in P$. Since $j^{-1}(y^2) = y$ for $y \in P$, we have

$$\begin{aligned} j_*(X) f(y^2) &= X(j^* f)(y) = \frac{d}{dt} f\big(y(1 + t\Phi(y))^2 y\big)\Big|_{t=0} \\ &= \frac{d}{dt} f\big(y^2(1 + t(\operatorname{Ad} y)^{-1}\Psi(y^2))\big)\Big|_{t=0} \end{aligned} \tag{13}$$

(note that $t \mapsto (1 + t\Phi(y))\theta(1 + t\Phi(y))^{-1}$ is tangent to $t \mapsto 1 + 2t\Phi(y)$ at $t = 0$). Equation (13) uniquely determines $j_*(X)$, since the map $y \mapsto y^2$ is surjective on P.

When $X \in \mathfrak{X}(G/K)^K$ is horizontally flat, it determines two vector fields on P, namely $X^\natural$ and $j_*(X)$. It is evident from equation (13) that these vector fields are not the same. However, the two notions of flatness are related as follows.

Lemma 3.3. *Let $X \in \mathfrak{X}(G/K)^K$. Then X is horizontally flat if and only if $j_*(X) \in \mathfrak{X}(P)^K_{\mathrm{flat}}$.*

Proof. Suppose $j_*(X) \in \mathfrak{X}(P)^K_{\mathrm{flat}}$. Then $j_*(X) = \tilde{\Psi}$, where $\Psi : P \to \mathfrak{p}$ is a regular K-covariant map such that $\operatorname{Ad}(y)\Psi(y) = \Psi(y)$ for $y \in P$. From equation (13) we see that

$$2\Phi(y) = \Psi(y^2) \quad \text{for all } y \in P.$$

It follows that $\operatorname{Ad}(y)\Phi(y) = \Phi(y)$ for $y \in P$, so X is horizontally flat by Lemma 3.2.

Conversely, if X is horizontally flat, then $\operatorname{Ad}(y)\Phi(y) = \Phi(y)$ for all $y \in P$. Let the map Ψ be as in equation (13). Since $\operatorname{Ad}(y)\Psi(y^2) = \Psi(y^2)$ for all $y \in P$, we see that $j_*(X) = \tilde{\Psi}$ by equation (13). The right K-covariance of Φ and the surjectivity of the map $y \mapsto y^2$ on P imply that

$$\Psi(kyk^{-1}) = \operatorname{Ad}(k)\Psi(y) \quad \text{for } k \in K \text{ and } y \in P.$$

Thus $\tilde{\Psi} \in \mathfrak{X}(P)^K_{\mathrm{flat}}$. □

In light of Lemma 3.3, we shall simply use the term *flat* in the rest of the paper to refer either to a horizontally flat vector field $X \in \mathfrak{X}(G/K)^K$ or to an element in $\mathfrak{X}(P)^K_{\mathrm{flat}}$.

Theorem 3.1. *Assume that G is simply connected and $\mathfrak{g}$ is simple. The following are equivalent:*

(i) *$\mathfrak{k}$ is semisimple.*
(ii) *Every K-invariant regular vector field on G/K is flat.*

Furthermore, when (ii) *holds, then $\mathfrak{X}(P)^K = X(P)^K_{\mathrm{flat}}$.*

Proof. Let $\varphi_1, \ldots, \varphi_r$ be a set of algebraically independent generators for $\mathcal{J}$. If $\mathfrak{k}$ is semisimple, the K-invariant vector fields corresponding to $\operatorname{grad}\varphi_1, \ldots, \operatorname{grad}\varphi_r$ are a $\mathcal{J}_\psi$-module basis for $\mathfrak{X}(G/K)^K_\psi$ by Theorem 2.2. These vector fields are flat by Proposition 3.4. Hence all K-invariant vector fields on G/K are flat by Proposition 3.3 (the property of flatness is invariant under localization). On the other hand, if $\mathfrak{k}$ is not semisimple, then $\operatorname{ad}(J)\operatorname{grad}\varphi_i(A) \not\subset \mathfrak{a}$, so the corresponding vector field Y_i is not flat by Lemma 3.2. The last statement follows from Lemma 3.3. □

Remark 3.3. When $\mathfrak{k}$ is not semisimple, the space G/K is the complexification of a hermitian symmetric space. From Theorem 2.2 we have a direct sum decomposition

$$\mathfrak{X}(G/K)^K_\psi = \mathfrak{X} + \mathfrak{Y},$$

where $\mathfrak{X}$ is the Lie algebra of flat rational vector fields generated over $\mathcal{J}_\psi$ by the gradient fields X_i, while $\mathfrak{Y}$ is generated over $\mathcal{J}_\psi$ by the (nonflat) fields Y_i. We have not determined the commutation relations between X_i and Y_j.

We finish this section with an easy example where the K-action is trivial, yielding the Witt algebra of algebraic vector fields on the one-sphere.

Example 3.1 (Witt algebra)**.** The (complexified) one-sphere $\mathbf{C}^*$ is, in Cartan's classification, a symmetric space of type BDI with the following involution,

$$\mathbf{C}^* \;=\; \mathrm{SO}(2,\mathbf{C})/S(\mathrm{O}(1,\mathbf{C})\times\mathrm{O}(1,\mathbf{C}))\,,\quad \theta\begin{bmatrix} a & b\\ -b & a\end{bmatrix} = \begin{bmatrix} a & -b\\ b & a\end{bmatrix}.$$

One checks that $K = \{\mathbf{1},-\mathbf{1}\}$, $P = \mathrm{SO}(2,\mathbf{C})$ and thus $\mathfrak{g} = \mathfrak{p} = \mathfrak{so}(2,\mathbf{C}) \cong \mathbf{C}$. Since the K-action on P is by conjugation, it is trivial, so *any* regular map $P \cong \mathbf{C}^* \to \mathfrak{p} \cong \mathbf{C}$ induces a K-invariant vector field on the sphere. Those are spanned by $f_n(x) = x^n$ for $n \in \mathbf{Z}$, with differential $(df_n)_x(a) = nax^{n-1}$. Since the projection π_x is trivial, the commutator formula (9) gives

$$f_n \circledast f_m(x) \;=\; (df_m)_x(x\cdot f_n(x)) - (df_n)_x(x\cdot f_m(x)) \;=\; (m-n)x^{n+m} \;=\; (m-n)f_{m+n}(x)\,.$$

This is the well-known commutator relation of the Witt algebra. The linear combinations $k_n = f_n - f_{-n}$ and $p_n = p_n + p_{-n}$ satisfy the relations

$$k_n \circledast k_m \;=\; (m-n)k_{n+m} - (n+m)k_{m-n},\quad p_n \circledast p_m \;=\; (n-m)k_{n+m} + (n+m)k_{n-m},$$

$$p_n \circledast k_m \;=\; (n-m)p_{n+m} - (n+m)p_{n-m}\,.$$

The Witt algebra thus carries a $\mathbf{Z}_2$-graduation which we shall encounter again later (Theorem 4.5).

4. The conjugation action

4.1. **Conjugation-invariant vector fields.** Consider the conjugation action of a connected reductive algebraic group G on itself. It fits into the general scheme by choosing $\bar{G} = G\times G$ with the involution $\theta(g,h) = (h,g)$. Then $K = \{(g,g) : g\in G\}$ is the diagonal embedding of G in $G\times G$, and the Cartan embedding

$$j : \bar{G}/K \;=\; (G\times G)/G \longrightarrow G\times G,\quad (g,h)K \longmapsto (g,h)\theta(g,h)^{-1} \;=\; (gh^{-1},hg^{-1})$$

realizes P as $\{(g,g^{-1}) : g\in G\}$, to which there corresponds $\mathfrak{p} = \{(X,-X) : X\in\mathfrak{g}\}$ on the Lie algebra side. The regular functions on $\bar{G}/K$ are of the form $\varphi(g,h) = f(gh^{-1})$, where $f\in\mathbf{C}[G]$. In particular, on P the function φ is given by $\varphi(g,g^{-1}) = f(g^2)$ (cf. the proof of Lemma 3.3).

The K action on P is by conjugation in each component, so that we may restrict attention to the first component. Thus $\mathbf{C}[G]\cong\mathbf{C}[P]$, where $f\in\mathbf{C}[G]$ gives the function $F(g,g^{-1}) = f(g)$. Conjugation-invariant algebraic vector fields then correspond to conjugation-equivariant regular maps from G to $\mathfrak{g}$, and we denote them by $\mathfrak{X}(G)^{\mathrm{Ad}\,G}$. With this identification, the spherical functions become the irreducible characters of G and the representation σ becomes the adjoint representation of G on $\mathfrak{g}$. The algebra $\mathcal{J}$ consists of the regular class functions on G.

Theorem 4.1. *Assume G is simply connected and $\mathfrak{g}$ is semisimple of rank r. Let $\varphi_1,\dots,\varphi_r$ be the characters of the fundamental representations of G. Then the vector fields $X_1,\dots,X_r$ on G corresponding to* $\operatorname{grad}\varphi_1,\dots,\operatorname{grad}\varphi_r$ *are a $\mathcal{J}$-module basis for $\mathfrak{X}(G)^{\mathrm{Ad}\,G}$. Furthermore, all conjugation-invariant vector fields are flat.*

Proof. Let $T \subset G$ be a maximal torus. We may take

$$A = \{(t, t^{-1}) \,:\, t \in T\}, \quad M = \{(t,t) \,:\, t \in T\}.$$

The action of M on $\mathfrak{p}$ is equivalent to the adjoint action of T on $\mathfrak{g}$, hence dim $\mathfrak{p}^M = \dim T = r$. By [Ste65, Theorem 8.1] the vector fields $X_1, \dots, X_r$ are linearly independent on the set of regular elements of G. Hence the function ψ in Theorem 2.2 never vanishes on the set of regular elements, so its zero set is contained in the set Q of irregular elements of G. But Q is a Zariski closed set of codimension 3 by [Ste65, Theorem 1.3]. Hence ψ must be constant. For $\Phi : G \to \mathfrak{g}$ a conjugation equivariant map, we have $\mathrm{Ad}\,(y)\Phi(y) = \Phi(y^2 y^{-1}) = \Phi(y)$ for all $y \in G$. Thus the vector field $\tilde{\Phi}$ is flat. □

Remark 4.1. Let $N \cong \mathbf{C}^r$ be the cross-section for the set of regular elements of G constructed in [Ste65, Theorem 1.4]. Then Theorem 4.1 applies to any set $\{\varphi_1, \dots, \varphi_r\}$ of generators for $\mathcal{J}$ if it is known that $\{d\varphi_1, \dots, d\varphi_r\}$ is linearly independent at every point of N.

4.2. **Invariant vector fields on** $\mathrm{SL}(n, \mathbf{C})$**.** We now apply some of our general results to $\mathrm{SL}(n, \mathbf{C})$. The same method applies to other classical groups and symmetric spaces using Theorems 2.2 and 3.1 and the generators for the invariant polynomials given in [GW97, Section 12.4.2].

Theorem 4.2. *Let $G = \mathrm{SL}(n, \mathbf{C})$. Define maps $\Phi_k : G \to \mathfrak{g}$ by*

$$\Phi_k(g) \;=\; g^k - (1/n)\,\mathrm{tr}(g^k) \cdot \mathbf{1} \quad \textit{for } g \in G.$$

Then $\mathfrak{X}(G)^{\mathrm{Ad}\,G}$ is generated (as a module over $\mathbf{C}[G]^{\mathrm{Ad}\,G}$) by the vector fields $\tilde{\Phi}_1, \dots, \tilde{\Phi}_{n-1}$.

Proof. Define $\varphi_k(g) = (1/k)\,\mathrm{tr}(g^k)$ for $g \in G$. Then for $X \in \mathfrak{g}$ we calculate that

$$d\varphi_k(g)(X) = \frac{d}{dt}\varphi_k(g(1+tX))\bigg|_{t=0} = \mathrm{tr}(g^k X) = \mathrm{tr}(\Phi_k(g)X).$$

Using the trace form to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, we see that $\mathrm{grad}\,\varphi_k = \Phi_k$. The restriction of φ_k to the diagonal is a multiple of the power sum of degree k, so $\varphi_1, \dots, \varphi_{n-1}$ generate the G-invariant regular functions. The matrices

$$X = \begin{bmatrix} c_1 & -c_2 & \cdots & (-1)^{n-2}c_{n-1} & (-1)^{n-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

give a cross-section N for the regular elements of G as $[c_1, c_2, \dots, c_{n-1}]$ ranges over $\mathbf{C}^{n-1}$ [Ste65, Section 7.4]. It is easy to see that $X, X^2, \dots, X^{n-1}$ are linearly independent. Hence the maps $\Phi_1, \dots, \Phi_{n-1}$ are linearly independent at all points of N. The result now follows from Remark 4.1. □

We compute the commutators of the vector fields in Theorem 4.2. Since all the conjugation invariant vector fields are flat (by Theorem 4.1), it suffices by Proposition 3.3 to calculate the maps $\Phi_k \circledast \Phi_l$.

Theorem 4.3. *The maps Φ_k satisfy the commutation relations*

$$\Phi_k \circledast \Phi_l(g) \;=\; (l-k)\cdot\Phi_{k+l}(g) + \frac{k}{n}\cdot\mathrm{tr}(g^l)\,\Phi_k(g) - \frac{l}{n}\cdot\mathrm{tr}(g^k)\,\Phi_l(g)\,.$$

Proof. One obtains for the differential

$$(d\Phi_k)_g(X) \;=\; Xg^{k-1} + gXg^{k-2} + \cdots + g^{k-1}X - \frac{k}{n}\mathrm{tr}(Xg^{k-1})\,,$$

which implies

$$\begin{aligned}(d\Phi_k)_g\big(g\cdot\Phi_l(g)\big) &= k\left(g^{l+1}-\frac{1}{n}\,\mathrm{tr}(g^l)\,g\right)g^{k-1}-\frac{k}{n}\,\mathrm{tr}\left(\big(g^{l+1}-\frac{1}{n}\mathrm{tr}(g^l)g\big)g^{k-1}\right)\\ &= k\big(g^{k+l}-\frac{1}{n}\,\mathrm{tr}(g^{k+l})\big)-\frac{k}{n}\,\mathrm{tr}(g^l)\big(g^k-\frac{1}{n}\,\mathrm{tr}(g^k)\big)\\ &= k\cdot\Phi_{k+l}-\frac{k}{n}\,\mathrm{tr}(g^l)\Phi_k\,.\end{aligned}$$

Now apply formula (9). □

In particular, the relation

$$\Phi_1 \circledast \Phi_{-1}(g) \;=\; \frac{1}{n}\big(\mathrm{tr}(g)\,\Phi_{-1}(g)+\mathrm{tr}(g^{-1})\,\Phi_1(g)\big)$$

shows that Φ_1 and Φ_{-1} generate a finite Lie ring over the ring of invariants.

4.3. **Invariant vector fields on** $\mathrm{SL}(2,\mathbf{C})$**.** We consider the case $G=\mathrm{SL}(2,\mathbf{C})$ in more detail.

Theorem 4.4. *Every conjugation invariant map* $\Psi : G=\mathrm{SL}(2,\mathbf{C})\to\mathfrak{g}=\mathfrak{sl}(2,\mathbf{C})$ *is a multiple of the map* $\Psi_1 : g\mapsto g-g^{-1}$ *by an element of* $\mathbf{C}[G]^{\mathrm{Ad}\,G}$.

Proof. The representation of G on $\mathbf{C}^2$ is self-dual, so its character χ satisfies

$$2\chi(g)=\mathrm{tr}(g+g^{-1})$$

Hence $2d\chi=\Psi_1$ by the calculation in the proof of Theorem 4.2. The result now follows from Theorem 4.1. □

In order to get a $\mathbf{C}$-basis of the space $\mathfrak{X}\big(\mathrm{SL}(2,\mathbf{C})\big)^{\mathrm{Ad}\,\mathrm{SL}(2,\mathbf{C})}=\mathfrak{X}_2$, it thus suffices to choose any convenient basis of the space of invariants. The traces on symmetric tensor powers of the fundamental representation V of G turned out to yield the simplest formulas.

Proposition 4.1. *Let* g *be an element of* $G=\mathrm{SL}(2,\mathbf{C})$ *and denote by* S^kV *the* $(k+1)$*-dimensional irreducible representation of* G*. Then*

$$g^{k+1}-g^{-k-1} \;=\; \mathrm{tr}(g)\big|_{S^kV}\cdot(g-g^{-1}).$$

Furthermore,

$$\mathrm{tr}(g)\big|_{S^kV} \;=\; \mathrm{tr}(g^k)+\mathrm{tr}(g^{k-2})+\cdots+\begin{cases}1 & k\text{ even}\\ \mathrm{tr}(g) & k\text{ odd.}\end{cases}$$

Proof. We first prove the second formula on the maximal torus T of G, chosen as before. For $h=\mathrm{diag}(x,1/x)\in T$, one has

$$\mathrm{tr}(h)\big|_{S^kV} \;=\; x^k+x^{k-2}+\cdots+x^{2-k}+x^{-k}\,.$$

Since $\mathrm{tr}(h^n)=x^n+x^{-n}$, the formula follows immediately on T. The case distinction for the last term depends on whether the number of summands is even or odd. Since the trace is conjugation invariant, the formula is valid on the dense set of all conjugates of T, and therefore also holds on G. For the first formula, we note that

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix}+\begin{bmatrix} d & -b\\ -c & a\end{bmatrix} \;=\; (a+d)\cdot\mathbf{1}$$

implies $g^n+g^{-n}=\mathrm{tr}(g^n)\cdot\mathbf{1}$, so the algebraic identity

$$g^{k+1}-g^{-k-1} \;=\; (g^k+g^{k-2}+\cdots+g^{2-k}+g^{-k})\cdot(g-g^{-1})$$

finishes the proof. □

Corollary 4.1. *The vector fields defined by the maps* $\Psi_k(g)=g^k-g^{-k}$ *for* $k\geq 1$ *are a basis for* $\mathfrak{X}_2$ *as a vector space over* $\mathbf{C}$.

We compute the commutation relations for this basis. For notational simplicity, we write Ψ_k for the conjugation invariant vector field defined by the map $g \mapsto \Psi_k(g)$.

Theorem 4.5. *The vector fields Ψ_k satisfy the commutator relations*

$$[\Psi_k, \Psi_l] \;=\; (l-k)\Psi_{k+l} - (k+l)\Psi_{l-k}\,.$$

In particular, the algebra $\mathfrak{X}_2$ of conjugation invariant vector fields on $\mathrm{SL}(2,\mathbf{C})$ *is isomorphic to a subalgebra of the Witt algebra, and the vector fields with even index $\{\Psi_{2k}\}_{k\geq 1}$ span a subalgebra of $\mathfrak{X}_2$.*

Proof. We compute the differential

$$\begin{aligned}(d\Psi_k)_g(X) \;&=\; Xg^{k-1} + gXg^{k-2} + \cdots + g^{k-1}X \\ &\quad + g^{-1}Xg^{-k} + g^{-2}Xg^{-k+1} + \cdots + g^{-k}Xg^{-1}\,,\end{aligned}$$

from which we obtain

$$(d\Psi_k)_g\big(g\cdot\Psi_l(g)\big) \;=\; k\cdot(g^{k+l} - g^{-k-l} + g^{l-k} - g^{k-l}) \;=\; k\cdot(\Psi_{k+l}(g) + \Psi_{l-k}(g))\,.$$

The commutator formula (9) and Proposition 3.3 then imply the result. □

The action of the vector fields Ψ_k on the invariants is of particular interest. The three most important bases for the invariant functions are:

$$\chi_m(g) \;=\; \mathrm{tr}(g)\big|_{S^mV}, \quad I_m(g) \;=\; \mathrm{tr}(g^m), \quad J_m(g) \;=\; \mathrm{tr}(g)^m.$$

Only the action of Ψ_k on the power sum I_m is given by a simple formula. For this reason, we restrict our attention to $k=1$ in the other two cases.

Theorem 4.6. *The vector field Ψ_k acts on invariants in $\mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,\mathrm{SL}(2,\mathbf{C})}$ as follows:*

$$\Psi_k(I_m) \;=\; m(I_{m+k} - I_{m-k}),$$

$$\Psi_1(\chi_m) \;=\; m\,\chi_{m+1} - (m+2)\,\chi_{m-1}, \quad \Psi_1(J_m) \;=\; m\,(J_{m+1} - 4\,J_{m-1})\,.$$

Proof. For the invariant I_m, the computation is straightforward using formula (1)

$$\begin{aligned}\Psi_k(I_m) \;&=\; \frac{d}{dt}I_m(g+tg\Psi_k(g))\Big|_{t=0} \;=\; \frac{d}{dt}\mathrm{tr}\big[(g+t(g^{1+k}-g^{1-k}))^m\big]\Big|_{t=0} \\ &=\; \frac{d}{dt}\mathrm{tr}\big[g^m + t\,m\,g^{m-1}(g^{1+k}-g^{1-k}) + \cdots\big]\Big|_{t=0} \\ &=\; \frac{d}{dt}\mathrm{tr}\big[g^m + t\,m(g^{m+k}-g^{m-k})\big]\Big|_{t=0} \\ &=\; m\,\big(\mathrm{tr}(g^{m+k}) - \mathrm{tr}(g^{m-k})\big) \;=\; m(I_{m+k}-I_{m-k})\,.\end{aligned}$$

Since $\chi_m = I_m + I_{m-2} + \cdots$ by Proposition 4.1, the second formula is easily proved by induction. The last formula is shown with a similar argument than the first and requires at one stage the identity $\mathrm{tr}(g^2) = \mathrm{tr}(g)^2 - 2$, which is immediately verified on matrices. □

From the point of representation theory, the infinite dimensional Lie algebra $\mathfrak{X}_2$ comes with two natural representations (and in fact many more, see Section 5). The commutator formula (Theorem 4.5) describes the structure of the adjoint representation of $\mathfrak{X}_2$ and shows in particular that it has no (non-trivial) finite-dimensional subalgebras. The action on invariants contains a trivial summand (the constant function, annihilated by all Ψ_k), the rest is indecomposable in the following sense: for any fixed $m \neq 0$, the linear hull V_1 of the invariants

$$I_m,\, I_{m\pm 2}, I_{m\pm 4},\, \ldots, I_{m+1} - I_{m-1},\, I_{m\pm 3} - I_{m\pm 1},\, I_{m\pm 5} - I_{m\pm 3},\, \ldots$$

is invariant under the action of $\mathfrak{X}_2$, but its complement V_2 spanned by

$$I_{m+1} + I_{m-1},\, I_{m\pm 3} + I_{m\pm 1},\, I_{m\pm 5} + I_{m\pm 3},\, \ldots$$

is not. The second claim is immediately clear, since Ψ_1 maps V_2 into V_1. For the first, Theorem 4.6 implies that Ψ_k maps I_m into a multiple of $I_{m+k} - I_{m-k}$, which is a linear combination of elements of V_1. The same applies to the image of all differences $I_{m+k} - I_{m-k}$.

Remark 4.2. The example $G = \mathrm{SL}(2, \mathbf{C})$ is treated in detail in Section 3 of the paper [KM01] and the authors obtain similar formulas.

5. A separation of variables theorem for $\mathrm{SL}(2, \mathbf{C})$

5.1. **Harmonic cofree actions.** We recall that an action of a reductive group G on an irreducible affine variety M is called *cofree* if there exists a G-invariant subspace H of $\mathbf{C}[M]$ such that the multiplication map

$$H \otimes \mathbf{C}[M]^G \longrightarrow \mathbf{C}[M], \quad h \otimes f \longmapsto h \cdot f \tag{14}$$

is an isomorphism of vector spaces. Let $M//G$ be the *algebraic quotient* of M by G (the affine variety such that $\mathbf{C}[M//G] \cong \mathbf{C}[M]^G$), and let $\pi : M \mapsto M//G$ be the canonical projection (see [Kra85]). By using the solution to the Serre conjecture concerning algebraic vector bundles on $\mathbf{C}^n$, Richardson ([Ric81]) was able to establish a general algebraic criterion for an action to be cofree.

Theorem 5.1 (Richardson)**.** *Let G be an algebraic group with reductive identity component and M a smooth irreducible affine G-variety. Then this action is cofree whenever the following two conditions are satisfied:*

(i) *the algebra of invariants $\mathbf{C}[M]^G$ is a polynomial ring;*
(ii) *the fiber $\pi^{-1}(x)$ has dimension* $\dim M - \dim M//G$ *for all $x \in M//G$.*

Let G be a simply connected semisimple algebraic group, T a maximal torus in G and W the Weyl group of G relative to T. Then in particular the following group actions are cofree:

(a) *the conjugation action of G on itself;*
(b) *the action of W on T;*
(c) *the K-action on the symmetric space G/K, where K is the fixed point set of some involution θ of G.*

However, Richardson's proof gives no explicit realization of the space H.

Classical results by Kostant and Kostant-Rallis ([Kos63], [KR71]) state (among others) that the isotropy representation $\mathfrak{p}$ of a symmetric space G/K is always cofree. Furthermore, in the factorization (14) in this case, the K-invariant subspace H may always be realized as the intersection of the kernels of a finite number of K-invariant differential operators with constant coefficients, thus generalizing the notion of harmonic polynomials for $\mathrm{SO}(n)$. This justifies the following definition.

Definition 5.1. A cofree action of a reductive algebraic group K on an irreducible affine variety M will be called *harmonic* if there exist K-invariant differential operators $D_1, \ldots, D_n$ on M such that the linear space

$$H \;=\; \bigcap_{i=1}^{n} \ker D_i$$

realizes the isomorphism (14).

Example 5.1. We start with an easy example of a harmonic Weyl group action.

Theorem 5.2. *The action of the Weyl group $W = \mathcal{S}_2$ on the maximal torus $T \cong \mathbf{C}^*$ of $G = \mathrm{SL}(2, \mathbf{C})$ is harmonic.*

Proof. The ring of regular functions of T is isomorphic to $\mathbf{C}[e^z, e^{-z}]$ and the non trivial element of $\mathcal{S}_2$ acts hereon as the inversion $e^{nz} \mapsto e^{-nz}$. Thus the invariant ring is exactly the polynomial ring generated by $e^z + e^{-z}$, and one easily shows that ∂_z and $\mathbf{C}[e^z, e^{-z}]$ together generate the ring of algebraic differential operators on T. The operator

$$D \;=\; (e^z - e^{-z})\partial_z + (e^z + e^{-z})\partial_z^2$$

is obviously W-invariant and an easy calculation shows that its kernel H consists exactly of the functions 1 and $e^z - e^{-z}$. Since on the other hand the affine ring $\mathbf{C}[T]$ splits into the isotypic components of the trivial and the signum representation, one gets

$$\mathbf{C}[T] \;=\; 1 \cdot \mathbf{C}[T]^W + (e^z - e^{-z}) \cdot \mathbf{C}[T]^W \;=\; H \otimes \mathbf{C}[T]^W$$

and the action is therefore harmonic. □

Notice that $(e^z - e^{-z})\partial_z$ is just the W-invariant vector field induced by the W-equivariant mapping $T \to \mathfrak{h}$, $h \mapsto h - h^{-1}$. It should be possible to extend this example to wide classes of Weyl group actions.

5.2. **Harmonicity of the $\mathrm{SL}(2,\mathbf{C})$ conjugation action.** The remainder of this section is devoted to the proof that the conjugation action of $\mathrm{SL}(2,\mathbf{C})$ on itself is harmonic. The strategy is to guess a good candidate for the space H of harmonics (this is the easy part) and to explicitely construct a conjugation invariant differential operator with kernel H.

Under the simultaneous left and right action of G, the affine ring of $\mathrm{SL}(2,\mathbf{C})$ decomposes by Frobenius reciprocity into

$$\mathbf{C}[\mathrm{SL}(2,\mathbf{C})] \;\cong\; \bigoplus_{d\geq 0} S^dV \otimes (S^dV)^* \,. \tag{15}$$

The decomposition of $\mathbf{C}[\mathrm{SL}(2,\mathbf{C})]$ under the conjugation action of G then amounts to decomposing each occurring tensor product into G-irreducibles. By the Clebsch-Gordon formula, we know that $S^dV \otimes (S^dV)^* = S^{2d}V \oplus S^{2(d-1)}V \oplus \cdots \oplus S^0V$, where the trivial representation corresponds to the trace over S^dV. Thus, we obtain

$$\mathbf{C}[\mathrm{SL}(2,\mathbf{C})] \;\cong\; \bigoplus_{d\geq 0} \mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,N}(d) \otimes S^dV \,,$$

where the first factor denotes the matrix functions invariant under the lower diagonal unipotent matrices N and of weight d. In [Agr01, Satz 2.2] it is shown that the infinite-dimensional multiplicity spaces of the appearing irreducible $\mathrm{SL}(2,\mathbf{C})$-modules are irreducible and pairwise inequivalent modules for the canonical action of the algebra of conjugation invariant differential operators; this applies in particular to the ring of invariants itself ($d = 0$). The space $\mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,N}(d)$ is spanned by the products of the d-th power of the matrix coefficient g_{12} with any invariant,

$$\mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,N}(d) \;=\; (g_{12})^d \cdot \mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,\mathrm{SL}(2,\mathbf{C})} \,.$$

Thus we may choose for H the sum of all $\mathrm{SL}(2,\mathbf{C})$-representations with highest weight vector $(g_{12})^d$ for $d = 0, 1, 2, \ldots$. The problem then is to construct an invariant differential operator D with

$$\mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,N} \cap \ker D \;=\; \mathbf{C}[g_{12}] \,.$$

Remark 5.1. The space H coincides with the pull back under the map $g \mapsto \Psi(g)$ of the harmonics on the Lie algebra $\mathfrak{g}$, defined as in [Kos63]. In the preprint [KM01, 6.2], it is shown that this choice of a subspace H of harmonics works for $G = \mathrm{SL}(n,\mathbf{C})$ for all n. Denote by $H(\mathfrak{g})$ the harmonics on $\mathfrak{g}$, and consider the conjugation invariant map $\Phi_1(g) = g - \mathrm{tr}(g) \cdot \mathbf{1}/n$ already encountered in Theorem 4.2. Then Kostant and Michor prove the isomorphism

$$\mathbf{C}[G] \;\cong\; \mathbf{C}[G]^{\mathrm{Ad}\,G} \otimes \Phi_1^* H(\mathfrak{g}) \,.$$

We now turn to the construction of the differential operator D. Besides the vector field $\Psi = \Psi_1$ from Theorem 4.4, the Casimir operator Δ that generates the center of $U(\mathfrak{g})$ will also play a crucial role. We normalize Δ such that

$$\Delta\big|_{S^dV\otimes(S^dV)^*} \;=\; d(d+2)\cdot \mathrm{id}\,.$$

We need to keep track of the behavior of matrix functions under the transition from the left and right to the conjugation action. For this we will use the explicit isomorphism of equation (15). Let u, v be a basis of V and x, y the dual basis of V^*. We choose the monomials $u^k v^{d-k}$, $k = 0, \ldots, d$, as a basis for S^dV and realize the isomorphism $S^d(V^*) \cong (S^dV)^*$ by

$$\varphi_1\cdots\varphi_d \longmapsto \Big[v_1\cdots v_d \longmapsto \frac{1}{d!}\sum_{\sigma\in S_d}\varphi_{\sigma(1)}(v_1)\cdots\varphi_{\sigma(d)}(v_d)\Big]\,.$$

Then $\binom{d}{k}x^k y^{d-k}$ is the basis vector in $(S^dV)^*$ dual to $u^k v^{d-k}$. Since confusions cannot occur, we shall henceforth omit the tensor product sign for elements of $S^dV\otimes(S^dV)^*$. With this choice of dual bases, the trace over S^dV is the total contraction and may be written

$$\mathrm{tr}|_{S^dV} \;=\; (ux+vy)^d \;=\; \sum_{k=0}^{d}\binom{d}{k}x^k y^{d-k}\cdot u^k v^{d-k}\,.$$

Elements of $\mathrm{SL}(2,\mathbf{C})$ will be parameterized as $g = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$. They act on V and V^* by

$$g\cdot\begin{bmatrix}u\\ v\end{bmatrix} \;=\; \begin{bmatrix}\alpha u+\beta v\\ \gamma u+\delta v\end{bmatrix},\quad g^{-t}\begin{bmatrix}x\\ y\end{bmatrix} \;=\; \begin{bmatrix}\delta x-\gamma y\\ -\beta x+\alpha y\end{bmatrix}.$$

For illustration, we check that $(ux+vy)^d$ is G-invariant, thus reproving its identification with the trace,

$$\begin{aligned} g\cdot(ux+vy)^d &= [(\alpha u+\beta v)(\delta x-\gamma y)+(\gamma u+\delta v)(-\beta x+\alpha y)]^d\\ &= [(\alpha\delta-\beta\gamma)(ux+vy)]^d = (ux+vy)^d \end{aligned}$$

and compute the function on G corresponding to the tensor $(uy)^d \in S^dV\otimes S^dV^*$:

$$\begin{aligned} f(g) = y^d(g\cdot u^d) &= y^d\big((\alpha u+\beta v)^d\big) = y^d(\beta^d v^d + d\alpha\beta^{d-1}uv^{d-1}+\ldots)\\ &= \beta^d\cdot 1 + \alpha\beta^{d-1}\cdot 0+\ldots+0 = \beta^d\,. \end{aligned}$$

This is just the highest weight function $(g_{12})^d$.

Theorem 5.3. *The kernel of the conjugation invariant differential operator*

$$D \;=\; -\mathrm{tr}(g)^3\Delta + \mathrm{tr}(g)\Psi^2 + (\mathrm{tr}(g)^2+4)\Psi$$

on $G = \mathrm{SL}(2,\mathbf{C})$, intersected with $\mathbf{C}[\mathrm{SL}(2,\mathbf{C})]^{\mathrm{Ad}\,N}$, consists of the linear hull of all the functions $(g_{12})^n$, $n\in\mathbf{N}$. Hence the conjugation action of $\mathrm{SL}(2,\mathbf{C})$ is harmonic.

Proof. The proof consists of a tedious computation; we only give an outline here. As in Theorem 4.6, one shows that the vector field Ψ acts on the matrix function $J_{m,n}(g) = J_m\cdot g_{12}^n = (\alpha+\delta)^m\beta^n$ by

$$\Psi(J_{m,n}) \;=\; (n+m)\,J_{m+1,n} - 4m\,J_{m-1,n}\,.$$

Thus we obtain for the square of its action

$$\Psi^2(J_{m,n}) \;=\; (n+m)(n+m+1)J_{m+2,n} - 4(2m^2+mn+n)J_{m,n} + 16m(m-1)J_{m-2,n}\,.$$

For $m = 0$ this means in particular

$$D(J_{0,n}) \;=\; -\mathrm{tr}(g)^3 n(n+2)J_{0,n} + \mathrm{tr}(g)\,[n(n+1)J_{2,n} - 4nJ_{0,n}] + (\mathrm{tr}(g)^2+4)nJ_{1,n} \;=\; 0\,,$$

as needed. The proof that $DJ_{m,n}\neq 0$ for $m\neq 0$ requires more work. The problem is to determine the function on G corresponding to the tensor $(uy)^m(ux+vy)^n \in S^{m+n}V\otimes S^{m+n}V^*$, in order to deduce the eigenvalue of Δ on $J_{m,n}$. In fact, a full formula can only be proved for $n = 0$. In this situation, one first shows on the maximal torus T of G the validity of the formula

$$\operatorname{tr}(g)\big|_{S^m V} \;=\; \sum_{k=0}^{[m/2]} (-1)^k \binom{m-k}{k} \operatorname{tr}(g)^{m-2k}\,.$$

Then, using $\Delta \operatorname{tr}(g)\big|_{S^m V} \;=\; m(m+2)\operatorname{tr}(g)\big|_{S^m V}$, a lengthy induction proof yields

$$\Delta \operatorname{tr}(g)^m \;=\; m(m+2)\operatorname{tr}(g)^m - 4m(m-1)\operatorname{tr}(g)^{m-2}\,.$$

The explicit calculation may be found in [Agr00, p. 54-55]. Since the action of Ψ and Ψ^2 was determined before, one gets for the action of D

$$D \operatorname{tr}(g)^m \;=\; -4m(m+1)\operatorname{tr}(g)^{m+1} + 16m(m-2)\operatorname{tr}(g)^{m-1}\,.$$

The right hand side vanishes exactly for $m = 0$, as it should. For the general case, we show that the matrix function $f_{m,n}(g)$ corresponding to $(uy)^n(ux + vy)^m$ has the form

$$f_{m,n}(g) \;=\; \beta^n \left[\operatorname{tr}(g)^m + \frac{m(1-m)}{n+m}\operatorname{tr}(g)^{m-2} + R\right],$$

where the remainder R is a sum of $\operatorname{tr}(g)$ to the powers $m-4, m-6, \ldots$. The main point here is in fact the precise value of the coefficient of $\operatorname{tr}(g)^{m-2}$, since the general form of this Ansatz is obviously correct. For $n = 0$, we recover for the second coefficient the old result $1 - m = -\binom{m-1}{1}$. For the computation, we may restrict $f_{m,n}(g)$ to the Borel subgroup B of all group elements with $\gamma = 0$. Then one has for $b \in B$

$$\begin{aligned} f_{m,n}(b) \;&=\; (\alpha u + \beta v)^n y^n \left[(\alpha u + \beta v)x + \alpha^{-1} v y\right]^m \\ &=\; \left(\beta^n v^n + n\alpha\beta^{n-1} u v^{n-1} + \cdots\right) y^n \\ &\quad\cdot \left[\frac{v^m y^m}{\alpha^m} + m\frac{v^{m-1}y^{m-1}}{\alpha^{m-1}}(\alpha u + \beta v)x + \frac{m(m-1)}{2}\frac{v^{m-2}y^{m-2}}{\alpha^{m-2}}(\alpha u + \beta v)^2 x^2 + \cdots\right]. \end{aligned}$$

We sort this product by increasing powers of α, starting with α^{-m}. The product of the first summands in every factor yields the only contribution to α^{-m}. The two mixed products of the first summand in one factor and the second summand in the other factor both yield contributions to $\alpha^{-(m-2)}$ and $\alpha^{-(m-1)}$. However, the contribution to $\alpha^{-(m-1)}$ is zero, because $v^{n+m}xy^{n+m-1} = 0$, these two basis elements are not dual to each other. Similarly, the product of $\beta^m v^m$ in the first factor with the third summand in the second factor gives no contribution to $\alpha^{-(m-2)}$, because the basis vectors do not match. To summarize, one gets the expansion

$$f_{m,n}(b) \;=\; \beta^n \left[\frac{v^{n+m}y^{n+m}}{\alpha^m} + (m + nm)\frac{uv^{n+m-1}xy^{n+m-1}}{\alpha^{m-2}} + \cdots\right].$$

The vector xy^{n+m-1} is dual to uv^{n+m-1} up to a correction factor of $n+m$, so we finally get

$$f_{m,n}(b) \;=\; \beta^n \left[\frac{1}{\alpha^m} + \frac{m(n+1)}{n+m}\frac{1}{\alpha^{m-2}} + \cdots\right].$$

By its nature, $f_{m,n}(b)$ has to be a product of β^n times an invariant. Thus the expression inside the brackets is a linear combination of powers of $(\alpha + \alpha^{-1})$. Since $(\alpha + \alpha^{-1})^m = \alpha^{-m} + m\alpha^{-(m-2)} + \cdots$, there exists a rearrangement of terms such that

$$f_{m,n}(b) \;=\; \beta^n \left[(\alpha + \alpha^{-1})^m + \left(\frac{m(n+1)}{n+m} - m\right)(\alpha + \alpha^{-1})^{m-2} + \cdots\right].$$

This is the desired expression, on which we can now study the action of the Casimir operator. The function $f_{m,n}$ is an eigenfunction of Δ with eigenvalue $(n+m)(n+m+2)$, hence

$$\Delta\beta^n \operatorname{tr}(g)^m \;=\; (n+m)(n+m+2)f_{m,n} - \frac{m(1-m)}{n+m}\Delta\beta^n\operatorname{tr}(g)^{m-2} - \Delta R\,,$$

and again $\Delta\beta^n \mathrm{tr}(g)^{m-2} = (n+m-2)(n+m)\beta^n \mathrm{tr}(g)^{m-2} +$ lower order terms. Sorting by powers of $\mathrm{tr}(g)$, we get

$$\Delta\beta^n \mathrm{tr}(g)^m \;=\; (n+m)(n+m+2)\beta^n \mathrm{tr}(g)^m + 4m(1-m)\beta^n \mathrm{tr}(g)^{m-2} + \tilde{R}\,.$$

We conjecture that $\tilde{R} = 0$, but we do not need this here. Notice that the second coefficient does *not* depend on n. Going back to the definition of the operator D, we sort the result again by decreasing powers of $\mathrm{tr}(g)$ to obtain

$$D\beta^n \mathrm{tr}(g)^m \;=\; -4m(n+m+1)\beta^n \mathrm{tr}(g)^{m+1} + \cdots\,.$$

As a polynomial in $\mathrm{tr}(g)$, $D\beta^n \mathrm{tr}(g)^m$ vanishes if and only if every coefficient is zero, and a look on the second factor shows that this happens precisely for $m = 0$. Thus we showed that $D\beta^n \mathrm{tr}(g)^m \neq 0$ for $m \neq 0$. □

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agricola@mathematik.hu-berlin.de
Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Sitz: WBC Adlershof
D-10099 Berlin, Germany

goodman@math.rutgers.edu
Rutgers University
Department of Mathematics
110 Frelinghuysen Rd
Piscataway NJ 08854-8019, USA