KILLING SPINORS IN SUPERGRAVITY WITH 4-FLUXES

ILKA AGRICOLA AND THOMAS FRIEDRICH

Abstract. We study the spinorial Killing equation of supergravity involving a torsion 3-form T as well as a flux 4-form F. In dimension seven, we construct explicit families of compact solutions out of 3-Sasakian geometries, nearly parallel G_2-geometries and on the homogeneous Aloff-Wallach space. The constraint $F \cdot \Psi = 0$ defines a non empty subfamily of solutions. We investigate the constraint $T \cdot \Psi = 0$, too, and show that it singles out a very special choice of numerical parameters in the Killing equation, which can also be justified geometrically.

Contents

1. Introduction

Supergravity models can be described geometrically by a tuple $(M^n, g, \mathrm{T}, \mathrm{F}, \Psi)$ consisting of a Riemannian manifold, a 3-form T, a 4-form flux F and a spinor field Ψ. The link between these geometric objects is the so called Killing equation (see [8], [9])

$$\nabla^g_X \Psi + \frac{1}{4} \cdot (X \lrcorner \mathrm{T}) \cdot \Psi + \frac{1}{144} \cdot (X \lrcorner \mathrm{F} - 8 \cdot X \wedge \mathrm{F}) \cdot \Psi \;=\; 0\,.$$

This equation should be satisfied for any tangent vector X. Consequently, we deal with a highly overdetermined system of first order partial differential equations. The 3-form T has an interpretation as the torsion of a linear, metric connection ∇ with totally skew-symmetric torsion. 11-dimensional space-time solutions are interesting and the models with a maximal number of supersymmetries have been classified (see [9]). The Kaluza-Klein reduction of $\mathcal{M}$-theory (see [3], [5], [6] and [17]) yields that dimensions $4 \leq n \leq 8$ are of interest, too. However, then additional algebraic constraints occur, for example, an algebraic coupling between the torsion 3-form or the flux 4-form and the spinor field Ψ,

$$\mathrm{T} \cdot \Psi \;=\; 0 \quad \text{or} \quad \mathrm{F} \cdot \Psi \;=\; 0\,.$$

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The aim of this note is to present a geometric method for solving the equation under consideration. The main idea of our approach is easy to explain. We start with a Riemannian manifold admitting a spinor field Ψ of some special type. For this, there are many possibilities. The spinor field may be a Riemannian Killing spinor (see [10]) on some irreducible Einstein space,

$$\nabla^g_X \Psi \;=\; \lambda \cdot X \cdot \Psi \,.$$

The spinor field may be a Kählerian Killing spinor (see [16]) defined on some special Kähler manifold. In odd dimensions, we can start with an η-Einstein-Sasakian manifold and its contact Killing spinor (see [15]). On a reductive space, the spinor field may be an invariant spinor of the isotropy representation. In any case, triples $(M^n,\, g,\, \Psi)$ of the type we need have been studied very intensively in mathematics since more then 20 years. In particular, the dimensions $n \leq 8$ and the corresponding special geometries play a crucial role. The books [4], [11] contain the results as well as the relevant references in detail. Let us moreover assume that there exists a "canonical" family $(\mathrm{T}_\omega,\, \mathrm{F}_\omega)$ of forms on M^n depending on some parameter $\omega \in \Omega$. We consider the system

$$\Big(\lambda \cdot X \,+\, \frac{1}{4}\cdot (X \,\lrcorner\, \mathrm{T}_\omega) \,+\, \frac{1}{144}(X \,\lrcorner\, \mathrm{F}_\omega \,-\, 8 \cdot X \wedge \mathrm{F}_\omega)\Big)\cdot \Psi \;=\; 0\,,$$

which is a highly overdetermined system of $n \cdot 2^{[n/2]}$ algebraic equations in the parameters $\omega \in \Omega$. Furthermore, we can add the equations $\mathrm{T}_\omega \cdot \Psi = 0$ or $\mathrm{F}_\omega \cdot \Psi = 0$. It is a matter of fact – quite surprising to the mathematician – that the solutions often constitute a non empty subset of the parameter space. We solve the corresponding equations with the help of standard math computer programs. In this way, we obtain families of 3- and 4-forms solving the equation on the manifold we started with. The first interesting dimension in presence of a 4-flux is seven. Therefore, we apply the described method to nearly parallel G_2-manifolds, to 3-Sasakian manifolds and to the 7-dimensional Aloff-Wallach space. The G_2-case has been already investigated in $\mathcal{M}$-theory compactifications to dimension four (see [6]). On a nearly parallel G_2-manifold, there is a natural flux form. The second and third case are more flexible and interesting. We will construct families of torsion and flux forms with Killing spinors out of a 3-Sasakian structure. The underlying metric of a 3-Sasakian manifold is still Einstein (see [7]). On the Aloff-Wallach space $N(1,1)$ we obtain families of non Einstein metrics equipped with torsion forms, flux forms and Killing spinors. The method we use can probably be applied to other dimensions and special geometries, too.

Let us consider a slightly more general equation depending on two real parameters $(p, q) \in \mathbb{R}^2$,

$$\nabla^g_X \Psi \,+\, \frac{1}{4}\cdot (X \,\lrcorner\, \mathrm{T})\cdot \Psi \,+\, p \cdot (X \,\lrcorner\, \mathrm{F})\cdot \Psi \,+\, q \cdot (X \wedge \mathrm{F})\cdot \Psi \;=\; 0\,.$$

We will construct 7-dimensional solutions of this field equation for *any* pair $(p, q) \in \mathbb{R}^2$ of parameters. Obviously, if the flux form is non trivial, then the ratio between the parameters p and q is important. It turns out that in any dimension n there is one distinguished pair of parameters, namely $4\,p \,-\, (n-4)\,q = 0$. This coupling of the parameters plays a special role for our solutions. It can be motivated as well by the observation that in this case the action of the Riemannian Dirac operator on the spinor depends only on the torsion form T, but not on the flux form F. Remark that this is *not* the ratio appearing in the original equation of supergravity, since we consider positive definite metrics. The equation in arbitrary dimension reads then as

$$\nabla^g_X \Psi + \frac{1}{4} \cdot (X \lrcorner \mathrm{T}) \cdot \Psi + \frac{n-4}{4} \cdot (X \lrcorner \mathrm{F}) \cdot \Psi + (X \wedge \mathrm{F}) \cdot \Psi = 0 .$$

In Section 5, we discuss in more detail the 7-dimensional solutions of this special equation which we constructed.

2. Killing Spinors with 4-Fluxes on 3-Sasakian Manifolds

The structure group of a 3-Sasakian geometry is the subgroup $\mathrm{SU}(2) \subset \mathrm{G}_2 \subset \mathrm{SO}(7)$, the isotropy group of four spinors in dimension seven. We describe the subgroup $\mathrm{SU}(2)$ in such a way that the vectors $e_1, e_2, e_7 \in \mathbb{R}^7$ are fixed. More precisely, let the Lie algebra $\mathfrak{su}(2)$ be generated by the following 2-forms in $\mathbb{R}^7$:

$$e_{34} + e_{56}, \quad e_{35} - e_{46}, \quad e_{36} + e_{56}.$$

The real spin representation Δ_7 splits under the action of $\mathrm{SU}(2)$ into a 4-dimensional trivial representation Δ_7^0 and the unique non trivial 4-dimensional representation Δ_7^1. We use the standard realization of the 8-dimensional $\mathrm{Spin}(7)$-representation as given in [4, p.97] or [11, p.13] (see [2], too). Denote by $\Psi_1, \ldots, \Psi_8$ its basis. The space Δ_7^0 is spanned by the spinors $\Psi_3, \Psi_4, \Psi_5, \Psi_6$. We consider the following $\mathrm{SU}(2)$-invariant 2-forms on $\mathbb{R}^7$:

$$de_1 := e_{35} + e_{46}, \quad de_2 := e_{45} - e_{36}, \quad de_7 := e_{34} - e_{56}.$$

Using this notation, we introduce a family of invariant 3-forms in $\mathbb{R}^7$ depending on 10 parameters,

$$\mathrm{T} = \sum_{i,j=1,2,7} t_{ij} \cdot e_i \wedge de_j + t \cdot e_1 \wedge e_2 \wedge e_7.$$

The space of $\mathrm{SU}(2)$-invariant 4-forms on $\mathbb{R}^7$ has also dimension ten,

$$\mathrm{F} = \sum_{i,j,k=1,2,7} f_{ijk} \cdot e_i \wedge e_j \wedge de_k + f \cdot e_3 \wedge e_4 \wedge e_5 \wedge e_6.$$

All together, on a 3-Sasakian manifold there exists a canonical family Ω of forms depending on 20 parameters. The key point of our considerations is the following algebraic observation.

Proposition 2.1. *Fix parameters $(p, q) \in \mathbb{R}^2$. For any spinor $0 \neq \Psi \in \Delta_7^0$, the set of all pairs (T, F) consisting of $\mathrm{SU}(2)$-invariant forms and satisfying, for any vector $X \in \mathbb{R}^7$, the equation*

$$\Big(\frac{1}{2} \cdot X + \frac{1}{4} \cdot (X \lrcorner \mathrm{T}) + p \cdot (X \lrcorner \mathrm{F}) + q \cdot (X \wedge \mathrm{F})\Big) \cdot \Psi = 0$$

is a 7-dimensional affine space. The condition $\mathrm{F} \cdot \Psi = 0$ defines a 6-dimensional subspace. If $4p - 3q \neq 0$, the condition $\mathrm{T} \cdot \Psi = 0$ defines a 6-dimensional affine subspace, too. If $4p - 3q = 0$, we have $\mathrm{T} \cdot \Psi = (14/3) \cdot \Psi$ for all torsion forms in the family. Both constraints $\mathrm{T} \cdot \Psi = 0$ and $\mathrm{F} \cdot \Psi = 0$ cannot be fulfilled simultaneously.

Proof. Given a spinor $\Psi = a\,\Psi_3 + b\,\Psi_4 + c\,\Psi_5 + d\,\Psi_6$, we solve the overdetermined system with respect to the coefficients of the 3-form T and the coefficients of the 4-form F. It turns out that solutions exist and can be given explicitely. For simplicity, we provide the formulas in case that $\Psi = \Psi_3$ is one of the basic spinors. The set of all solutions is an affine space parameterized by f_{127}, f_{172}, f_{177}, f_{271}, f_{272}, f_{277}, f. The other coordinates are given by the formulas

$$\begin{aligned}
t_{12} &= -4\,(p+q)\,f_{272} = -t_{21}, \quad t_{17} = -4\,(p+q)\,f_{277} = -t_{71},\\
t_{27} &= 4\,(p+q)\,f_{177} = t_{72},\\
t_{11} &= -\frac{2}{3} + \frac{4p}{3}f - \frac{8p}{3}f_{127} + \frac{8p}{3}f_{172} - \frac{4}{3}(p+3q)\,f_{271},\\
t_{22} &= \frac{2}{3} - \frac{4p}{3}f + \frac{8p}{3}f_{127} - \frac{8p}{3}f_{271} + \frac{4}{3}(p+3q)\,f_{172},\\
t_{77} &= \frac{2}{3} - \frac{4p}{3}f - \frac{8p}{3}f_{172} - \frac{8p}{3}f_{271} - \frac{4}{3}(p+3q)\,f_{127},\\
t &= \frac{2}{3} + \frac{8p}{3}f_{127} - \frac{8p}{3}f_{172} - \frac{8p}{3}f_{271} + \frac{4}{3}(2p+3q)\,f,\\
f_{122} &= -f_{177}, \quad f_{121} = -f_{277}, \quad f_{171} = f_{272}\,.
\end{aligned}$$

The equation $\mathrm{F}\cdot\Psi_3 = 0$ describes a 6-dimensional linear subspace,

$$f - 2\,f_{127} + 2\,f_{172} + 2\,f_{271} = 0.$$

On the other side, the equation $\mathrm{T}\cdot\Psi_3 = 0$ defines a 6-dimensional affine subspace

$$-7 + (8\,p - 6\,q)\big(f - 2\,f_{127} + 2\,f_{172} + 2\,f_{271}\big) = 0.$$

The intersection of these spaces is empty. □

Remark 2.1. The cases $\mathrm{T}=0$ and $\mathrm{F}=0$ have been discussed already in [2]. Under this constraint the algebraic system has a unique solution.

Remark 2.2. From the geometric point of view, there is an interesting case, namely $4\,p - 3\,q = 0$. This is *not* the ratio of the parameters p, q appearing in supergravity. In this case, the constraint $\mathrm{T}\cdot\Psi = 0$ is never satisfied for a non trivial spinor. In Section 5, we will discuss this family of solutions in more detail.

Consider a simply connected 3-Sasakian manifold M^7 of dimension seven and denote its three contact structures by η_1, η_2, and η_7. It is known that M^7 is then an Einstein space, and examples (also non homogeneous ones) can be found in the paper [7]. The tangent bundle of M^7 splits into the 3-dimensional part spanned by η_1, η_2, η_7 and its 4-dimensional orthogonal complement. We restrict the exterior derivatives $d\eta_1, d\eta_2$ and $d\eta_7$ to this complement. In an adapted orthonormal frame, these forms coincide with the algebraic forms de_1, de_2 and de_7. The space of Riemannian Killing spinors

$$\nabla^g_X\Psi = \frac{1}{2}X\cdot\Psi$$

is non trivial and has at least dimension three (see [13]). Moreover, the proof of this fact shows that all the Riemannian Killing spinors are sections in the subbundle corresponding to the SU(2)-representation Δ_7^0. Now we apply Proposition 2.1 and we obtain the following result.

Theorem 2.1. *Let M^7 be 3-Sasakian manifold in dimension seven and fix a Riemannian Killing spinor Ψ. Then there exists a 7-dimensional family of torsion forms* T *and flux forms* F *defined by the contact structures such that*

$$\nabla^g_X\Psi + \frac{1}{4}\cdot(X\lrcorner\mathrm{T})\cdot\Psi + p\cdot(X\lrcorner\mathrm{F})\cdot\Psi + q\cdot(X\wedge\mathrm{F})\cdot\Psi = 0\,.$$

The condition $\mathrm{F}\cdot\Psi = 0$ restricts to a subfamily of dimension six. If $4\,p - 3\,q \neq 0$, the condition $\mathrm{T}\cdot\Psi = 0$ defines again a 6-dimensional subfamily. If $4\,p - 3\,q = 0$, then

$\mathrm{T} \cdot \Psi = (14/3) \cdot \Psi$ *for any torsion form in the family. Both constraints together imply that the spinor field* Ψ *is necessarily zero.*

All 3-Sasakian manifolds are Einstein. In Section 4, we will generalize this family of solutions. In particular, we construct homogeneous solutions on certain non-Einstein manifolds. The qualitative behavior of these solutions does not differ from the torsion and flux forms in the 3-Sasakian case, but the metric is allowed to depend on several parameters, and is hence more flexible.

3. Killing Spinors with 4-Fluxes on Nearly Parallel G_2-Manifolds

Fix a spinor $\Psi \in \Delta_7$ in the 7-dimensional spin representation, and consider the corresponding 3-form $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ defined by the formula

$$\omega^3(X,Y,Z) \ := \ -\big(X \cdot Y \cdot Z \cdot \Psi\,,\,\Psi\big).$$

The 3-form acts on the spinor by $\omega^3 \cdot \Psi \ = \ -\,7 \cdot \Psi$ (see [14]). The pair $(\omega^3\,,\,*\,\omega^3)$ generates a 2-dimensional parameter space.

Proposition 3.1. *The equation*

$$\Big(X \,+\, \frac{r}{4} \cdot (X \lrcorner\, \omega^3) \,+\, p \cdot (X \lrcorner *\omega^3) \,+\, q \cdot (X \wedge *\,\omega^3)\Big) \cdot \Psi \ = \ 0$$

holds for all vectors $X \in \mathbb{R}^7$ *if and only if* $16\,p = -\,4 \,+\, 12\,q \,-\, 3\,r$.

Proof. A direct computation using the matrices of the spin representation yields the result. □

Corollary 3.1. *Fix parameters* $(p,q) \in \mathbb{R}^2$*. There a 1-parameter family of admissible pairs, namely*

$$\mathrm{T} \ = \ \Big[\frac{12q - 16p}{3f} \,-\, \frac{4}{3}\Big] \cdot \omega^3 \quad \text{and} \quad \mathrm{F} \ = \ f \cdot (*\,\omega^3).$$

If $4\,p \,-\, 3\,q = 0$*, then the torsion form does not depend on the flux form,*

$$\mathrm{T} \ = \ -\,\frac{4}{3} \cdot \omega^3 \quad \text{and} \quad \mathrm{F} \ = \ f \cdot (*\,\omega^3).$$

Consider a simply connected, nearly parallel G_2-manifold M^7. It is an Einstein space and we normalize the metric by the condition that the scalar curvature equals 168. There exists a Riemannian Killing spinor (see [14])

$$\nabla^g_X \Psi \ = \ X \cdot \Psi\ .$$

The triple $(\mathrm{T},\,\mathrm{F},\,\Psi)$ defined above is a solution of the Killing equation

$$\nabla^g_X \Psi \,+\, \frac{1}{4} \cdot (X \lrcorner\, \mathrm{T}) \cdot \Psi \,+\, p \cdot (X \lrcorner\, \mathrm{F}) \cdot \Psi \,+\, q \cdot (X \wedge \mathrm{F}) \cdot \Psi \ = \ 0\ .$$

In particular, nearly parallel G_2-manifolds admit a torsion form T and a flux form F such that its Riemannian Killing spinor is a Killing spinor with respect to the pair (T, F). The 1-parameter family has been computed in the Corollary.

4. Killing Spinors with 4-Fluxes on a Aloff-Wallach space

The goal of this section is to construct on the Aloff-Wallach manifold $N(1,1) = \mathrm{SU}(3)/S^1$ a two-parameter family of metrics $g = g_{s,y}$ that admits, for every $g_{s,y}$, a large family of torsion and flux forms making a fixed spinor parallel. We use the computations available in [4, p.109 ff], which we hence shall not reproduce here. Consider the embedding $S^1 \to \mathrm{SU}(3)$ given by $e^{i\theta} \mapsto \mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{-2i\theta})$. The Lie algebra $\mathfrak{su}(3)$ splits into $\mathfrak{su}(3) = \mathfrak{m} + \mathbb{R}$, where $\mathbb{R}$ denotes the Lie algebra of S^1 deduced from the given embedding. The space $\mathfrak{m}$ has a preferred direction, namely the subspace $\mathfrak{m}_0$ generated by the matric $L := \mathrm{diag}(3i, -3i, 0)$. Let E_{ij} $(i < j)$ be the matrix with 1 at the place (i,j) and zero elsewhere, and define $A_{ij} = E_{ij} - E_{ji}$, $\tilde{A}_{ij} = i(E_{ij} + E_{ji})$. We set $\mathfrak{m}_1 := \mathrm{Lin}\{A_{12}, \tilde{A}_{12}\}$, $\mathfrak{m}_2 := \mathrm{Lin}\{A_{13}, \tilde{A}_{13}\}$ and $\mathfrak{m}_3 := \mathrm{Lin}\{A_{23}, \tilde{A}_{23}\}$. The sum $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ is an algebraic complement of $\mathfrak{m}_0$ inside $\mathfrak{m}$, and in fact all spaces $\mathfrak{m}_i$ are pairwise perpendicular with respect to the Killing form $B(X,Y) := -\mathrm{Re}(\mathrm{tr}XY)/2$. Hence, the following formula

$$\mathfrak{g}_{s,y} \;:=\; \frac{1}{s^2}\,B\big|_{\mathfrak{m}_0} + B\big|_{\mathfrak{m}_1} + \frac{1}{y}B\big|_{\mathfrak{m}_2} + \frac{1}{y}B\big|_{\mathfrak{m}_3}$$

defines a two-parameter family of metrics on $N(1,1) := \mathrm{SU}(3)/S^1$. It is a subfamily of the family considered in [4, p.109 ff]; in particular, $(s = 1, y = 2)$ corresponds to the 3-Sasakian metric that has three Riemannian Killing spinors with Killing number $1/2$, and $(s = 1, y = 2/5)$ is the Einstein metric with one Killing spinor with Killing number $-3/10$ (see [4, Thm 12, p.116]). An orthonormal basis of $\mathfrak{m}$ is given by

$$X_1 = A_{12},\; X_2 = \tilde{A}_{12},\; X_3 = \sqrt{y}A_{13},\; X_4 = \sqrt{y}\tilde{A}_{13},\; X_5 = \sqrt{y}A_{23},\; X_6 = \sqrt{y}\tilde{A}_{23},$$

and $X_7 = s \cdot L/3$. The isotropy representation $\mathrm{Ad}\,(\theta)$ leaves the vectors X_1, X_2 and X_7 invariant, and acts as a rotation by 3θ in the (X_3, X_4)-plane and in the (X_5, X_6)-plane. We use the realization of the 8-dimensional $\mathrm{Spin}(7)$-representation Δ_7 as given in Section 2. One then checks that Ψ_3, Ψ_4, Ψ_5 and Ψ_6 are fixed under the lift $\tilde{\mathrm{Ad}}\,(\theta)$ of the isotropy representation to $\mathrm{Spin}(7)$. Thus, they define constant sections in the spinor bundle $S = \mathrm{SU}(3) \times_{\tilde{\mathrm{Ad}}} \Delta_7$. The Levi-Civita connection of $N(1,1)$ is described by a map $\Lambda :\ \mathfrak{m} \mapsto \mathfrak{so}(7)$, whose lift $\tilde{\Lambda} :\ \mathfrak{m} \mapsto \mathfrak{spin}(7)$ can be found either in [4, p.112] or in [2].

In order to define a global form on $N(1,1)$, an algebraic form on $\mathfrak{m}$ needs to be invariant under the isotropy representation. It turns out that there are precisely 13 isotropy invariant 3-forms on $\mathfrak{m}$, hence the most general 3-form we can consider is

$$\begin{aligned}
\mathrm{T} \;:=\;& \alpha_3\,(X_{135} + X_{146}) + \beta_3\,(X_{235} + X_{246}) + \gamma_3\,(X_{357} + X_{467}) + \delta_3\,(X_{145} - X_{136}) \\
+\;& \varepsilon_3\,(X_{245} - X_{236}) + \xi_3\,(X_{457} - X_{367}) + \mu_3\,X_{127} + \nu_3\,X_{347} \\
+\;& \lambda_3\,X_{567} + \eta_3\,X_{134} + \omega_3\,X_{234} + \pi_3\,X_{156} + \varrho_3\,X_{256}.
\end{aligned}$$

For notational convenience, we shall write X_{ijk} for $X_i \wedge X_j \wedge X_k$, and similarly for forms of any degree. By Hodge duality, the Ansatz for a 4-form is

$$\begin{aligned}
\mathrm{F} \;:=\;& \alpha_4\,X_{1234} + \beta_4\,X_{1256} + \gamma_4\,X_{3456} + \delta_4\,X_{1347} + \varepsilon_4\,X_{1567} + \xi_4\,X_{2347} + \mu_4\,X_{2567} \\
+\;& \nu_4\,(X_{1235} + X_{1246}) + \lambda_4\,(X_{1357} + X_{1467}) + \eta_4\,(X_{1245} - X_{1236}) \\
+\;& \omega_4\,(X_{1457} - X_{1367}) + \pi_4\,(X_{2457} - X_{2367}) + \varrho_4\,(X_{2357} + X_{2467}).
\end{aligned}$$

In particular, the parameter space Ω of pairs (T,F) of possible 3- and 4-forms has now six dimensions more than in the 3-Sasakian case. Notice that for the 3-form T, $X \lrcorner \mathrm{T} = -(X\cdot\mathrm{T}+\mathrm{T}\cdot X)/2$, whereas the 4-form F satisfies

$$X \lrcorner \mathrm{F} \,=\, -\frac{1}{2}(X\cdot\mathrm{F}-\mathrm{F}\cdot X), \quad X\wedge\mathrm{F} \,=\, \frac{1}{2}(X\cdot\mathrm{F}+\mathrm{F}\cdot X).$$

Theorem 4.1. *For every metric $g_{s,y}$ on $N(1,1)$ and pair $(p,q)\in\mathbb{R}^2$, there exists a 10-dimensional affine space Ω' of forms (T,F) such that the spinor field Ψ_3 satisfies the Killing spinor equation*

$$\nabla_X\Psi := \nabla^g_X\Psi + \frac{1}{4}(X\lrcorner\mathrm{T})\Psi + p\,(X\lrcorner\mathrm{F})\Psi + q\,(X\wedge\mathrm{F})\Psi = 0.$$

Furthermore, the additional condition $\mathrm{F}\cdot\Psi_3 = 0$ singles out a 9-dimensional affine subspace of Ω'. For $4\,p-3\,q\neq 0$, the set of forms inside Ω' satisfying $\mathrm{T}\cdot\Psi_3=0$ is again a 9-dimensional affine subspace, but its intersection with forms such that $\mathrm{F}\cdot\Psi_3=0$ is empty. For $4\,p-3\,q=0$, there are no 3-forms in Ω' such that $\mathrm{T}\cdot\Psi=0$.

Proof. Evaluating the Killing spinor equation in all directions $X = X_1,\ldots,X_7$, one observes that of the resulting seven 8-dimensional spinorial equations, half is trivial, hence the linear system in $\alpha_3,\ldots,\varrho_3,\alpha_4,\ldots,\varrho_4$ to be solved consists only of $7\times 4 = 28$ equations (with 4 parameters s,y,p,q). This system turns out to be highly redundant. In order to state its general solution, we decided to express it as functions of the parameters of F. The 10 coefficients α_4, β_4, γ_4, ε_4, ξ_4, ν_4, η_4, ω_4, π_4, ϱ_4 can be chosen freely, the three remaining ones are given by

$$(*)\qquad \delta_4=\varepsilon_4-2\eta_4,\quad \mu_4=\xi_4+2\nu_4,\quad \lambda_4=\pi_4.$$

The coefficients of T are then expressed as functions of the coefficients of F, hence yielding 13 formulas. These are of two types: the first set is independent of the metric and relatively simple,

$$\beta_3=4(p+q)\pi_4,\quad \gamma_3=-4(p+q)\nu_4,\quad \delta_3=-4(p+q)\pi_4,\quad \xi_3=-4(p+q)\eta_4,$$

$$\eta_3=4(p+q)(\xi_4+2\nu_4),\ \omega_3=-4(p+q)\varepsilon_4,\ \pi_3=4(p+q)\xi_4,\ \varrho_3=-4(p+q)(\varepsilon_4-2\eta_4).$$

The second set of formulas is more complicated and, in particular, dependent on the metric parameters s,y,

$$\begin{aligned}
\alpha_3 &= -\frac{1}{3s}\left[-2-6s+3ys+4s^2-2y-4ps(-\alpha_4+\beta_4+\gamma_4+2\omega_4-\varrho_4)+12qs\,\varrho_4\right],\\
\varepsilon_3 &= -\frac{1}{3s}\left[2-6s+3ys-4s^2+2y-4ps(\alpha_4-\beta_4-\gamma_4+\omega_4-2\varrho_4)-12qs\,\omega_4\right],\\
\mu_3 &= +\frac{2}{3s}\left[-1-4s^2+2y+2ps(\alpha_4-\beta_4+2\gamma_4-2\omega_4-2\varrho_4)+6qs\,\gamma_4\right],\\
\nu_3 &= +\frac{1}{3s}\left[4-8s^2+y-4ps(-\alpha_4-2\beta_4+\gamma_4+2\omega_4+2\varrho_4)+12qs\,\beta_4\right],\\
\lambda_3 &= -\frac{1}{3s}\left[4-8s^2+y-4ps(2\alpha_4+\beta_4+\gamma_4+2\omega_4+2\varrho_4)-12qs\,\alpha_4\right].
\end{aligned}$$

This shows the main part of the Theorem. The equation $\mathrm{F}\cdot\Psi_3=0$ yields for the coefficients of F four conditions; three of them coincide with the equations $(*)$, whilst the last one is the linear equation

$$-\alpha_4+\beta_4+\gamma_4+2\omega_4+2\varrho_4=0\,.$$

Surprisingly, none of the parameters s, y, p, q occurs. The constraint $\mathrm{T}\cdot\Psi_3 = 0$ gives only one condition, namely,

$$s(6q-8p)(-\alpha_4+\beta_4+\gamma_4+2\omega_4+2\varrho_4) = 1+y+4s^2\,.$$

Since $1+y+4s^2>0$, all remaining claims follow. □

5. Solutions for the special (p,q)-coupling

The coupling $4\,p-3\,q=0$ between the different parts involving the flux term of the Killing equation plays a special role (see Theorem 2.1 and Theorem 4.1). Let us discuss the solutions in this case in more detail. The Killing equation reads as ($n=7$)

$$\nabla^g_X\Psi + \frac{1}{4}\cdot(X\,\lrcorner\,\mathrm{T})\cdot\Psi + \frac{3}{4}\cdot(X\,\lrcorner\,\mathrm{F})\cdot\Psi + (X\wedge\mathrm{F})\cdot\Psi \;=\; 0\,.$$

The first series of examples are nearly parallel G_2-manifolds. We normalize the scalar curvature by the condition $\mathrm{Scal}=168$. Then there exists a Riemannian Killing spinor Ψ corresponding to the G_2-structure ω^3,

$$\nabla^g_X\Psi \;=\; X\cdot\Psi\,,\quad \omega^3\cdot\Psi \;=\; -\,7\cdot\Psi\,.$$

The pair $3\cdot\mathrm{T}=-\,4\cdot\omega^3$ and $\mathrm{F}=f\cdot(*\omega^3)$ together with the spinor Ψ solves the equation, where $f\in\mathbb{R}^1$ is an arbitrary real parameter. The torsion form has a geometric meaning. It defines the unique linear, metric connection $\nabla=\nabla^g+(1/2)\cdot\mathrm{T}$ preserving the nearly parallel G_2-structure (see [12, Example 5.2]). Moreover, the spinor field Ψ is ∇-parallel and the Killing equation decouples into

$$\nabla^g_X\Psi + \frac{1}{4}\cdot(X\,\lrcorner\,\mathrm{T})\cdot\Psi \;=\; 0 \quad\text{and}\quad 3\cdot(X\,\lrcorner\,\mathrm{F})\cdot\Psi + 4\cdot(X\wedge\mathrm{F})\cdot\Psi \;=\; 0\,.$$

Compact nearly parallel G_2-manifolds are studied, for example, in [14]. The 1-parameter family of flux forms associated with a nearly parallel G_2-manifold has already been investigated in supergravity (see [6]).

A larger family of solutions arises from a 7-dimensional 3-Sasakian manifold M^7. It is an Einstein space, and the scalar curvature is normalized automatically to $\mathrm{Scal}=42$. There exist four Riemannian Killing spinors (see [13]). Let us fix one of them. In the family of torsion and flux forms considered in Section 3, there exists a 7-dimensional affine subspace of solutions. The torsion forms are completely determined by the flux forms ($p=3/4$ and $q=1$ in the notation of Section 2),

$$\begin{aligned}
t_{12} &= -\,7\,f_{272} \;=\; -\,t_{21},\quad t_{17} \;=\; -\,7\,f_{277} \;=\; -\,t_{71},\\
t_{27} &= 7\,f_{177} \;=\; t_{72},\\
t_{11} &= -\frac{2}{3}+f-2\,f_{127}+2\,f_{172}-5\,f_{271},\\
t_{22} &= \frac{2}{3}-f+2\,f_{127}-2\,f_{271}+5\,f_{172},\\
t_{77} &= \frac{2}{3}-f-2\,f_{172}-2\,f_{271}-5\,f_{127},\\
t &= \frac{2}{3}+2\,f_{127}-2\,f_{172}-2\,f_{271}+6\,f,\\
f_{122} &= -\,f_{177},\quad f_{121} \;=\; -\,f_{277},\quad f_{171} \;=\; f_{272}\,.
\end{aligned}$$

All torsion forms in the 7-dimensional family of solutions act on the spinor by the formula $\mathrm{T}\cdot\Psi = (14/3)\cdot\Psi$. The equation $\mathrm{F}\cdot\Psi = 0$ defines a 6-dimensional affine subspace,

$$\gamma \ := \ f \,-\, 2\,f_{127} \,+\, 2\,f_{172} \,+\, 2\,f_{271} \ = \ 0\,.$$

We compute the action of the symmetric endomorphisms T and F on the 8-dimensional space of spinors explicitely. In order to formulate the result, let us introduce the following (3×3)-matrix F^*,

$$\begin{bmatrix} f-2f_{127}-2f_{172}-2f_{271} & -4\,f_{177} & -4\,f_{277} \\ -4\,f_{177} & f+2f_{127}+2f_{172}-2f_{271} & 4\,f_{272} \\ -4\,f_{277} & 4\,f_{272} & f+2f_{127}-2f_{172}+2f_{271} \end{bmatrix}.$$

We order the basis in Δ_7 in such a way that Ψ is the last element in the basis. Then the traceless and symmetric endomorphisms T and F are given by the matrices

$$\mathrm{F} \ = \ \begin{bmatrix} -f\cdot\mathrm{Id}_4 & 0 & 0 \\ 0 & \mathrm{F}^* & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \mathrm{T} \ = \ \begin{bmatrix} (\gamma-\frac{2}{3})\cdot\mathrm{Id}_4 & 0 & 0 \\ 0 & (\gamma-\frac{2}{3})\cdot\mathrm{Id}_3 & 0 \\ 0 & 0 & \frac{14}{3}-7\,\gamma \end{bmatrix} \,+\, 7\,\mathrm{F}\,.$$

Remark that F^* is an arbitrary symmetric (3×3)-matrix. It acts in the 3-plane generated by the Riemannian Killing spinors orthogonal to the fixed Riemannian Killing Ψ. Let us look at the family of solutions from the point of view of G_2-structures. The spinor Ψ defines such a structure on M^7 (see Section 2). Since it is a Riemannian Killing spinor, the G_2-structure is nearly parallel (see [14]). On the other side, a 3-Sasakian structure on M^7 is topologically a $\mathrm{SU}(2)$-reduction of the frame bundle. Since $\mathrm{SU}(2)\subset\mathrm{G}_2\subset\mathrm{SO}(7)$, any 3-Sasakian manifold induces a family of G_2-structures. The spinor Ψ singles out one of them. In any case, we have an underlying G_2-structure ω^3 on M^7. In our parametrization of the family (T,F) the case $\mathrm{F} = 0$ yields again the canonical torsion form of the unique connection preserving the nearly parallel G_2-structure (see again [12, Example 5.1]). Moreover, the condition

$$\mathrm{T} \ = \ \begin{bmatrix} -\frac{2}{3}\cdot\mathrm{Id}_4 & 0 & 0 \\ 0 & -\frac{2}{3}\cdot\mathrm{Id}_3 & 0 \\ 0 & 0 & \frac{14}{3} \end{bmatrix}$$

defines a 1-parameter subfamily of flux forms. This is exactly the above mentioned solution line of the nearly parallel G_2-structure. Consequently, if the nearly parallel G_2-structure arises from an underlying 3-Sasakian geometry, we can embed the canonical solution $(\mathrm{T}=\omega^3,\ \mathrm{F}=f\cdot(*\omega^3))$ into a larger family of solutions. Then the Killing equation does not decouple anymore. We have the same picture for the solutions on $N(1,1)$. In this case, the underlying G_2-structure is not nearly parallel, but only cocalibrated and additional parameters for the metric occur.

A special coupling between the (p,q)-parameters in the Killing equation with fluxes occurs in any dimension. We explain one way to understand this effect. First of all, one easily verifies the following algebraic formulas concerning the action of exterior forms of degree three and four on spinors:

$$\sum_{i=1}^{n} e_i\cdot\big(e_i\,\lrcorner\,\mathrm{T}\big) = 3\cdot\mathrm{T}, \quad \sum_{i=1}^{n} e_i\cdot\big(e_i\,\lrcorner\,\mathrm{F}\big) = 4\cdot\mathrm{F}, \quad \sum_{i=1}^{n} e_i\cdot\big(e_i\wedge\mathrm{F}\big) = -\,(n\,-\,4)\cdot\mathrm{F}\,.$$

Contracting the equation

$$\nabla^g_X \Psi \,+\, \frac{1}{4}\cdot (X \lrcorner \mathrm{T})\cdot \Psi \,+\, p\cdot (X \lrcorner \mathrm{F})\cdot\Psi \,+\, q\cdot (X\wedge \mathrm{F})\cdot \Psi \;=\; 0\,,$$

we obtain

$$D^g\Psi \,+\, \frac{3}{4}\,\mathrm{T}\cdot\Psi \,+\, \big(4\,p \,-\, (n-4)\,q\big)\cdot \mathrm{F}\cdot\Psi \;=\; 0\,.$$

If $4\,p - (n-4)\,q \,=\, 0$, the action of the Riemannian Dirac operator D^g on the spinor Ψ depends only on the torsion form, but not on the flux form. In this case we obtain a link between the spectrum of the Riemannian Dirac operator and the admissible algebraic constraints given by the torsion form. We can apply well-known estimates for the Dirac spectrum of a Riemannian manifold in order to exclude some of these solutions. For example, we obtain (see [10])

Proposition 5.1. *Let $(M^n,\, g,\, \mathrm{T},\, \mathrm{F},\, \Psi)$ be a compact solution of the equation*

$$\nabla^g_X \Psi \,+\, \frac{1}{4}\cdot (X \lrcorner \mathrm{T})\cdot \Psi \,+\, \frac{n-4}{4}\cdot (X \lrcorner \mathrm{F})\cdot\Psi \,+\, (X\wedge \mathrm{F})\cdot \Psi \;=\; 0$$

with the constraint $\mathrm{T}\cdot\Psi \,=\, c\cdot\Psi$. Then the eigenvalue c is bounded by the minimum Scal_0 of the scalar curvature of the Riemannian manifold,

$$c^2 \;\geq\; \frac{4\,n}{9\,(n-1)}\cdot \mathrm{Scal}_0\;.$$

The solutions on 3-Sasakian manifolds discussed before realize the lower bound, since they come from a Riemannian Killing spinor. In case of the homogeneous solutions on the Aloff-Wallach space $N(1,1)$ the eigenvalue c^2 is strictly greater then the lower bound.

Remark 5.1. On an 8-dimensional manifold the equation corresponding to the special (p,q)-parameters simplifies,

$$\nabla^g_X \Psi \,+\, \frac{1}{4}\cdot (X \lrcorner \mathrm{T})\cdot \Psi \,+\, \mathrm{F}\cdot X\cdot\Psi \;=\; 0\,.$$

Our method for the construction of torsion and flux forms solving the equation at hand applies in dimension eight, too. The key point is the following Proposition. Its proof relies on a computer computation. We have to solve a system of 128 linear equations in 126 variable, and turns out to have sufficiently many solutions.

Proposition 5.2. *Let $\Psi = \Psi^+ + \Psi^-$ be an 8-dimensional spinor with non trivial positive and negative part, $\Psi^{\pm}\neq 0$. Then there exists a family depending on 25 parameters of 3-forms $\mathrm{T}\in\Lambda^3(\mathbb{R}^8)$ and 4-forms $\mathrm{F}\in\Lambda^4(\mathbb{R}^8)$ such that, for any vector $X\in\mathbb{R}^8$, the following equation holds:*

$$\frac{1}{4}\cdot (X \lrcorner \mathrm{T})\cdot \Psi \,+\, \mathrm{F}\cdot X\cdot\Psi \;=\; 0\;.$$

Consider an 8-dimensional Lie group $(\mathrm{G}^8,\, g)$ equipped with a biinvariant Riemannian metric. The formula

$$\mathrm{T}_0(X,Y,Z) \;:=\; -\,g([X\,,\,Y]\,,\,Z)$$

defines the canonical torsion form of the Lie group. The action of the Levi-Civita connection on a spinor field $\Psi:\mathrm{G}^8\to\Delta_8$ is given by the formula

$$\nabla^g_X\Psi \;=\; d\Psi(X) \,+\, \frac{1}{4}\,(X\lrcorner \mathrm{T}_0)\cdot\Psi\;.$$

Consequently, any constant spinor field on the Lie group admits a family depending on 25 parameters of torsion and flux forms solving the equation.

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