THE GEODESICS OF METRIC CONNECTIONS WITH VECTORIAL TORSION

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ABSTRACT. The present note deals with the dynamics of metric connections with vectorial torsion, as already described by E. Cartan in 1925. We show that the geodesics of metric connections with vectorial torsion defined by gradient vector fields coincide with the Levi-Civita geodesics of a conformally equivalent metric. By pullback, this yields a systematic way of constructing invariants of motion for such connections from isometries of the conformally equivalent metric, and we explain in as much this result generalizes the Mercator projection which maps sphere loxodromes to straight lines in the plane. An example shows that Beltrami's theorem fails for this class of connections. We then study the system of differential equations describing geodesics in the plane for vector fields which are not gradients, and show among others that the Hopf-Rinow theorem does also not hold in general.

1. **Introduction**

The present note deals with the dynamics of metric connections on a given Riemannian manifold (M^n, g) of the form

$$\nabla_X Y \;=\; \nabla^g_X Y + g(X,Y)V - g(V,Y)X,$$

where V denotes a fixed vector field on M and ∇^g is the usual Levi-Civita connection. This is one of the three basic types of metric connections, as already described by E. Cartan in [Car25], and will be called by us a metric connection *with vectorial torsion*. The case of a surface ($n = 2$) is special in as much that any metric connection has to be of this type. In fact, classical topics of surface theory like the Mercator projection which maps loxodromes on the sphere to straight lines in the plane can be understood in a different light with their help. We will show that the geodesics of metric connections with vectorial torsion defined by gradient vector fields coincide with the Levi-Civita geodesics of a conformally equivalent metric. By pullback, this yields a systematic way of constructing invariants of motion for such connections from isometries of the conformally equivalent metric. Furthermore, the discussion of the catenoid will show that there exist geodesic mappings for this larger class of connections between surfaces of constant and non-constant Gaussian curvature, hence Beltrami's theorem becomes wrong in this situation. In the last Section, we will treat the euclidian plane with an arbitrary vector field in great detail and show that the behaviour of geodesics is, already in this low dimension, dominated by a highly non-trivial system of ordinary differential equations. In particular, the Hopf-Rinow theorem can fail, too.

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For some reasons, these connections have not attracted as much attention in the past as we believe they deserve. Correspondingly, an overview over the existing literature (that we are aware of) is quickly given. In [TV83], Tricerri and Vanhecke were led to the study of such connections in the context of the classification problem of homogeneous structures on manifolds. They showed that if M is connected, complete, and simply connected and V is parallel, i. e. $\nabla V = 0$, then (M, g) has to be isometric to hyperbolic space. Vicente Miquel studied in [Miq82] and [Miq01] the growth of geodesic balls of such connections, but did not investigate the detailed shape of geodesics. Connections with vectorial torsion on spin manifolds may also play a role in superstring theory (see [AF03] and the literature cited therein), but this aspect will not be discussed in the present paper. Both authors wish to thank Thomas Friedrich and Paweł Nurowski for valuable discussions and suggestions.

We finish this introduction with a short review of the possible classes of metric connections. For details, we refer to [Car25], [TV83] and [AF03]. In any point p of an n-dimensional Riemannian manifold (M^n, g), the difference between its Levi-Civita connection ∇^g and any linear connection ∇ is a $(2,1)$-tensor A,

$$\nabla_X Y \ = \ \nabla^g_X Y + A(X, Y), \quad X, Y \in T_pM .$$

We identify T_pM with $\mathbb{R}^n$, on which the real orthogonal group $\mathrm{O}(n, \mathbb{R})$ then acts in the standard way. The connection ∇ is metric if and only if the difference tensor A (viewed as a $(3,0)$-tensor) belongs in every point to the $n^2(n-1)/2$-dimensional space

$$\mathcal{A}^g \ := \ \mathbb{R}^n \otimes \wedge^2\mathbb{R}^n \ = \ \{A \in \otimes^3\mathbb{R}^n \mid A(X, V, W) + A(X, W, V) \ = \ 0\} .$$

Proposition 1.1. *For $n \geq 3$, the space $\mathcal{A}^g$ of possible metric difference tensors splits under $\mathrm{O}(n, \mathbb{R})$ into the sum of three irreducible representations, $\mathcal{A}^g \cong \mathbb{R}^n \oplus \wedge^3\mathbb{R}^n \oplus \mathcal{A}'$. For $n = 2$, $\mathcal{A}^g$ is an irreducible $\mathrm{O}(2, \mathbb{R})$-module isomorphic to $\mathbb{R}^2$.*

The torsion of ∇ is then constructed from A by the usual formula $T(X, Y) = A(X, Y) - A(Y, X)$. It has become customary to call metric connections such that $A \in \wedge^3\mathbb{R}^n$ *metric connections with totally skew-symmetric torsion*, and we shall use the denomination *metric connections with vectorial torsion* in case $A \in \mathbb{R}^n$. The correspondence can be made explicit: for a vector field V on M, A and T are then given by

$$A(X, Y) \ = \ g(X, Y)V - g(V, Y)X, \quad T(X, Y, Z) \ = \ g\big(g(V, X)Y - g(V, Y)X, Z\big).$$

The metric connection ∇ has the same geodesics as the Levi-Civita connection precisely if it has totally skew-symmetric torsion.

2. Basic remarks on geodesics of metric connections with vectorial torsion

Let (M, g) be a Riemannian manifold, V a vector field on M and ∇ the metric connection with vectorial torsion defined by V. A curve $\gamma(t)$ is a geodesic of ∇ if it satisfies the differential equation

$$\nabla^g_{\dot\gamma}\dot\gamma + g(\dot\gamma, \dot\gamma)V - g(V, \dot\gamma)\dot\gamma \ = \ 0 .$$

Taking the scalar product of this equation with $\dot\gamma$ yields $g(\nabla^g_{\dot\gamma}\dot\gamma, \dot\gamma) = 0$, that is, $\dot\gamma$ has constant length $E > 0$, which reflects of course just the fact that ∇ was metric. Hence, the geodesic equation can be written

$$\nabla^g_{\dot\gamma}\dot\gamma + E^2\, V - g(V, \dot\gamma)\dot\gamma \ = \ 0 . \tag{1}$$

In fact, there are qualitatively two cases to be distinguished. If $\dot\gamma$ is parallel to V at the origin, $\dot\gamma(0) = \alpha \cdot V(\gamma(0))$, we conclude that $\nabla^g_{\dot\gamma(0)}\dot\gamma(0) = 0$ and $\gamma(t)$ coincides locally with a classical geodesic of the Levi-Civita connection. In particular, a ∇-geodesic which stays parallel to V for all times is exactly a ∇^g-geodesic. Generic ∇-geodesics are those for which $\dot\gamma$ is never parallel to V; their shape will be qualitatively very different from that of their Levi-Civita "cousins". First, we express the curvature of a geodesic curve through V and $\dot\gamma$:

Lemma 2.1. *The Riemannian geodesic curvature κ of a ∇-geodesic γ is given by $\kappa^2 = ||V||^2 - g(V,\dot\gamma)^2/E^2$ for general V, and by $\kappa^2 = -E^{-2}\, d/dt\, g(V,\dot\gamma)$ if V is a Killing vector field.*

Proof. Observe that $\ddot\gamma = \nabla^g_{\dot\gamma}\dot\gamma$, and $\kappa = ||\ddot\gamma||/E^2$ in our normalization. Differentiating the geodesic equation (1) once more and taking the scalar product with $\dot\gamma$, we obtain

$$g(\dddot\gamma,\dot\gamma) \;=\; E^2 g(V,\ddot\gamma) \;=\; E^2[g(V,\dot\gamma)^2 - E^2||V||^2]\,.$$

But from $g(\dot\gamma,\ddot\gamma) = 0$, we conclude that $g(\ddot\gamma,\ddot\gamma) + g(\dot\gamma,\dddot\gamma) = 0$, hence the first claim follows. Suppose now that V is a Killing vector field, $g(\nabla^g_X V, Y) + g(\nabla^g_Y V, X) = 0$ for all X and Y. This implies in particular $g(\dot\gamma,\dot V) = g(\dot\gamma,\nabla^g_{\dot\gamma}V) = 0$, and so

$$\frac{d}{dt}g(V,\dot\gamma) \;=\; g(V,\ddot\gamma) \;=\; g(V,\dot\gamma)^2 - E^2||V||^2 \;=\; -E^2\kappa^2. \qquad \square$$

In examples, this allows to determine a differential equation for κ, which one then needs to study in detail. Even without this, the lemma is useful for proving results of the following type: periodic ∇-geodesics of Killing vector fields have to be of non-generic type. More precisely:

Corollary 2.1. *For a Killing vector field V, any periodic ∇-geodesic is automatically a Levi-Civita geodesic and, up to a constant, an integral curve of V.*

Proof. If the curve γ is periodic, the functions $g(V,\dot\gamma)$ and $d/dt\, g(V,\dot\gamma)$ are periodic too. But since the latter has to be non-positive by Lemma 2.1, we conclude that $d/dt\, g(V,\dot\gamma) = 0$, hence $\nabla^g_{\dot\gamma}\dot\gamma = 0$. Furthermore, the geodesic equation yields $E^2 V = c\,\dot\gamma$, where c is the constant $g(V,\dot\gamma)$, and $(c/E^2)\gamma$ becomes an integral curve of V. $\square$

In the next step, we show that isometries leaving V invariant generate symmetries of the ∇-geodesics.

Proposition 2.1. *If X is a Killing vector field commuting with V, its flow Φ_s maps ∇-geodesics to ∇-geodesics.*

Proof. Let $\gamma(t)$ be a ∇-geodesic, and set $\gamma^*(t) := \Phi_s\big(\gamma(t)\big)$, for which one has $\dot\gamma^* = d\Phi_s(\dot\gamma)$. Since Φ_s consists of isometries, $||\dot\gamma|| = ||\dot\gamma^*|| = E$, and the invariance of V under Φ_s implies $g(V,\dot\gamma) = g(V,\dot\gamma^*)$. In addition, $d\Phi_s(\nabla^g_{\dot\gamma}\dot\gamma) = \nabla^g_{d\Phi_s(\dot\gamma)} d\Phi_s(\dot\gamma)$, hence applying $d\Phi_s$ to the geodesic equation for $\gamma(t)$ yields that $\gamma^*(t)$ satisfies the geodesic equation, too. Notice that by the same argument, the first curvatures of γ and γ^* coincide. $\square$

Remark 2.1. In Section 3, we will construct a geodesic mapping for metric connections with vectorial torsion that is not an isometry leaving V invariant. Also, the main problem in the study of the geodesic equation is that Proposition 2.1 does *not* yield invariants of motion of Noether type: if γ is a ∇-geodesic and W a vector field whose flow Φ_s consists of isometries and leaves V invariant, the quantity $g(W,\dot\gamma)$ is *not* a first integral (easy examples will again be discussed in Section 3). This reflects the fact that the geodesic equation is not the Euler-Lagrange equation of a Lagrange function $\mathcal{L}(\dot\gamma) = g_{ij}\dot x^i \dot x^j/2 - V(\gamma)$ for some potential function V.

3. Geodesics for gradient vector fields and conformal mappings

The motivation for this section was Cartan's example of a metric connection with vectorial torsion on the sphere. Up to our knowledge, this is the only instance where some geodesics of metric connections with vectorial torsion are described in the literature.

Example 3.1 (Cartan's example)**.** In [Car23, § 67, p. 408–409], Cartan describes the two-dimensional sphere with its flat metric connection, and observes (without proof) that "on this manifold, the straight lines are the *loxodromes*, which intersect the meridians at a constant angle. The only straight lines realizing shortest paths are those which are normal to the torsion in every point: *these are the meridians*[1]".

This suggests that there exists a class of metric connections on surfaces of revolution whose geodesics admit a generalization of Clairaut's theorem, yielding loxodromes in the case of the flat connection. Furthermore, it is well known that the Mercator projection maps loxodromes to straight lines in the plane (i. e., Levi-Civita geodesics of the euclidian metric), and that this mapping is conformal. Theorem 3.1 provides the right setting to understanding both effects, as explained in the introduction. First, let us recall without proof a standard formula for a conformal change $\tilde g = e^{2\sigma} g$ of a metric g that we will need in the sequel:

Proposition 3.1. *The Levi-Civita connection behaves as follows under conformal change of the metric:*

$$\nabla^{\tilde g}_X Y \;=\; \nabla^g_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)\mathrm{grad}(\sigma).$$

Theorem 3.1. *Let σ be a function on the Riemannian manifold (M,g), ∇ the metric connection with vectorial torsion defined by $V = -\mathrm{grad}(\sigma)$, and consider the conformally equivalent metric $\tilde g = e^{2\sigma} g$. Then:*

1. *Any ∇-geodesic $\gamma(t)$ is, up to a reparametrisation τ, a $\nabla^{\tilde g}$-geodesic, and the function τ is the unique solution of the differential equation $\ddot\tau + \dot\tau\dot\sigma = 0$, where we set $\sigma(t) := \sigma\circ\gamma\circ\tau(t)$;*
2. *If X is a Killing field for the metric $\tilde g$, the function $e^\sigma g(\dot\gamma, X)$ is a constant of motion for the ∇-geodesic $\gamma(t)$.*

Proof. Assume that $\gamma : I \to M$ is a ∇-geodesic, i. e. it satisfies

$$\nabla^g_{\dot\gamma}\dot\gamma \;=\; -E^2\,V + g(V,\dot\gamma)\dot\gamma \;=\; E^2\,\mathrm{grad}(\sigma) - g(\mathrm{grad}(\sigma),\dot\gamma)\dot\gamma.$$

Then Proposition 3.1 implies

$$(2)\quad \nabla^{\tilde g}_{\dot\gamma(t)}\dot\gamma(t) \;=\; 2\dot\gamma(\sigma)\dot\gamma - g(\mathrm{grad}(\sigma),\dot\gamma)\dot\gamma \;=\; +\,g(\mathrm{grad}(\sigma),\dot\gamma)\dot\gamma \;=\; \frac{d}{dt}(\sigma\circ\gamma)(t)\cdot\dot\gamma(t).$$

We claim that there exists a reparametrisation $\gamma^*(t) := \gamma\circ\tau(t)$ of the curve $\gamma(t)$ which satisfies $\nabla^{\tilde g}_{\dot\gamma^*}\dot\gamma^* = 0$. For a still arbitrary function $\tau : I \to I$, we have $\dot\gamma^*(t) \;=$

[1]Sur cette variété, les lignes droites sont les *loxodromies*, qui font un angle constant avec les méridiennes. Les seules lignes droites qui réalisent les plus courts chemins sont celles qui sont normales en chaque point à la torsion : *ce sont les méridiennes.* loc. cit.

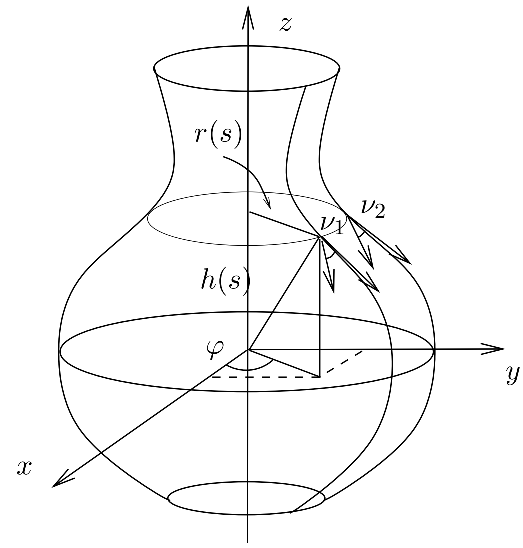


Figure 1. Surface of revolution generated by a curve α.

$\dot\tau(t)\cdot\dot\gamma(\tau(t))$, hence

$$\begin{aligned}\nabla^{\tilde g}_{\dot\gamma^*(t)}\dot\gamma^*(t) &= \nabla^{\tilde g}_{\dot\gamma^*(t)}\big(\dot\tau(t)\cdot\dot\gamma(\tau(t))\big) = \dot\gamma^*(\dot\tau)\cdot\dot\gamma(\tau(t)) + \dot\tau(t)\,\nabla^{\tilde g}_{\dot\gamma^*(t)}\dot\gamma(\tau(t))\\ &= \dot\tau\dot\gamma(\dot\tau)\cdot\dot\gamma + \dot\tau^2\,\nabla^{\tilde g}_{\dot\gamma(\tau(t))}\dot\gamma(\tau(t)) = \left[\dot\tau\dot\gamma(\dot\tau) + \dot\tau^2\,\frac{d}{ds}\Big|_{s=\tau(t)}(\sigma\circ\gamma)(s)\right]\dot\gamma(\tau(t)),\end{aligned}$$

where we used equation (2) in the last step. It remains to be shown that τ can be chosen such that the expression in parentheses vanishes. For this, we first rewrite this expression as

$$\dot\tau(t)\frac{d}{ds}\Big|_{s=\tau(t)}(\dot\tau\circ\gamma)(s) + \dot\tau^2(t)\,\frac{d}{ds}\Big|_{s=\tau(t)}(\sigma\circ\gamma)(s) = \frac{d}{ds}\Big|_{s=t}(\dot\tau\circ\gamma\circ\tau)(s) + \dot\tau\,\frac{d}{ds}\Big|_{s=t}(\sigma\circ\gamma\circ\tau)(s),$$

leading to the differential equation

$$\ddot\tau(t) + \dot\tau(t)\dot\sigma(t) = 0 \tag{3}$$

for the function τ. Here, we defined $\sigma(t) := (\sigma\circ\gamma\circ\tau)(t)$ and viewed $\dot\tau$ as a function on $\gamma(I)\subset M$ if necessary. It cannot be solved explicitely, since σ depends implicitly on τ; but by Picard-Lindelöf, equation (3) does always admit a solution. Observe that, formally, the differential equation can be integrated once, yielding the relation $\dot\tau(t) = e^{-\sigma(t)}$. For the second part of the proof, let us now assume that X is a Killing vector field relatively to the metric $\tilde g = e^{2\sigma}g$. Thus, the $\nabla^{\tilde g}$-geodesic γ^* satisfies $\tilde g(\dot\gamma^*, X\circ\gamma^*) = \text{const} =: c$. By the integrated form of equation (3), this implies

$$c = e^{2\sigma(t)}g\big(\dot\tau(t)\dot\gamma(\tau(t)), X\big|_{\gamma(\tau(t))}\big) = e^{2\sigma(t)}e^{-\sigma(t)}g\big(\dot\gamma(\tau(t)), X\big) = e^{\sigma}g(\dot\gamma, X). \qquad \square$$

Since one is most often only interested in the set of points of a curve and not in the parametrisation itself, it is rarely necessary to determine τ explicitely.

Example 3.2 (Loxodromes and Mercator projection)**.** We discuss Cartan's example in the light of Theorem 3.1. Let $\alpha = (r(s), h(s))$ be a curve in natural parametrisation, and $M(s,\varphi) = (r(s)\cos\varphi, r(s)\sin\varphi, h(s))$ the surface of revolution generated by it.

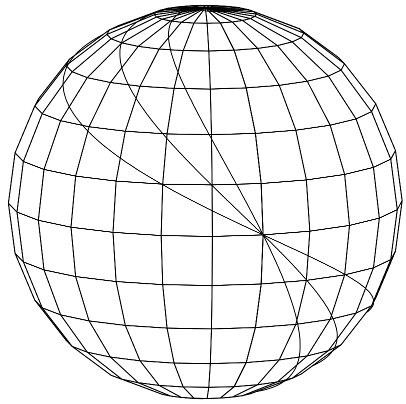


Figure 2. Loxodromes on the sphere.

The first fundamental form is then $g = \mathrm{diag}(1, r^2(s))$, and we fix the orthonormal frame $e_1 = \partial_s$, $e_2 = (1/r)\partial_\varphi$ with dual 1-forms $\sigma^1 = ds$, $\sigma^2 = r\,d\varphi$. We convene to call two tangential vectors v_1 and v_2 parallel if their angles ν_1 and ν_2 with the meridian through that point coincide (see Figure 1). Hence $\nabla e_1 = \nabla e_2 = 0$, and the connection ∇ is flat. But for a flat connection, the torsion T is can be derived from $d\sigma^i(e_j, e_k) = \sigma^i(T(e_j, e_k))$. Since $d\sigma^1 = 0$ and $d\sigma^2 = (r'/r)\sigma^1 \wedge \sigma^2$, one obtains

$$T(e_1, e_2) \;=\; \frac{r'(s)}{r(s)}\, e_2 \;\text{ and }\; V \;=\; \frac{r'(s)}{r(s)}\, e_1 \;=\; -\mathrm{grad}\big(-\ln r(s)\big).$$

Thus, the metric connection ∇ with vectorial torsion T is determined by the gradient of the function $\sigma := -\ln r(s)$. By Theorem 3.1, we conclude that its geodesics are the Levi-Civita geodesics of the conformally equivalent metric $\tilde{g} = e^{2\sigma} g = \mathrm{diag}(1/r^2, 1)$. This coincides with the standard euclidian metric if one performs the change of variables $x = \varphi$, $y = \int ds/r(s)$. For example, the sphere is obtained for $r(s) = \sin s, h(s) = \cos s$, hence $y = \int ds/\sin s = \ln\tan(s/2)$ $(|s| < \pi/2)$, and this is precisely the coordinate change of the Mercator projection. Furthermore, $X = \partial_\varphi$ is a Killing vector field for $\tilde{g}$, hence the second part of Theorem 3.1 yields for a ∇-geodesic γ the invariant of motion

$$\mathrm{const} \;=\; e^{\sigma} g(\dot\gamma, X) \;=\; \frac{1}{r(s)} g(\dot\gamma, \partial_\varphi) \;=\; g(\dot\gamma, e_2),$$

and this is just the cosine of the angle between γ and a parallel circle. This shows that γ is a loxodrome on M, as claimed (see Figure 2 for loxodromes on the sphere). In the same way, one obtains a "generalized Clairaut theorem" for any gradient vector field on a surface of revolution. Figure 3 illustrates loxodromes on a pseudosphere and a catenoid. For the pseudosphere, one chooses

$$r(s) \;=\; e^{-s}, \quad h(s) \;=\; \mathrm{arctanh}\sqrt{1 - e^{-2s}} - \sqrt{1 - e^{-2s}},$$

hence $V = -e_1$ and $\nabla V = 0$, in accordance with the results by [TV83] cited in the introduction. Notice that X is also a Killing vector field for the metric g and does commute with V; nevertheless, $g(\dot\gamma, X)$ is *not* an invariant of motion (compare this to Proposition 2.1 and Remark 2.1).

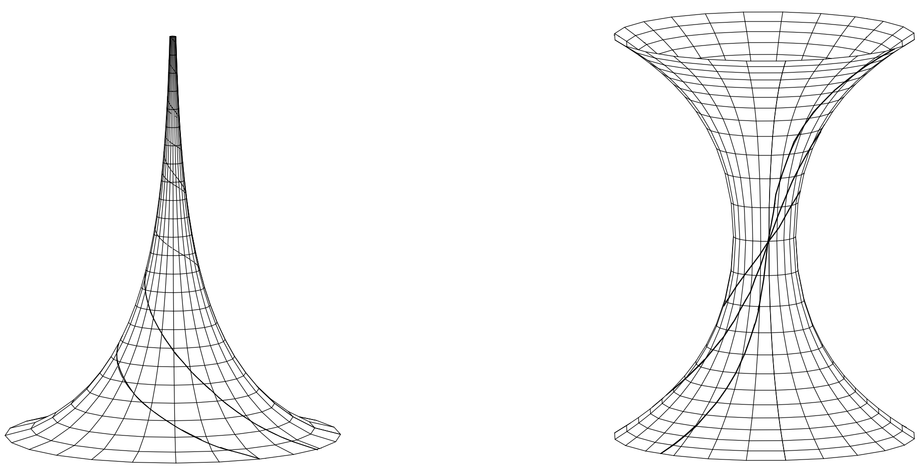


Figure 3. Loxodromes on the pseudosphere and the catenoid.

Remark 3.1. The catenoid is interesting for another reason: since it is a minimal surface, the Gauss map to the sphere is a conformal mapping, hence it maps loxodromes to loxodromes. Thus, Beltrami's theorem ("If a portion of a surface S can be mapped LC-geodesically onto a portion of a surface S^* of constant Gaussian curvature, the Gaussian curvature of S must also be constant", see for example [Kre91, §95]) does *not* hold for metric connections with vectorial torsion – the sphere is a a surface of constant Gaussian curvature, but the catenoid is not.

Remark 3.2. The unique flat metric connection ∇ does not have to be of vectorial type. For example, on the compact Lie group $\mathrm{SO}(3,\mathbb{R})$, its torsion is a 3-form: Fix an orthonormal basis e_1, e_2, e_3 with commutator relations $[e_1, e_2] = e_3, [e_2, e_3] = e_1$ and $[e_1, e_3] = -e_2$. Cartan's structural equations then read $d\sigma^1 = \sigma^2 \wedge \sigma^3, d\sigma^2 = -\sigma^1 \wedge \sigma^3, d\sigma^3 = \sigma^1 \wedge \sigma^2$, from which we deduce $T = 2A = \sigma^1 \wedge \sigma^2 \wedge \sigma^3$. In particular, ∇ has the same geodesics as ∇^g.

Remark 3.3. The spirit of Theorem 3.1 reflects a well-known fact from Weyl geometry. Recall that a *Weyl manifold* is a manifold M^n with a conformal structure c and a torsion-free connection on the $\mathrm{CO}(n)$-reduction of the frame bundle of M^n. For fixed c, the set of all such connections is an affine space modelled over the vector space of all 1-forms. The choice of a metric g in the conformal class c simply identifies the connection with a 1-form $\theta \in \Omega^1(M^n)$. For a second metric $g' = e^{-2f}g$ in the conformal class c, the characteristic 1-form θ' is then given by $\theta' = \theta + df$.

4. **Geodesics of vectorial connections in the plane**

We consider the plane $M = \mathbb{R}^2$ endowed with its standard euclidian metric $g = dx^2 + dy^2$, and the metric connection ∇ with vectorial torsion defined by the vector field

$V = f(x,y)\,\partial_x + g(x,y)\,\partial_y$, where f and g may be any smooth functions of x and y. Then one computes for the standard orthonormal basis $e_1 = \partial_x$, $e_2 = \partial_y$ that

$$\nabla_{e_1} e_1 \;=\; g\,e_2, \quad \nabla_{e_1} e_2 \;=\; -g\,e_1, \quad \nabla_{e_2} e_1 \;=\; -f\,e_2, \quad \nabla_{e_2} e_2 \;=\; f\,e_1,$$

and hence the connection form ω_{12} and the curvature form $d\omega_{12}$ are given by

$$\omega_{12} \;=\; g(x,y)\,dx - f(x,y)\,dy, \quad d\omega_{12} \;=\; -(\partial_x f + \partial_y g)\,dx \wedge dy.$$

The geodesics $\gamma(t) = (x(t), y(t))$ of ∇ are the solutions of the differential equations

$$\ddot{x} - g(x,y)\,\dot{x}\dot{y} + f(x,y)\,\dot{y}^2 \;=\; 0, \quad \ddot{y} - f(x,y)\,\dot{x}\dot{y} + g(x,y)\,\dot{x}^2 \;=\; 0.$$

Since a plane curve admits a notion of curvature endowed with a sign and is, up to euclidian motions, uniquely determined by it, the general considerations of Section 2 can be strengthened in this case. The curvature κ is defined through the requirement $\ddot{\gamma} = \kappa \cdot i \cdot \dot{\gamma}$, hence we obtain at once:

Lemma 4.1. *The equations for a geodesic γ of the metric connection ∇ with vectorial torsion defined by $V = f(x,y)\,\partial_x + g(x,y)\,\partial_y$ are*

$$\ddot{x} \;=\; -\kappa\,\dot{y}, \quad \ddot{y} \;=\; \kappa\,\dot{x},$$

and $\kappa = f(x,y)\,\dot{y} - g(x,y)\,\dot{x}$ is the curvature of the plane curve γ.

If $\partial_y f = \partial_x g$, the vector field V is (locally) a gradient vector field and its geodesics are best treated with the methods of Section 3. The form of the differential equations does already suggest that the system will, in general, not admit an easy solution. In fact, even a discussion of its qualitative behaviour turns out to be rather difficult.

Yet, there is a second case which turns out to be manageable, namely, the flat case. By the general form of $d\omega_{12}$ given above, the connection ∇ will be flat if there exists a smooth function $p(x,y)$ such that $f = \partial_y p$ and $g = -\partial_x p$. The time derivative of p is then $\dot{p} = \partial_x p\,\dot{x} + \partial_y p\,\dot{y} = -g\,\dot{x} + f\,\dot{x} = \kappa$, so the two geodesic equations of Lemma 4.1 can be restated in one single equation for the complex variable $z = x + iy$,

$$\ddot{z} \;=\; i\,\dot{p}\dot{z}.$$

If $\dot{z} \neq 0$, we divide both sides by $\dot{z}$ and obtain a second invariant of motion (besides E)

$$\frac{d}{dt}(\ln \dot{z} - ip) \;=\; \text{const} \in \mathbb{C}.$$

By choosing the constants appropriatly, we finally arrive at

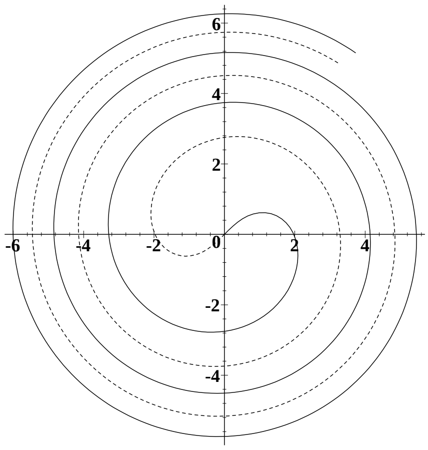
$$\dot{z} \;=\; z_0 e^{ip(x,y)} \quad \text{for some } z_0 \in \mathbb{C}.$$

For given p and z_0, the solution of this differential equation may now be drawn with any standard ODE computer package. Notice that, strictly speaking, only the modulus of this equation yields a new invariant of motion; taking its absolute value, we arrive at $|\dot{z}| = |z_0| = 1$ if we assume—as we always did—that γ is given in natural parametrisation. Obviously, this is not an invariant of Noether type. We give two examples which illustrate how the behaviour of geodesics can differ for different functions p.

Example 4.1. For $p = -(x^2 + y^2)/2$, one has $f = -y, g = +x$, and V is the winding vector field in the plane. The geodesic equations can be written

$$\ddot{x} \;=\; k\,\dot{y}, \quad \ddot{y} \;=\; -k\,\dot{x},$$

where $k = -\kappa = x\dot{x} + y\dot{y} = d/dt\,||\gamma(t)||^2/2$: the curvature of γ coincides, up to sign, with half the derivative of the distance between the point $\gamma(t)$ and the origin. This illustrates that the origin plays indeed a special role for this vector field. V is the



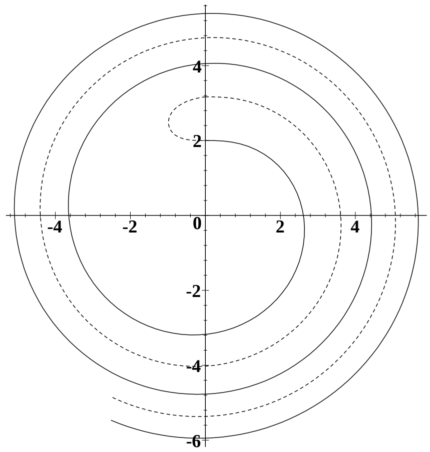


Figure 4. Geodesic through $(0, 0)$ with slope $(1, 1)$ and through $(0, 2)$ with slope $(1, 0)$ for $V = x\partial_y - y\partial_x$ ($t > 0$ solid, $t < 0$ dashed).

Killing vector field generating rotations r_φ by any angle φ around the origin, hence Proposition 2.1 implies that the curve $r_\varphi(\gamma)$ is again a geodesic if γ was one. Figure 4 shows the shape of two typical geodesics of ∇ for this vector field; the first goes through the origin, the second through the point $(0, 2)$. Observe that for the first geodesic, its extension in negative time coincides with its development in positive time rotated by π. The geodesic through $(0, 0)$ with zero slope resembles an archimedean spiral, but turns out not to be one.

Example 4.2. The vector field $V = y\partial_x$ ($f = y, g = 0$) is obtained for $p = y^2/2$. The geodesic equations read

$$\ddot{x} + y\,\dot{y}^2 \;=\; 0, \quad \ddot{y} - y\,\dot{x}\dot{y} \;=\; 0.$$

In this particular case, the second invariant of motion can be stated and discussed rather explicitly. For simplicity, set $E = 1$. Inserting $\dot{x} = \pm\sqrt{1 - \dot{y}^2}$ into the second geodesic equation, we obtain an ordinary differential equation for y alone,

$$\ddot{y} \;=\; \pm y\dot{y}\sqrt{1 - \dot{y}^2} \;\text{ for } \dot{x} \;=\; \pm\sqrt{1 - \dot{y}^2}.$$

Lemma 4.2. *The ordinary differential equation $\ddot{y} = \pm y\dot{y}\sqrt{1 - \dot{y}^2}$ has the invariant of motion $c := \pm\, y^2/2 - \arcsin \dot{y}$.* □

Formally, this allows us to fully integrate the differential equation,

$$t + \tilde{c} \;=\; \int_{y_0}^{y} \frac{dy}{\sin(\pm y^2/2 - c)}.$$

The integrand becomes singular whenever there exists a $k \in \mathbb{Z}$ such that $\pm\, y^2 = 2c + 2k\pi$. For a given y_0, two cases can be distinguished. Either c is such that y_0 is a zero of the denominator, then the integral has to be interpreted in such a way that t growths to $\pm\infty$ for this value $y = y_0$. Thus, y is the constant function y_0, and one checks that indeed, $(at + b, y_0)$ is a solution of the geodesic equations. Otherwise, a given y_0 lies in some interval between two singularities y_1, y_2 of the integrand, $y_0 \in (y_1, y_2)$; then, t will

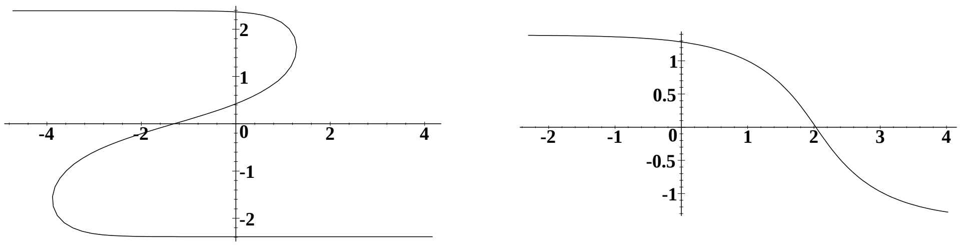


Figure 5. Geodesics through $(1,1)$ with slopes $(1,1)$ and $(-1,1/2)$ for the vector field $V = y\partial_x$.

become infinitely large as y approaches y_1 and y_2, hence meaning that $y(\pm\infty) \to y_1, y_2$. A generic geodesic lies thus in a horizontal strip, which eventually can degenerate to a line. Figure 5 illustrates two typical geodesics of this connection. The vector field V commutes with ∂_x, hence horizontal translates of geodesics are again geodesics by Lemma 2.1.

Remark 4.1. Hence, the Hopf-Rinow theorem does not necessarily hold for metric connections with vectorial torsion: for $V = y\partial_x$, two points lying in different "geodesic strips" cannot be joined by a geodesic arc.

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