THE CASIMIR OPERATOR OF A METRIC CONNECTION WITH SKEW-SYMMETRIC TORSION

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ABSTRACT. For any triple (M^n, g, ∇) consisting of a Riemannian manifold and a metric connection with skew-symmetric torsion we introduce an elliptic, second order operator Ω acting on spinor fields. In case of a reductive space and its canonical connection our construction yields the Casimir operator of the isometry group. Several non-homogeneous geometries (Sasakian, nearly Kähler, cocalibrated G_2-structures) admit unique connections with skew-symmetric torsion. We study the corresponding Casimir operator and compare its kernel with the space of ∇-parallel spinors.

Contents

1. Introduction

Consider a Riemannian manifold (M^n, g, ∇) equipped with a metric connection with skew-symmetric torsion T, and denote by $(D^{1/3})^2$ the square of the Dirac operator corresponding to the connection with torsion form T/3. We introduce a second order differential operator Ω that differs from $(D^{1/3})^2$ by a zero order term. This parameter shift has been already used by Bismut in the proof of the local index theorem for hermitian manifolds. Later, generalizing the well-known Parthasarathy formula for the square of the Dirac operator of a symmetric space, Kostant noticed a simple algebraic formula for some element in the tensor product of the universal enveloping algebra by the Clifford algebra of a naturally reductive space. The geometric interpretation of Kostant's "cubic Dirac operator" as a 1/3-parameter shifted Dirac operator for such a space endowed with its canonical connection as well as the formula for the square of any operator D^s in the family have been discussed in the paper [1]. It turns out that in the homogenous situation, our operator coincides with the Casimir operator of

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the reductive space, hence motivating its name. The integral formulas for $(D^{1/3})^2$ are then used in order to study the new operator Ω in greater detail. In general, the kernel of the operator Ω contains all ∇-parallel spinors. In case that the torsion form T is ∇-parallel, the formula for Ω simplifies to

$$\Omega \;=\; (D^{1/3})^2 \,-\, \frac{1}{16}\big(2\,\mathrm{Scal}^g \,+\, ||\mathrm{T}||^2\big)\,,$$

and then the operators Ω and $(D^{1/3})^2$ commute with the action of the torsion form on spinors. Triples (M^n, g, ∇) occur in the study of non integrable special Riemannian manifolds in a natural way. For example, any Sasakian manifold in odd dimensions, any hermitian manifold with skew-symmetric Nijenhuis tensor in even dimensions, any cocalibrated G_2-manifold in dimension seven and any Spin(7)-manifold in dimension eight admit a unique metric connection with skew-symmetric torsion and preserving the additional geometric structure (see [10] and [9]). The torsion forms of these connections are models for the B-field in the string equations and their parallel spinor fields are the supersymmetries of the models. From the mathematical point of view, the basic role of these connections is closely related to the fact that many of the geometric data of the non integrable geometric structure can be read of its unique torsion form.

We study the Casimir operator of a Riemannian manifold equipped with a metric connection. In particular, we compare its kernel with the space of ∇-parallel or with the space of Riemannian Killing spinors. The low dimensions are specially interesting. Therefor we investigate Sasakian manifolds in dimension five, nearly Kähler manifolds in dimension six, and cocalibrated G_2-manifolds in dimension seven in detail. In case that a non integrable geometric structure admits a transitive automorphism group and the space is reductive, then its unique geometric connection coincides with the canonical connection of the reductive space. Henceforth, our geometric Casimir operator is the group-theoretical Casimir operator acting on spinors and we can understand some of its properties in a geometric way, for example vanishing theorems.

2. An overview of Schrödinger-Lichnerowicz type formulas for Dirac operators

Consider a Riemannian spin manifold (M^n, g, T) with a 3-form T. Then we obtain a metric connection with torsion T,

$$\nabla_X Y \;:=\; \nabla^g_X Y \,+\, \frac{1}{2}\,\mathrm{T}(X, Y, -)\,.$$

We denote by Scal^g and Scal the scalar curvature of the Levi-Civita connection and the connection ∇, respectively. The connection can be lifted to a connection on the spinor bundle S of M, where it takes the expression

$$\nabla_X \psi \;:=\; \nabla^g_X \psi \,+\, \frac{1}{4}\,(X \lrcorner\, \mathrm{T})\cdot\psi\,.$$

There is a formula for the square of the Dirac operator D associated with the connection ∇. In order to state it, let us introduce the first order differential operator

$$\mathcal{D}\psi \;:=\; \sum_{k=1}^{n}(e_k \lrcorner\, \mathrm{T})\cdot\nabla_{e_k}\psi \;=\; \mathcal{D}^g\psi \,+\, \frac{1}{4}\sum_{k=1}^{n}(e_k \lrcorner\, \mathrm{T})\cdot(e_k \lrcorner\, \mathrm{T})\cdot\psi\,,$$

where $e_1, \ldots, e_n$ denotes an orthonormal basis. It will be convenient to introduce a 4-form derived from T,

$$\sigma_{\mathrm{T}} \ := \ \frac{1}{2}\sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \wedge (e_k \lrcorner \mathrm{T}) \, .$$

Acting on spinors, the difference between the endomorphisms σ_{T} and $(\mathcal{D} - \mathcal{D}^g)$ is given by the formula

$$\sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \cdot (e_k \lrcorner \mathrm{T}) \ = \ 2\,\sigma_{\mathrm{T}} \ - \ 3\,||\mathrm{T}||^2 \ = \ \sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \wedge (e_k \lrcorner \mathrm{T}) \ - \ 3\,||\mathrm{T}||^2 \ .$$

Theorem 2.1 ([10, Thm 3.1, 3.3]). *Let (M^n, g, ∇) be an n-dimensional Riemannian manifold with a metric connection ∇ of skew-symmetric torsion* T. *Then, the square of the Dirac operator D associated with ∇ acts on an arbitrary spinor field ψ as*

$$(1) \qquad D^2\psi \ = \ \Delta_{\mathrm{T}}(\psi) \ + \ \frac{3}{4}\,d\mathrm{T} \cdot \psi \ - \ \frac{1}{2}\,\sigma_{\mathrm{T}} \cdot \psi \ + \ \frac{1}{2}\,\delta\mathrm{T} \cdot \psi \ - \ \mathcal{D}\psi + \frac{1}{4}\,\mathrm{Scal} \cdot \psi,$$

where Δ_{T} is the spinor Laplacian of ∇,

$$\Delta_{\mathrm{T}}(\psi) \ = \ (\nabla)^*\nabla\psi \ = \ -\sum_{k=1}^{n} \nabla_{e_k}\nabla_{e_k}\psi \ + \ \nabla_{\nabla^g_{e_i} e_i}\psi \, .$$

Furthermore, the anti-commutator of D and T *is*

$$(2) \qquad D \circ \mathrm{T} + \mathrm{T} \circ D \ = \ d\mathrm{T} \ + \ \delta\mathrm{T} \ - \ 2\,\sigma_{\mathrm{T}} - 2\,\mathcal{D}.$$

This formula for D^2 has the disadvantage of still containing a first order differential operator. By shifting the parameter in the torsion of the connection ∇, we can simplify the formula essentially. For the computations, we need the square of T inside the Clifford algebra. The proof of the following proposition is completely similar to that of Proposition 3.1 in [1] and will hence be omitted.

Proposition 2.1. *Let* T *be a 3-form, and denote by the same symbol its associated $(2,1)$-tensor, the two being related by $\mathrm{T}(X, Y, Z) = \langle \mathrm{T}(X, Y), Z\rangle$. Then its square inside the Clifford algebra has no contribution of degree 6, and its scalar and fourth degree part are given by*

$$\mathrm{T}_0^2 \ = \ \frac{1}{6}\sum_{i,j=1}^{n} ||\mathrm{T}(e_i, e_j)||^2 \ = \ ||\mathrm{T}||^2, \quad \mathrm{T}_4^2 \ = \ -\,2\,\sigma_{\mathrm{T}}.$$

With these preparations in hand, we can state a more useful Schrödinger-Lichnerowicz type formula. It links the Dirac operator $D^{1/3}$ of the connection with torsion $\mathrm{T}/3$ and the Laplacian of the connection with torsion T. The remainder is a zero order operator. Similar formulas can be found in [4].

Theorem 2.2 ([2, Thm 6.2]). *The spinor Laplacian Δ_{T} and the square of the Dirac operator $D^{1/3}$ are related by*

$$(D^{1/3})^2 \ = \ \Delta_{\mathrm{T}} \ + \ \frac{1}{4}\,d\mathrm{T} \ + \ \frac{1}{4}\,\mathrm{Scal}^g \ - \ \frac{1}{8}\,\mathrm{T}_0^2 \ .$$

Integrating the latter formula on a compact manifold M^n, we obtain

$$\int_{M^n} ||D^{1/3}\psi||^2 \ = \ \int_{M^n} \Big[||\nabla\psi||^2 + \frac{1}{4}\langle d\mathrm{T} \cdot \psi, \psi\rangle + \frac{1}{4}\mathrm{Scal}^g\,||\psi||^2 - \frac{1}{8}\,\mathrm{T}_0^2\,||\psi||^2 \Big] \, .$$

Finally, we state the Kostant-Parthasarathy formula for $(D^{1/3})^2$ in the homogeneous case, as it is the main motivation for what follows.

Theorem 2.3 ([1, Thm 3.3])**.** *Let $M = G/H$ be a naturally reductive homogeneous space, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then its canonical connection ∇ has skew-symmetric torsion $\mathrm{T}(X,Y,Z) = -g([X,Y]_{\mathfrak{m}},Z)$ $(X,Y,Z \in \mathfrak{m})$, T is ∇-parallel and $D^{1/3}$ satisfies the identity*

$$(D^{1/3})^2 \;=\; \Omega_{\mathfrak{g}} + \frac{1}{8}\mathrm{Scal}^g + \frac{1}{16}\mathrm{T}_0^2,$$

where $\Omega_{\mathfrak{g}}$ denotes the Casimir operator of $\mathfrak{g}$.

In this formula, $D^{1/3}$, Scal^g and T_0^2 appear as geometric invariant objects, whereas $\Omega_{\mathfrak{g}}$ heavily relies on the concrete realization of the homogeneous space M as a quotient. In fact, it is well known that one and the same space may or may not be naturally reductive depending on its realization, and that it typically needs to be represented as a quotient of some *larger* groups in order to make it naturally reductive. Hence it was our goal to find a tool similar to $\Omega_{\mathfrak{g}}$ which has more intrinsic geometric meaning.

3. The Casimir operator of a triple (M^n, g, ∇)

We consider a Riemannian spin manifold (M^n, g, ∇) with a metric connection ∇ and skew-symmetric torsion T. Denote by Δ_{T} the spinor Laplacian of the connection.

Definition 3.1. The *Casimir operator* of the triple (M^n, g, ∇) is the differential operator acting on spinor fields by

$$\begin{aligned}\Omega \;&:=\; (D^{1/3})^2 + \frac{1}{8}(d\mathrm{T} - 2\,\sigma_{\mathrm{T}}) + \frac{1}{4}\,\delta(\mathrm{T}) - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{16}\mathrm{T}_0^2 \\ &=\; \Delta_{\mathrm{T}} + \frac{1}{8}(3\,d\mathrm{T} - 2\,\sigma_{\mathrm{T}} + 2\,\delta(\mathrm{T}) + \mathrm{Scal})\,.\end{aligned}$$

Remark 3.1. For a naturally reductive space $M^n = G/H$, $\Omega = \Omega_{\mathfrak{g}}$ by Theorem 2.3.

Example 3.1. For the Levi-Civita connection $(\mathrm{T} = 0)$ of an arbitrary Riemannian manifold, we obtain

$$\Omega \;=\; (D^g)^2 - \frac{1}{8}\mathrm{Scal}^g \;=\; \Delta^g + \frac{1}{8}\mathrm{Scal}^g\,.$$

The second equality is just the classical Schrödinger-Lichnerowicz formula for the Riemannian Dirac operator, whereas the first one is in case of a symmetric space the classical Parthasarathy formula.

Example 3.2. Consider a 3-dimensional manifold of constant scalar curvature, a constant $a \in \mathbb{R}$ and the 3-form $\mathrm{T} = 2\,a\,dM^3$. Then

$$\Omega \;=\; (D^g)^2 - a\,D^g - \frac{1}{8}\mathrm{Scal}^g\,.$$

The kernel of the Casimir operator corresponds to eigenvalues $\lambda \in \mathrm{Spec}(D^g)$ of the Riemannian Dirac operator such that

$$8\,(\lambda^2 - a\,\lambda) - \mathrm{Scal}^g \;=\; 0\,.$$

In particular, the kernel of Ω is in general larger then the space of ∇-parallel spinors. Indeed, such spinors exist only on space forms. More generally, fix a real-valued smooth function f and consider the 3-form $\mathrm{T} := f \cdot dM^3$. If there exists a ∇-parallel spinor

$$\nabla^g_X\psi + (X \lrcorner \mathrm{T})\cdot\psi \;=\; \nabla^g_X\psi + f\cdot X\cdot\psi \;=\; 0\,,$$

then, by a theorem of A. Lichnerowicz (see [18]), f is constant and (M^3, g) is a space form.

Let us collect some elementary properties of the Casimir operator of a triple (M^n, g, ∇).

Proposition 3.1. *The kernel of the Casimir operator contains all ∇-parallel spinor.*

Proof. By Theorem 2.1, one of the integrability conditions for a ∇-parallel spinor field ψ is

$$\big(3\, d\mathrm{T} \,-\, 2\,\sigma_{\mathrm{T}} \,+\, 2\,\delta(\mathrm{T}) \,+\, \mathrm{Scal}\big)\cdot\psi \;=\; 0\,. \qquad \square$$

If the torsion form T is ∇-parallel, the formulas for the Casimir operator simplify. Indeed, in this case we have (see [10])

$$d\mathrm{T} \;=\; 2\,\sigma_{\mathrm{T}}, \quad \delta(\mathrm{T}) \;=\; 0\,.$$

Moreover, the Ricci tensor Ric^{∇} is symmetric and we have

$$\mathrm{Scal} \;=\; \mathrm{Scal}^g \,-\, \frac{3}{2}\,\mathrm{T}_0^2.$$

Using the formula of Proposition 2.1 as well as the formulas of Theorem 2.1 and Theorem 2.2, we obtain a simpler expressions for the Casimir operator.

Proposition 3.2. *The Casimir operator of a triple (M^n, g, ∇) with $\nabla\mathrm{T} = 0$ can equivalently be written as:*

$$\begin{aligned}\Omega \;&=\; (D^{1/3})^2 \,-\, \frac{1}{16}\big(2\,\mathrm{Scal}^g \,+\, \mathrm{T}_0^2\big) \;=\; \Delta_{\mathrm{T}} \,+\, \frac{1}{16}\big(2\,\mathrm{Scal}^g \,+\, \mathrm{T}_0^2\big) \,-\, \frac{1}{4}\,\mathrm{T}^2\\ &=\; \Delta_{\mathrm{T}} \,+\, \frac{1}{8}\big(2\,d\mathrm{T} \,+\, \mathrm{Scal}\big)\,.\end{aligned}$$

Integrating these formulas, we obtain a vanishing theorem for the kernel of the Casimir operator.

Proposition 3.3. *Let (M^n, g, ∇) be a compact triple such that the torsion form is ∇-parallel. If one of the conditions*

$$2\,\mathrm{Scal}^g \;\leq\; -\,\mathrm{T}_0^2 \quad \text{or} \quad 2\,\mathrm{Scal}^g \;\geq\; 4\,\mathrm{T}^2 \,-\, \mathrm{T}_0^2,$$

holds, the Casimir operator is non-negative in $\mathrm{L}^2(S)$.

Example 3.3. For a naturally reductive space $M = G/H$, the first condition can never hold, since a representation theoretic argument [1, Lemma 3.6] shows that $2\,\mathrm{Scal}^g + \mathrm{T}_0^2$ is strictly positive. In concrete examples, the second condition typically singles out the normal homogeneous metrics among the naturally reductive ones. Notice a small mistake in Lemma 3.5 of [1]: in general, the fact that the negative definite contribution of the scalar product comes from an abelian summand in $\mathfrak{g}$ is *not* enough to conclude that $\Omega_{\mathfrak{g}}$ is non-negative.

Proposition 3.4. *If the torsion form is ∇-parallel, then the Casimir operator Ω and the square of the Dirac operator $(D^{1/3})^2$ commute with the endomorphism T,*

$$\Omega\circ\mathrm{T} \;=\; \mathrm{T}\circ\Omega\,, \quad (D^{1/3})^2\circ\mathrm{T} \;=\; \mathrm{T}\circ(D^{1/3})^2.$$

The endomorphism T acts on the spinor bundle as a symmetric endomorphism with *constant* eigenvalues.

Theorem 3.1. *Let (M^n, g, ∇) be a compact Riemannian spin manifold equipped with a metric connection ∇ with parallel, skew-symmetric torsion, $\nabla \mathrm{T} = 0$. The endomorphism T and the Riemannian Dirac operator D^g act in the kernel of the Dirac operator $D^{1/3}$. In particular, if, for all $\mu \in \mathrm{Spec}(\mathrm{T})$, the number $-\mu/4$ is not an eigenvalue of the Riemannian Dirac operator, then the kernel of $D^{1/3}$ is trivial.*

Proof. On a compact manifold, the kernels of $D^{1/3}$ and $(D^{1/3})^2$ coincide. □

If ψ belongs to the kernel of $D^{1/3}$ and is an eigenspinor of the endomorphism T, we have $4 \cdot D^g \psi = -\mu \cdot \psi$, $\mu \in \mathrm{Spec}(\mathrm{T})$. Using the estimate of the eigenvalues of the Riemannian Dirac operator (see [8]) we obtain an upper bound for the minimum $\mathrm{Scal}^g_{\mathrm{min}}$ Riemannian scalar curvature in case that the kernel of the operator $D^{1/3}$ is non trivial.

Proposition 3.5. *Let (M^n, g, ∇) be a compact Riemannian spin manifold equipped with a metric connection ∇ with parallel, skew-symmetric torsion, $\nabla \mathrm{T} = 0$. If the kernel of the Dirac operator $D^{1/3}$ is non trivial, then the minimum of the Riemannian scalar curvature is bounded by*

$$\max\left\{\mu^2 \,:\, \mu \in \mathrm{Spec}(\mathrm{T})\right\} \;\geq\; \frac{4n}{n-1}\,\mathrm{Scal}^g_{\mathrm{min}}\,.$$

Remark 3.2. If $(n-1)\,\mu^2 = 4n\,\mathrm{Scal}^g$ is in the spectrum of T and there exists a spinor field ψ in the kernel of $D^{1/3}$ such that $\mathrm{T} \cdot \psi = \mu \cdot \psi$, then we are in the limiting case of the inequality in [8]. Consequently, M^n is an Einstein manifold of non-negative scalar curvature and ψ is a Riemannian Killing spinor,

$$\nabla^g_X \psi \;-\; \frac{\mu}{4n} \cdot X \cdot \psi \;=\; 0\,.$$

Examples of this type are 7-dimensional 3-Sasakian manifolds. The possible torsion form has been discussed in [2], Section 9.

4. The Casimir operator of a 5-dimensional Sasakian manifold

Let $(M^5, g, \xi, \eta, \varphi)$ be a compact 5-dimensional Sasakian manifold and denote by ∇ its unique connection with skew-symmetric torsion and preserving the contact structure. Then we have (see [10])

$$\nabla \mathrm{T} \;=\; 0\,, \quad \mathrm{T} \;=\; \eta \wedge d\eta \;=\; 2\,(e_{12} \,+\, e_{34}) \wedge e_5\,, \quad \mathrm{T}^2 \;=\; 8 \,-\, 8\,e_{1234}$$

and

$$\Omega \;=\; (D^{1/3})^2 \,-\, \frac{1}{8}\,\mathrm{Scal}^g \,-\, \frac{1}{2} \;=\; \Delta_{\mathrm{T}} \,+\, \frac{1}{8}\,\mathrm{Scal}^g \,-\, \frac{3}{2} \,+\, 2\,e_{1234}.$$

We study the kernel of the Dirac operator $D^{1/3}$. The endomorphism T acts with eigenvalues ± 4 and 0 and, according to Theorem 3.1, we have to distinguish two cases. If $D^{1/3}\psi = 0$ and $\mathrm{T} \cdot \psi = 0$, the spinor field is harmonic and the formulas of Proposition 3.2 yield in the compact case the condition

$$\int_{M^5} \left(2\,\mathrm{Scal}^g \,+\, 8\right) ||\psi||^2 \;\leq\; 0\,.$$

Examples of that type are the 5-dimensional Heisenberg group with its left invariant Sasakian structure or certain S^1-bundles over a flat torus. On these spaces, there exist ∇-parallel spinors ψ_0 satisfying the algebraic equation $\mathrm{T} \cdot \psi_0 = 0$ (see [10], [11]). Their scalar curvature equals $\mathrm{Scal}^g = -4$. Let us describe the 5-dimensional Heisenberg group. Its Sasakian structure is given on $\mathbb{R}^5$ by the 1-forms:

$$e_1 := \frac{1}{2}dx_1\,,\quad e_2 := \frac{1}{2}dy_1\,,\quad e_3 := \frac{1}{2}dx_2\,,\quad e_4 := \frac{1}{2}dy_2,$$
$$e_5 = \eta := \frac{1}{2}\big(dz - y_1\cdot dx_1 - y_2\cdot dx_2\big)\,.$$

The space of all ∇-parallel spinors satisfying $\mathrm{T}\cdot\psi_0 = 0$ is a 2-dimensional subspace of the kernel of the operator $D^{1/3}$. In a left-invariant frame of M^5, spinors are simply functions $\psi : M^5 \to \Delta_5$ with values in the 5-dimensional spin representation. It turns out that the spinors ψ_0 are constant. Consequently, for any discrete subgroup Γ of the Heisenberg group, the manifold M^5/Γ equipped with its trivial spin structure is a Sasakian manifold admitting spinors in $\mathrm{Ker}(D^{1/3})$. The second case for spinors in the kernel is given by $D^{1/3}\psi = 0$ and $\mathrm{T}\cdot\psi = \pm 4\,\psi$. The spinor field is an eigenspinor for the Riemannian Dirac operator, $D^g\psi = \mp\,\psi$. The formulas of Proposition 3.2 and Proposition 3.5 yield in the compact case two conditions:

$$\int_{M^5}\big(\mathrm{Scal}^g - 12\big)||\psi||^2 \leq 0 \qquad\text{and}\qquad 5\,\mathrm{Scal}^g_{\mathrm{min}} \leq 16.$$

The paper [15] contains a construction of Sasakian manifolds admitting a spinor field of that algebraic type in the kernel of $D^{1/3}$. We describe the construction explicitely. Suppose that the Riemannian Ricci tensor of a simply-connected, 5-dimensional Sasakian manifold is given by the formula

$$\mathrm{Ric}^g = -2\cdot g + 6\cdot\eta\otimes\eta\,.$$

Its scalar curvature equals $\mathrm{Scal}^g = -4$. In the simply-connected and compact case, they are total spaces of S^1 principal bundles over 4-dimensional Calabi-Yau orbifolds (see [5]). There exist (see [15], Theorem 6.3) two spinor fields ψ_1, ψ_2 such that

$$\nabla^g_X\psi_1 = -\frac{1}{2}X\cdot\psi_1 + \frac{3}{2}\eta(X)\cdot\xi\cdot\psi_1\,,\quad \mathrm{T}\cdot\psi_1 = -4\,\psi_1,$$
$$\nabla^g_X\psi_2 = \frac{1}{2}X\cdot\psi_2 - \frac{3}{2}\eta(X)\cdot\xi\cdot\psi_2\,,\quad \mathrm{T}\cdot\psi_2 = 4\,\psi_2.$$

In particular, we obtain

$$D^g\psi_1 = \psi_1\,,\ \mathrm{T}\cdot\psi_1 = -4\,\psi_1\,,\quad\text{and}\quad D^g\psi_2 = -\,\psi_2,\ \mathrm{T}\cdot\psi_2 = 4\,\psi_2,$$

and therefore the spinor fields ψ_1 and ψ_2 belong to the kernel of the operator $D^{1/3}$.

Next, we investigate the kernel of the Casimir operator. Under the action of the torsion form, the spinor bundle S splits into three subbundles $\mathrm{S} = \mathrm{S}_0\oplus\mathrm{S}_4\oplus\mathrm{S}_{-4}$ corresponding to the eigenvalues of T. Since $\nabla\mathrm{T} = 0$, the connection ∇ preserves the splitting. The endomorphism e_{1234} acts by the formulas

$$e_{1234} = 1 \ \text{ on } \mathrm{S}_0\,,\quad e_{1234} = -1 \ \text{ on } \mathrm{S}_4\oplus\mathrm{S}_{-4}.$$

Consequently, the formula

$$\Omega = \Delta_{\mathrm{T}} + \frac{1}{8}\mathrm{Scal}^g - \frac{3}{2} + 2\,e_{1234}$$

shows that the Casimir operator splits into the sum $\Omega = \Omega_0\oplus\Omega_4\oplus\Omega_{-4}$ of three operators acting on sections in S_0, S_4 and S_{-4}. On S_0, we have

$$\Omega_0 = \Delta_{\mathrm{T}} + \frac{1}{8}\mathrm{Scal}^g + \frac{1}{2} = (D^{1/3})^2 - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{2}\,.$$

In particular, the kernel of Ω_0 is trivial if $\mathrm{Scal}^g \neq -4$. The Casimir operator on $\mathrm{S}_4 \oplus \mathrm{S}_{-4}$ is given by

$$\Omega_{\pm 4} \;=\; \Delta_{\mathrm{T}} + \frac{1}{8}\mathrm{Scal}^g - \frac{7}{2} \;=\; (D^{1/3})^2 - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{2}$$

and a non trivial kernel can only occur if $-4 \leq \mathrm{Scal}^g \leq 28$. A spinor field ψ in the kernel of the Casimir operator Ω satisfies the equations

$$(D^{1/3})^2 \cdot \psi \;=\; \frac{1}{8}\,(4 + \mathrm{Scal}^g)\,\psi\,, \quad \mathrm{T}\cdot\psi \;=\; \pm\,4\,\psi\,.$$

In particular, we obtain

$$\int_{M^5} \langle (D^g \pm 1)^2\,\psi\,,\,\psi\rangle \;=\; \frac{1}{8}\int_{M^5} \big(4 + \mathrm{Scal}^g\big)\,||\psi||^2,$$

and the first eigenvalue of the operator $(D^g \pm 1)^2$ is bounded by the scalar curvature,

$$\lambda_1(D^g \pm 1)^2 \;\leq\; \frac{1}{8}\big(4 + \mathrm{Scal}^g_{\max}\big).$$

Let us consider special classes of Sasakian manifolds. A first case is $\mathrm{Scal}^g = -4$. Then the formula for the Casimir operator simplifies,

$$\Omega_0 \;=\; \Delta_{\mathrm{T}} \;=\; (D^{1/3})^2\,, \quad \Omega_{\pm 4} \;=\; \Delta_{\mathrm{T}} - 4 \;=\; (D^{1/3})^2.$$

If M^5 is compact, the kernel of the operator Ω_0 coincides with the space of ∇-parallel spinors in the bundle S_0. A spinor field ψ in the kernel the operator $\Omega_{\pm 4}$ is an eigenspinor of the Riemannian Dirac operator,

$$D^g(\psi) \;=\; \mp\,\psi, \quad \mathrm{T}\cdot\psi \;=\; \pm\,4\,\psi\,.$$

Compact Sasakian manifolds admitting spinor fields in the kernel of Ω_0 are quotients of the 5-dimensional Heisenberg group (see [11], Theorem 4.1). Moreover, the 5-dimensional Heisenberg group and its compact quotients admit spinor fields in the kernel of $\Omega_{\pm 4}$, too. Indeed, the non trivial connection forms of the Levi-Civita connection are

$$\omega_{12} \;=\; e_5 \;=\; \omega_{34}, \quad \omega_{15} \;=\; e_2, \quad \omega_{25} \;=\; -\,e_2, \quad \omega_{35} \;=\; e_4, \quad \omega_{45} \;=\; -\,e_2,$$

and a computation of the Riemannian Dirac operator yields the formula

$$D^g(\psi) \;=\; \sum_{k=1}^{5} e_k \cdot e_k(\psi) \quad \text{on} \quad \mathrm{S}_0\,, \quad D^g(\psi) \;=\; \sum_{k=1}^{5} e_k \cdot e_k(\psi) \mp \psi \quad \text{on} \quad \mathrm{S}_{\pm 4}\,.$$

Spinors in the kernel of $\Omega_{\pm 4}$ occur on Sasakian η-Einstein manifolds of type $\mathrm{Ric}^g = -\,2\cdot g + 6\cdot\eta\otimes\eta$, too. This example has been discussed above.

A second case is $\mathrm{Scal}^g = 28$. Then

$$\Omega_0 \;=\; \Delta_{\mathrm{T}} + 4 \;=\; (D^{1/3})^2 - 4\,, \quad \Omega_{\pm 4} \;=\; \Delta_{\mathrm{T}} \;=\; (D^{1/3})^2 - 4\,.$$

The kernel of Ω_0 is trivial and the kernel of $\Omega_{\pm 4}$ coincides with the space of ∇-parallel spinors in the bundle $\mathrm{S}_{\pm 4}$. Sasakian manifolds admitting spinor fields of that type have been described in [10], Theorem 7.3 and Example 7.4.

If $-4 < \mathrm{Scal}^g < 28$, the kernel of the operator Ω_0 is trivial and the kernel of $\Omega_{\pm 4}$ depends on the geometry of the Sasakian structure. Let us discuss Einstein-Sasakian manifolds. Their scalar curvature equals $\mathrm{Scal}^g = 20$ and the Casimir operators are

$$\Omega_0 \ = \ \Delta_{\mathrm{T}} + 3 \, , \quad \Omega_{\pm 4} \ = \ \Delta_{\mathrm{T}} - 1 \ = \ (D^{1/3})^2 - 3 \ .$$

If M^5 is simply-connected, there exist two Riemannian Killing spinors (see [15])

$$\nabla^g_X \psi_1 \ = \ \frac{1}{2} X \cdot \psi_1, \quad D^g(\psi_1) \ = \ -\frac{5}{2}\psi_1, \quad \mathrm{T}\cdot\psi_1 \ = \ 4\,\psi_1,$$

$$\nabla^g_X \psi_2 \ = \ -\frac{1}{2} X \cdot \psi_2, \quad D^g(\psi_2) \ = \ \frac{5}{2}\psi_2, \quad \mathrm{T}\cdot\psi_2 \ = \ -4\,\psi_2.$$

We compute the Casimir operator

$$\Omega(\psi_1) \ = \ -\frac{3}{4}\psi_1, \quad \Omega(\psi_2) \ = \ -\frac{3}{4}\psi_2.$$

In particular, the Casimir operator of a Einstein-Sasakian manifold has *negative* eigenvalues. The Riemannian Killing spinors are parallel sections in the bundles $\mathrm{S}_{\pm 4}$ with respect to the flat connections $\nabla^{\pm}$

$$\nabla^+_X \psi \ := \ \nabla^g_X \psi - \frac{1}{2} X \cdot \psi \quad \text{in} \quad \mathrm{S}_4, \quad \nabla^-_X \psi \ := \ \nabla^g_X \psi + \frac{1}{2} X \cdot \psi \quad \text{in} \quad \mathrm{S}_{-4}.$$

We compare these connections with our canonical connection ∇:

$$\big(\nabla^{\pm}_X - \nabla_X\big)\cdot\psi^{\pm} \ = \ \pm\frac{i}{2} g(X,\xi)\cdot\psi^{\pm} \, , \quad \psi^{\pm} \in \mathrm{S}_{\pm 4} \ .$$

The latter equation means that the bundle $\mathrm{S}_4 \oplus \mathrm{S}_{-4}$ equipped with the connection ∇ is equivalent to the 2-dimensional trivial bundle with the connection form

$$\mathcal{A} \ = \ \frac{i}{2}\eta \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The curvature of ∇ on these bundles is given by the formula

$$\mathcal{R}^{\nabla} \ = \ \frac{i}{2} d\eta \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \ i\,(e_1 \wedge e_2 + e_3 \wedge e_4) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since the divergence $\mathrm{div}(\xi) = 0$ of the Killing vector field vanishes, the Casimir operator on $\mathrm{S}_4 \oplus \mathrm{S}_{-4}$ is the following operator acting on pairs of functions:

$$\Omega_4 \oplus \Omega_{-4} \ = \ \Delta_{\mathrm{T}} - 1 \ = \ \Delta - \frac{3}{4} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \xi \, .$$

Here Δ means the usual Laplacian of M^5 acting on functions and ξ is the differentiation in direction of the vector field ξ. In particular, the kernel of Ω coincides with solutions $f : M^5 \to \mathbb{C}$ of the equation

$$\Delta(f) - \frac{3}{4} f \pm i\,\xi(f) \ = \ 0 \ .$$

The L^2-symmetric differential operators Δ and $i\,\xi$ commute. Therefore, we can diagonalize them simultaneously. The latter equation is solvable if and only if there exists a common eigenfunction

$$\Delta(f) \ = \ \mu\, f, \quad i\xi(f) \ = \ \lambda\, f \, , \quad 4\,(\mu + \lambda) - 3 \ = \ 0 \, .$$

The Laplacian Δ is the sum of the *non-negative* horizontal Laplacian and the operator $(i\,\xi)^2$. Now, the conditions

$$\lambda^2 \;\leq\; \mu\,, \quad 4\,(\mu\,+\,\lambda)\,-\,3\;=\;0$$

restrict the eigenvalue of the Laplacian, $0\;\leq\;\mu\;\leq\;3$. On the other side, by the Lichnerowicz-Obata Theorem (see [3]) we have $5\;\leq\;\mu$, a contradiction. In particular, we proved

Theorem 4.1. *The Casimir operator of a compact* 5*-dimensional Sasakian-Einstein manifold has trivial kernel.*

The same argument estimates the eigenvalues of the Casimir operator. It turns out that the smallest eigenvalues of Ω is negative and equals $-3/4$. The eigenspinors are the Riemannian Killing spinors. The next eigenvalue of the Casimir operator is at least

$$\lambda_2(\Omega)\;\geq\;\frac{17}{4}\,-\,\sqrt{5}\approx\;2.014.$$

5. An explicit example: The 5-dimensional Stiefel manifold

The 5-dimensional Stiefel manifold $\mathrm{V}_{4,2}=\mathrm{SO}(4)/\mathrm{SO}(2)$ admits a homogeneous Einstein-Sasakian metric. This metric can be constructed via the Kaluza-Klein approach observing that $\mathrm{V}_{4,2}$ is a principal $\mathrm{SO}(2)$-bundle over the 4-dimensional Einstein-Kähler manifold $\mathrm{G}_{4,2}$ of all oriented 2-planes in $\mathbb{R}^4$. As a homogeneous space, the geometry and the Dirac operator of $\mathrm{V}_{4,2}$ have been described in [8]. We will use these formulas in our computation, with a slight change in normalization: we set the scalar curvature of a 5-dimensional Einstein-Sasakian manifold equal to 20, whereas the metric as described in the latter paper has scalar curvature 20/3. In order to fix the notation, let E_{ij} be the standard basis of the Lie algebra $\mathfrak{so}(4)$. The subalgebra $\mathfrak{so}(2)$ is generated by the matrix E_{34} and

$$X_1\;:=\;\sqrt{3}\,E_{13},\quad X_2\;:=\;\sqrt{3}\,E_{14},\quad X_3\;:=\;\sqrt{3}\,E_{23},\quad X_4\;:=\;\sqrt{3}\,E_{24},\quad \xi\;=\;X_5\;:=\;\frac{3}{2}\,E_{12}$$

constitute an orthonormal basis defining the metric of $\mathrm{V}_{4,2}$. The formula for the Riemannian Dirac operator has been computed in [8]:

$$D^g(\psi)\;=\;\sqrt{3}\sum_{i=1}^{5}X_i\cdot X_i(\psi)\;+\;S(\psi)\;,\quad S\;:=\;\frac{5i}{2}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}.$$

Using the commutator relations for $[X_i,X_j]$ as well as the matrix of the endomorphism $\mathrm{T}=\eta\wedge d\eta$ we compute the square of the operator $D^{1/3}$,

$$(D^{1/3})^2(\psi)\;=\;-\,3\sum_{i=1}^{5}X_i^2(\psi)\;+\;M_1\cdot\psi\;+\;M_2\cdot E_{34}(\psi)\;+\;M_3\cdot X_5(\psi).$$

Here the matrices M_1,M_2 and M_3 are given by

$$M_1:=\frac{9}{4}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\,,\;M_2:=6i\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\,,\;M_3\;:=\;\sqrt{3}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}.$$

According to the lift of the isotropy representation into the spin module (see [8]), a spinor field is a triple $\psi=(\psi_+,\psi_-,\psi_*)$ of maps $\psi_\pm:\mathrm{SO}(4)\to\mathbb{C}$ and $\psi_*:\mathrm{SO}(4)\to\mathbb{C}^2$

such that $E_{34}(\psi_\pm) = \pm i\,\psi_\pm$ and $E_{34}(\psi_*) = 0$. The map ψ_* is a section in the bundle $\mathrm{S}_4 \oplus \mathrm{S}_{-4}$ and (ψ_+, ψ_-) are sections in S_0. Specially over $\mathrm{V}_{4,2}$ the latter bundle splits into the sum of two line bundles. The Casimir operator $\Omega = \Omega_0 \oplus \Omega_4 \oplus \Omega_{-4}$ is equivalent to the operators

$$\Omega_0 \;=\; -3\sum_{\alpha=1}^{5} X_\alpha^2 + 3\,, \quad \Omega_4 \oplus \Omega_{-4} \;=\; -3\sum_{\alpha=1}^{5} X_\alpha^2 - \frac{3}{4} \pm \sqrt{3}i \cdot X_5$$

acting on functions $f : \mathrm{SO}(4) \to \mathbb{C}$ satisfying the quasi-periodicity conditions $E_{34}(f) = \pm\, i\, f$ and $E_{34}(f) = 0$, respectively.

6. **The Casimir operator of 6-dimensional nearly Kähler manifolds**

Let $(M^6, g, \mathcal{J})$ be a 6-dimensional nearly Kähler manifold. Then M^6 is an Einstein manifold of positive scalar curvature,

$$\mathrm{Ric}^g \;=\; \frac{5}{2}\,a\,g, \quad \mathrm{Scal}^g \;=\; 15\,a \;>\; 0\,.$$

The Nijenhuis tensor N does not vanish. There exists a unique connection ∇ with skew-symmetric torsion T. This connection is Gray's characteristic connection (see [16]) and its geometric data are given by

$$\nabla \mathrm{T} \;=\; 0\,, \quad 4\,\mathrm{T} \;=\; \mathrm{N}\,, \quad \mathrm{Ric}^\nabla \;=\; 2\,a\,g\,.$$

Moreover, we have

$$2\,\sigma_{\mathrm{T}} \;=\; d\mathrm{T} \;=\; a\,\big(\omega \wedge \omega\big) \;=\; 2\,a\,\big(e_{1234} + e_{1256} + e_{3456}\big), \quad \mathrm{T}_0^2 \;=\; 2\,a\,,$$

where ω denotes the fundamental form of the nearly Kähler structure. A general reference for all these formulas is the paper [10]. We compute the symmetric endomorphism $d\mathrm{T}$ in the spinor bundle :

$$2\,d\mathrm{T} + \mathrm{Scal} \;=\; 16\,a \cdot \mathrm{diag}(\,0\,,\,0\,,\,1\,,\,1\,,\,1\,,\,1\,,\,1\,,\,1).$$

Consequently, the Casimir operator

$$\Omega \;=\; \Delta_{\mathrm{T}} + \frac{1}{8}(2\,d\mathrm{T} + \mathrm{Scal}) \;=\; (D^{1/3})^2 - 2\,a$$

is non-negative. Its kernel coincides with the two-dimensional space of all ∇-parallel spinors. These spinor fields are the Riemannian Killing spinors on M^6. The Dirac operator $(D^{1/3})^2$ is bounded from below by

$$(D^{1/3})^2 \;\geq\; \frac{2}{15}\,\mathrm{Scal}^g \;>\; 0\,.$$

7. **The Casimir operator of 7-dimensional G_2-manifolds**

Let (M^7, g, ω^3) be a 7-dimensional cocalibrated G_2-manifold ($d * \omega^3 = 0$) such that the scalar product $(d\omega^3, *\omega^3)$ is constant. There exists a unique connection ∇ preserving the G_2-structure with skew-symmetric torsion

$$\mathrm{T} \;=\; -\,*\,d\omega^3 + \frac{1}{6}\,(d\omega^3, *\omega^3)\cdot\omega^3, \quad \delta(\mathrm{T}) \;=\; 0\,.$$

The Riemannian scalar curvature is given by the formula

$$\mathrm{Scal}^g \;=\; \frac{1}{18}\,(d\omega^3, *\omega^3)^2 - \frac{1}{2}\,\mathrm{T}_0^2 \;=\; 2\,(\mathrm{T}, \omega^3)^2 - \frac{1}{2}\,\mathrm{T}_0^2\,.$$

Moreover, there exists a parallel spinor field ψ_0 such that

$$\nabla\psi_0 \;=\; 0\,, \quad \mathrm{T}\cdot\psi_0 \;=\; -\,\frac{1}{6}\,(d\omega^3, *\omega^3)\cdot\psi_0\,.$$

A general reference for these facts are the papers [10] and [12]. The Casimir operator is given by the formula

$$\begin{aligned}\Omega \;&=\; (D^{1/3})^2 \,-\, \frac{1}{4}\,(\mathrm{T},\omega^3)^2 \,+\, \frac{1}{8}\,(d\mathrm{T} \,-\, 2\,\sigma_{\mathrm{T}})\\ &=\; \Delta_{\mathrm{T}} \,+\, \frac{1}{4}\,(\mathrm{T},\omega^3)^2 \,+\, \frac{1}{8}\,(3\,d\mathrm{T} \,-\, 2\,\sigma_{\mathrm{T}} \,-\, 2\,\mathrm{T}_0^2).\end{aligned}$$

There are two special types of cocalibrated G_2-structures. A *nearly parallel* G_2-manifold is characterized by the equation $d\omega^3 = -\,a\,(*\omega^3)$. The paper [14] contains examples of compact nearly parallel G_2-manifolds and their relation to Riemannian Killing spinors. The torsion form as well as the Riemannian Ricci tensor are given by the formulas

$$\mathrm{T} \;=\; -\,\frac{a}{6}\,\omega^3, \quad \mathrm{Ric}^g \;=\; \frac{3}{8}\,a^2\cdot g, \quad \mathrm{Scal}^g \;=\; \frac{21}{8}\,a^2, \quad \mathrm{T}_0^2 \;=\; \frac{7}{36}\,a^2.$$

The torsion form of a nearly parallel G_2-manifold is ∇-parallel (see [10] , Corollary 4.9) and $d\mathrm{T} = 2\,\sigma_{\mathrm{T}}$. The Casimir operator is given by

$$\Omega \;=\; (D^{1/3})^2 \,-\, \frac{49}{144}\,a^2.$$

The ∇-parallel spinor ψ_0 is the Riemannian Killing spinor and satisfies the equations (see [10])

$$D^g\psi_0 \;=\; -\,\frac{7}{8}\,a\,\psi_0, \quad \mathrm{T}\cdot\psi_0 \;=\; \frac{7}{6}\,a\,\psi_0.$$

In particular, ψ_0 belongs to the kernel of the Casimir operator. Consider now an arbitrary spinor field ψ in its kernel. Since the 3-form ω^3 acts in the spinor bundle with two eigenvalues -7 and $+1$, there are two possibilities. If

$$\Omega(\psi) \;=\; 0, \quad \mathrm{T}\cdot\psi \;=\; \frac{7}{6}\,a\,\psi,$$

we obtain in the compact case the equation

$$\frac{49}{144}\,a^2\int_{M^7}||\psi||^2 \;=\; \int_{M^7}||(D^g + \frac{7}{24}\,a)\,\psi||^2.$$

Consequently, there exists an eigenvalue $\lambda \in \mathrm{Spec}(D^g)$ of the Riemannian Dirac operator such that

$$\left(\lambda \,+\, \frac{7}{24}\,a\right)^2 \;\leq\; \frac{49}{144}\,a^2, \quad \frac{7}{8}\,a \;\leq\; |\lambda|.$$

The latter conditions imply that

$$\lambda \;=\; -\,\frac{7}{8}\,a$$

and we are in the limiting case of the well-known estimate for the eigenvalues of the Riemannian Dirac operator (see [8]). The spinor field ψ is a Riemannian Killing spinor, i.e., ψ is ∇-parallel. In a similar way, we discuss the second possibility

$$\Omega(\psi) \;=\; 0\,, \quad \mathrm{T}\cdot\psi \;=\; -\,\frac{1}{6}\,a\,\psi\,.$$

Then we obtain the inequalities

$$\left(\lambda - \frac{1}{24}a\right)^2 \leq \frac{49}{144}a^2, \quad \frac{7}{8}a \leq |\lambda|.$$

and a solution λ does not exist. Let us summarize the result:

Theorem 7.1. *Let (M^7, g, ω^3) be a compact, nearly parallel G_2-manifold ($d\omega^3 = -a \cdot (*\omega^3)$) and denote by ∇ its unique connection with skew-symmetric torsion. The kernel of the Casimir operator of the triple (M^7, g, ∇) coincides with the space of ∇-parallel spinors,*

$$\mathrm{Ker}(\Omega) = \big\{\psi : \nabla\psi = 0, \quad \mathrm{T}\cdot\psi = \frac{7}{6}a\cdot\psi\big\} = \mathrm{Ker}(\nabla).$$

A *cocalibrated* G_2*-structure of type* $\mathcal{W}_3$ in the Fernandez/Gray classification is characterized by the equations $d*\omega^3 = 0$ and $(d\omega^3, *\omega^3) = 0$ (see [7], [9]). The geometric data are

$$\mathrm{T} = -*d\omega^3, \quad \mathrm{Scal}^g = -\frac{1}{2}\mathrm{T}_0^2, \quad \nabla\psi_0 = 0, \quad \mathrm{T}\cdot\psi_0 = 0,$$

see [10], [12]. In contrast to the nearly parallel case, cocalibrated G_2-manifolds of type $\mathcal{W}_3$ do not satisfy the condition $d\mathrm{T} = 2\sigma_{\mathrm{T}}$. The Casimir operator is given by the formula

$$\Omega = (D^{1/3})^2 + \frac{1}{8}(d\mathrm{T} - 2\sigma_{\mathrm{T}}) = \Delta_{\mathrm{T}} + \frac{1}{8}(3\,d\mathrm{T} - 2\sigma_{\mathrm{T}} - 2\,\mathrm{T}_0^2)\ .$$

Examples of G_2-structures of type $\mathcal{W}_3$ on nilpotent Lie groups are discussed in the paper [10], on the Aloff-Wallach space $N(1,1)$ in [2]. We recall these examples and compute the relevant endomorphisms.

Example 7.1. There exists a G_2-structure of type $\mathcal{W}_3$ on the product of $\mathbb{R}^1$ by the Heisenberg group. In this case, we have $\mathrm{T}_0^2 = 4$ and

$$3\,d\mathrm{T} - 2\sigma_{\mathrm{T}} = \mathrm{diag}(8,0,8,-16,8,-16,8,0), \quad d\mathrm{T} - 2\sigma_{\mathrm{T}} = \mathrm{diag}(0,8,0,-8,0,-8,0,8).$$

A second example on the product of $\mathbb{R}^1$ by a 3-dimensional complex, solvable Lie group has been described in [10], too. Remark that in both examples $3\,d\mathrm{T} - 2\sigma_{\mathrm{T}} - 2\,\mathrm{T}_0^2$ is a non-positive endomorphism acting on spinors. Consequently, the Casimir operator is dominated by the spinorial Laplacian,

$$\int_{M^7}\langle\Omega(\psi), \psi\rangle \leq \int_{M^7}\langle\Delta_{\mathrm{T}}(\psi), \psi\rangle.$$

Example 7.2. In [2], we constructed on the Aloff-Wallach space $N(1,1) = \mathrm{SU}(3)/S^1$ a family of metrics depending on a parameter $0 < y < 1$ as well as G_2-structures of type $\mathcal{W}_3$ (see Proposition 8.8). In the notation of that paper, the spinor ψ_5 is the ∇-parallel spinor and algebraically the torsion form is given by $4\cdot\mathrm{T}_5$ with

$$\mathrm{T}_5 = -\frac{y+2}{4}[X_{135} + X_{146} + X_{245} - X_{236}] + \frac{3y}{y-1}X_{127} + \frac{2+2y-y^2}{2y-2}[X_{347} - X_{567}].$$

Using the structure equations of the underlying geometry, we compute the exterior derivative,

$$\begin{aligned} d\mathrm{T}_5 &= (2+4y)[X_{2357} + X_{2467} - X_{1457} + X_{1367}] + \frac{3y(-2-2y+y^3)}{(y-1)^2}X_{3456} \\ &+ \frac{10+9y+12y^2+5y^3}{(y-1)^2}[X_{1234} - X_{1256}]. \end{aligned}$$

Inserting the matrices of the 7-dimensional spin representation, we compute the endomorphism $3\,(4\,d\mathrm{T}_5) + (4\,\mathrm{T}_5)^2 - 3\,||4\,\mathrm{T}_5||^2$. It turns out that this endomorphism has the eigenvalues $\mathrm{diag}(a, a, b, b, 0, c, a, a)$, where $c := 64(7 + 10y + y^2) > 0$ and

$$a \,:=\, -\,\frac{72(2 + y + y^2 - y^3 + y^4)}{(y-1)^2} \;<\; 0, \quad b \,:=\, \frac{16(20 + 7y + 33y^2 + 13y^3 - y^4)}{(y-1)^2} \;>\; 0.$$

The endomorphism $4\,d\mathrm{T}_5 - 2\sigma_{4\,\mathrm{T}_5} = 4\,d\mathrm{T}_5 + (4\,\mathrm{T}_5)^2 - ||4\,\mathrm{T}_5||^2$ has the eigenvalues $\mathrm{diag}(a^*, a^*, b^*, b^*, 0, c^*, a^*, a^*)$, where $c^* := 64(5 + 6y + y^2) > 0$ and

$$a^* \,:=\, \frac{24(-2 + y)(1 + y)^2}{1 - y} \;<\; 0\,, \quad b^* \,:=\, \frac{16(4 - 7y - 10y^2 + y^3)}{(y-1)}\,.$$

Let us finally consider *arbitrary* cocalibrated G_2-structures. The following example on $N(1,1)$ is described in the paper [2], including the computation of the canonical connection and its geometric data.

Example 7.3. In [2], Proposition 8.5, we constructed on $N(1,1)$ a cocalibrated G_2-structure with some special symmetry property. Its torsion form is given by $4 \cdot \mathrm{T}$ with

$$\mathrm{T} \;=\; \frac{\sqrt{3}}{6}\,[X_{135} \,+\, X_{146} \,-\, X_{245} \,+\, X_{236}].$$

Using the structure equations of the underlying geometry we compute the exterior derivative,

$$d\mathrm{T} \;=\; -\,X_{2357} \,-\, X_{2467} \,-\, X_{1457} \,+\, X_{1367},$$

and finally the endomorphism

$$\frac{1}{4}\,(4\,\mathrm{T}, \omega^3)^2 \,+\, \frac{1}{8}\,(12\,d\mathrm{T} \,-\, 2\,\sigma_{4\,\mathrm{T}} \,-\, 2\,||4\,\mathrm{T}||^2) \;=\; \mathrm{diag}\Big(\frac{10}{3}, \frac{10}{3}, 0, 12, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}\Big).$$

In particular, the Casimir operator of this G_2-structure is non-negative,

$$\int_{N(1,1)} \langle \Omega(\psi)\,,\,\psi\rangle \;\geq\; \int_{N(1,1)} \langle \Delta_{\mathrm{T}}(\psi)\,,\,\psi\rangle \;\geq\; 0\;.$$

and its kernel coincides with the space of ∇-parallel spinors.

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