REPORT ON THE BOOK
'CLIFFORD ALGEBRAS AND LIE THEORY'
BY E. MEINRENKEN

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If mathematics is a landscape whose regions are its different research areas, this book lives in the algebra region, at the main junction between the highways to harmonic analysis & quantization, representation theory, global analysis and spin geometry, and from where smaller roads heading to topology (more precisely, the neighbourhoods of homology and cohomology), complex geometry, supergeometry, and index theorems depart. Thus, any motivating path to this book has the drawback to start somewhere outside its core topic, and may therefore present a subjective perspective on the book. Such an interdisciplinary approach, moreover, may mislead the travelling mathematician to believe that the book is 'only' a collection of algebraic tools needed for accessing these other areas.

Such point of view is wrong. Meinrenken's research monography is a self-contained book about a new fascinating research topic in non-commutative algebra, that has a number of interesting connections to other cutting-edge subjects; this partially explains why this area is so rich and beautiful. The writing of such a text is a considerable challenge on its own. In the present case the author took the decision not to leave his area; rather, he chose to show the routes to other topics whenever they appeared, as marked by an impressive number of side-turns and bibliographic comments. Additionally, one of the work's strong points is that it contains abundant original material that appears for the first time in a textbook.

One good way to understand the key achievements and the current status of a certain branch of pure mathematics is to follow its historical development. Therefore, I shall approach the topic not by starting in some area and following the highway to 'Clifford algebras and Lie theory', but rather by looking back in time when these highways were still meandering farm tracks, when many roads hadn't been yet constructed and when areas that are nowadays well connected seemed distant and unrelated. In fact, this is presumably quite close to how the author (and his long-lasting collaborator Anton Alekseev) came to think about the questions that triggered the research work exposed in this book. Along the way I will comment on the methods and tools needed to advance in the theory and indicate where in the book they are treated.

In the 1960s Alexander Kirillov proposed the so-called 'orbit method' to describe unitary representations of Lie groups. This boils down to that fact that given a Lie group G with Lie algebra $\mathfrak{g}$, the orbits of the coadjoint action of G on the dual $\mathfrak{g}^*$ are symplectic manifolds for the Kostant-Souriau form, and fibre bundles over them carry unitary representations of G. While this worked neatly for nilpotent groups, the technical difficulties were tremendous in virtually any other case, which spawned a large amount of research activity by many mathematicians. In my opinion what is important is that Kirillov conjectured a 'universal character formula' for such representations, which roughly says the following: given a coadjoint orbit M with Liouville measure β, the character of the associated Hilbert space representation T should look like

$$\mathrm{tr} T(\exp X) \;=\; \int_M e^{iF(X)} J(X)^{-1/2}\, d\beta(F).$$

This is meant as an identity of distributions on a suitable neighbourhood of 0 inside $\mathfrak{g}$, X is an element of the Lie algebra $\mathfrak{g}$, hence indeed $\exp X \in G$; since F belongs in M, a subset of $\mathfrak{g}^*$, its

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action on X is well defined. The most interesting part is the occurence of the mysterious function $J(X)$,

$$J(X) \;=\; \det(j(\mathrm{ad}_X)), \quad j(z) = \frac{\sinh(z/2)}{z/2}.$$

This turns out to be a smooth map on all of $\mathfrak{g}$; recall that $\mathrm{ad}_X Y = [X, Y]$, so this is meant as a formal power series in the operators ad_X. This function and its many relatives have various manifestations in mathematics—for example, $j(z)$ is basically the generating function of the $\hat{A}$ genus, an integer number on spin manifolds that coincides with the index of the Dirac operator by the Atiyah–Singer index theorem. This already brings us close to Clifford algebras, even if the link is not transparent! Besides, mathematical experience tells us that 'good' character formulas can (and should) be understood as index formulas in disguise.

The orbit method, albeit more a philosophical principle than a rigourous method, inspired another famous result. For any Lie group G with Lie algebra $\mathfrak{g}$ let us consider the following two important algebras: the universal algebra $\mathcal{U}(\mathfrak{g})$ – that is, the quotient of the tensor algebra $T(\mathfrak{g})$ by the relations $X \otimes Y - Y \otimes X = [X, Y]$) – and the symmetric algebra $S(\mathfrak{g})$, i.e. the polynomial ring on a basis of $\mathfrak{g}$ (Chapter 5; observe that this chapter contains an interesting alternative proof of the Poincaré-Birkhoff-Witt Theorem due to Emanuela Petracci, [Pe03]). The inclusion of the symmetric algebra $S(\mathfrak{g})$ in $T(\mathfrak{g})$ followed by the quotient map gives an isomorphism $\mathrm{sym}: \; S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, which, alas, is not compatible with the product structure of the two algebras. In 1977 Michel Duflo proved that the composition $\mathrm{sym} \circ J^{1/2}: \; S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, if interpreted in the right way, restricts to an *algebra isomorphism* between the algebra of G-invariant polynomials $S(\mathfrak{g})^G$ and the center $\mathcal{U}(\mathfrak{g})^G$ of the universal enveloping algebra. This looks like a purely algebraic statement at first sight; however, $\mathcal{U}(\mathfrak{g})$ can be identified in a natural way with the left-invariant differential operators on G, and $S(\mathfrak{g})$ with the differential operators on $\mathfrak{g}$ with constant coefficients. Duflo's theorem, therefore, yields an algebra isomorphism between *bi-invariant* differential operators on G and *G-invariant*, constant-coefficient differential operators on $\mathfrak{g}$, and both algebras are commutative. A set of generators of $\mathcal{U}(\mathfrak{g})^G \cong S(\mathfrak{g})^G$ is typically called a set of *Casimir operators*. If $\mathfrak{g}$ is semisimple, the number of Casimir operators is equal to the rank of $\mathfrak{g}$, and Duflo's map coincides with the Harish-Chandra isomorphism. For example, if $G = \mathrm{SU}(2) \cong S^3$ is the 3-dimensional Lie group of unitary 2×2 matrices with determinant 1, there exists exactly one Casimir operator C; as an element of $S(\mathfrak{g})^G$, it is a quadratic polynomial, and in the usual basis X_1, X_2, X_3 of $\mathfrak{g} = \mathfrak{su}(2)$ (the trace-free skew-hermitian 2×2 matrices), we can write it as $C = X_1^2 + X_2^2 + X_3^2$; viewed as bi-invariant differential operator on $G \cong S^3$ it coincides with the Laplacian, which is known to be invariant under unitary transformations. As one learns in any standard lecture on quantum mechanics, finite-dimensional representations of $G = \mathrm{SU}(2)$ can be characterized by the eigenvalues of the Casimir operator. Duflo's theorem is a beautiful generalization of this collection of classical results. Unfortunately, his original proof is very technical and sophisticated. Again, this theorem was the starting point for a wealth of subsequent developments that point in many different mathematical directions (see for example [CR11] for additional reading).

The starting point of Meinrenken's book is the observation that the situation can be drastically simplified if the Lie algebra $\mathfrak{g}$ carries a non-degenerate symmetric bilinear form $B: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is invariant under $\mathrm{Ad}\, G$, i.e. satisfying

$$B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) \;=\; B(X, Y) \quad \forall X, Y \in \mathfrak{g},\ \forall g \in G.$$

Recall that if G is a matrix group, the adjoint action of G on $\mathfrak{g}$ is just conjugation, $\mathrm{Ad}_g X = gXg^{-1}$, i.e. we require the scalar product to be constant on conjugacy classes of matrices; in the previous example $G = \mathrm{SU}(2)$ the form $B(X, Y) = \mathrm{tr}(XY)$ does the job. Lie algebras of this type are called *quadratic*, and they include many interesting instances (for example, all semisimple ones). Given the scalar product B, many further tools and results become available. One can consider a third algebra (besides the universal and symmetric algebras), namely, the Clifford algebra (Chapter 2). This is the quotient of the tensor algebra of $\mathfrak{g}$ by the relations

$X \otimes Y - Y \otimes X - 2\,B(X,Y)$,

$$\mathrm{Cl}(\mathfrak{g}) \;:=\; T(\mathfrak{g})/\langle X \otimes Y - Y \otimes X - 2\,B(X,Y) \;:\; X, Y \in \mathfrak{g}\rangle.$$

Since the exterior algebra $\Lambda(\mathfrak{g})$ of $\mathfrak{g}$ can be embedded in $T(\mathfrak{g})$ as totally antisymmetric tensors, we have a vector space isomorphism

$$q: \quad \Lambda(\mathfrak{g}) \longrightarrow \mathrm{Cl}(\mathfrak{g}),$$

called the *quantization map*. Roughly speaking, the Duflo factor $J^{1/2}(X)$ measures the difference between two possible ways to embed the exponential of skew-symmetric matrices into $\mathrm{Cl}(\mathfrak{g})$ (Chapter 4). In fancy argot, it can be interpreted as a 'Berezin integral' and is deeply linked to the quantization map q. The exterior and the symmetric algebra can be combined into a differential algebra $W(\mathfrak{g})$, the *Weil algebra*, which the factorwise map $\mathrm{sym} \otimes q$ maps to the *quantum Weil algebra* $\mathcal{W}(\mathfrak{g})$,

$$\mathrm{sym} \otimes q: \quad W(\mathfrak{g}) := S(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}) \longrightarrow \mathcal{W}(\mathfrak{g}) := \mathcal{U}(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}).$$

The algebra $\mathcal{W}(\mathfrak{g})$ was defined and investigated in the article [AM00]. The classical Weil algebra $W(\mathfrak{g})$ was introduced by Henri Cartan (1950) as an algebraic model for the algebra of differential forms on the classifying space EG (for G compact). Both Weil algebras carry a lot of structure—graduations, differentials, Lie derivatives and contraction operators—all of which is described in great detail in the book using the languague of abstract enveloping algebras, Hopf algebras, super spaces, and $\mathfrak{g}$-differential spaces (Chapters 6 and 7). Not surprisingly, in the light of Duflo's result the map $\mathrm{sym} \otimes q$ is not compatible with this extra data. Alekseev and Meinrenken prove that there is a suitable generalization of $J^{1/2}$ (let us denote it by the same symbol for simplicity) such that $\mathcal{Q} := (\mathrm{sym} \otimes q) \circ J^{1/2} :\ W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$ *is* compatible with the additional structure of these algebras. Since Duflo's theorem can now be obtained by restriction to $S(\mathfrak{g}) \subset W(\mathfrak{g})$, this approach gives a much simpler proof of this classical result for quadratic Lie algebras (Section 7.3). This illustrates neatly the power of the methods exposed in this book.

There exists a natural element in the center of $\mathcal{W}(\mathfrak{g})$, which one should understand as the non-commutative relative of the quadratic Casimir operator. Observe that the scalar product B allows to define a closed, bi-invariant 3-form on G,

$$T(X,Y,Z) \;=\; -\frac{1}{6}\,B([X,Y],Z) \quad \forall\ X, Y, Z \in \mathfrak{g}.$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to B, viewed both as a subset of $\mathcal{U}(\mathfrak{g})$ and of $\mathrm{Cl}(\mathfrak{g})$. We can then define 'Kostant's cubic Dirac operator' ([KS87], [Ko99]; Section 7.2)

$$D \;:=\; \sum_{i=1}^{n} e_i \otimes e_i + 1 \otimes T \;\in\; \mathcal{W}(\mathfrak{g})\,,$$

which has the remarkable property that D^2 lies in the center of $\mathcal{W}(\mathfrak{g})$. The kernel of the twisted cubic Dirac operator is described in terms of representations and used to prove a character formula that generalizes the classical Weyl character formula (Chapter 8). In fact, the element D can be understood as the symbol of a true Dirac operator on G. Its analogue on homogeneous spaces G/K (its algebraic counterpart is called 'relative Dirac operator' in the book) turns out to be the usual Dirac operator, although not with respect to the Levi-Civita connection, but rather for a metric connection with skew torsion induced by the K-principal fibre bundle $G \to G/K$ (Chapter 9; see also [Ag06]). Again, one sees that the presence of a scalar product plays a crucial role.

Invariant elements in the different non-commutative algebras associated with a Lie algebra are one possible road map to this book. The last space of invariants discussed is $\Lambda(\mathfrak{g})^{\mathfrak{g}}$ (for $\mathfrak{g}$ reductive). The exterior algebra $\Lambda(\mathfrak{g})$ carries a differential map d that makes it a complex (the Chevally-Eilenberg complex), thus it induces a cohomology theory $H^*(\mathfrak{g})$. It is shown that $\Lambda(\mathfrak{g})^{\mathfrak{g}}$ and $H^*(\mathfrak{g})$ are isomorphic, and that they can then be identified with the exterior algebra of a certain subspace of $\Lambda(\mathfrak{g})$, the so-called space of *primitive elements* $P(\mathfrak{g})$ (Hopf-Koszul-Samelson

Theorem). Primitivity is a notion coming from the theory of Hopf algebras that I will not explain here; I'll just observe that we already encountered a primitive element, namely the 3-form T. Remarkably, $P(\mathfrak{g})$ has a natural interpretation in terms of Weil algebras (Chapter 10). Furthermore, there is a Clifford analogue of this Theorem that identifies $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ with the Clifford algebra $\mathrm{Cl}(P(\mathfrak{g}))$, another result of Kostant ([Ko97]; Chapter 11).

At this juncture our mathematical *tour d'horizon* comes to an end. Meinrenken's book is far from a standard textbook on a standard topic, meaning that the reader will find a profusion of material presented from many different viewpoints. Even if all topics are cultivated, and thrive, within the gates of the garden of algebra, some are likely to be new or at least unfamiliar. Although the book starts with a review of basic results (symmetric bilinear forms, Clifford algebras, Lie algebras...), it assumes a good knowledge from these areas, for the exposition is elegantly short, with few motivating examples. The title might be slightly misleading in the sense that the text is not meant to be (nor include) a complete treatise on Clifford algebras and spin representations. The reader should be comfortable with the methods of abstract algebra, lest he feel lost already in the preparations to the most interesting results. But then he is rewarded with a concise account of a modern topic in non-commutative algebra, a fresh view over invariants and a large supply of pointers that lead directly to current research.

References

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