3-Sasakian manifolds and intrinsic torsion

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1 Intrinsic torsion

Let $T := \mathbb{R}^n$ and $G \subset GL(n, \mathbb{R})$ be a subgroup. The following map is clearly G-invariant.

$$\begin{aligned} \delta : T^* \otimes \mathfrak{g} \hookrightarrow T^* \otimes \mathfrak{gl}(n, \mathbb{R}) \\ = T^* \otimes T^* \otimes T \;\rightarrow\; \Lambda^2 T^* \otimes T \\ \beta \otimes \gamma \otimes x \;\mapsto\; (\beta \wedge \gamma) \otimes x \end{aligned}$$

Therefore $\operatorname{Ker}\delta$, $\operatorname{Im}\delta$ and $\Lambda^2 T^* \otimes T/\operatorname{Im}\delta$ are also representations of G and the projection $\pi : \Lambda^2 T^* \otimes T \rightarrow \Lambda^2 T^* \otimes T/\operatorname{Im}\delta$ is G-invariant.

So, if $P_G M \subset P_{GL(n,\mathbb{R})} M$ is a G-structure on a manifold M, then all the above spaces define corresponding associated with $P_G M$ bundles TM, T^*M, $\mathfrak{g}(M)$ ..., and δ and π induce correctly defined bundle maps.

Let ∇ and ∇' be connections in P_G. Then

$$\nabla' - \nabla \in \Gamma(T^*M \otimes \mathfrak{g}(M)),$$
$$\delta(\nabla' - \nabla) = T^{\nabla'} - T^{\nabla}$$
$$\Rightarrow T^{\nabla'} - T^{\nabla} \in \Gamma(\mathrm{Im}\,\delta)$$
$$\Rightarrow \pi\left(T^{\nabla'} - T^{\nabla}\right) = 0.$$

This shows that the following definition is independent of the choice of ∇.

Def. $T_{P_GM} := \pi\left(T^{\nabla}\right) \in \Gamma(\Lambda^2 T^* \otimes T/\mathrm{Im}\,\delta)$ is the *intrinsic torsion* of the G-structure P_GM.

We have furthermore

$T^{\nabla'} = T^{\nabla} \Leftrightarrow \delta(\nabla' - \nabla) = 0 \Leftrightarrow \nabla' - \nabla \in \Gamma(\mathrm{Ker}\,\delta)$

$\Leftrightarrow \nabla' = \nabla + A$ for some $A \in \Gamma(\mathrm{Ker}\,\delta)$.

Thus, given ∇, the connections ∇' satisfying $T^{\nabla'} = T^{\nabla}$ are parametrized by $\Gamma(\mathrm{Ker}\,\delta)$.

Def. Let $W \subset \Lambda^2 T^* \otimes T/\mathrm{Im}\,\delta$ be a G-invariant subspace. P_GM is said to be of (Gray-Hervella) type W if $T_{P_GM} \in \Gamma(W(M))$.

E.g., $T_{P_GM} = 0$ iff there exists a connection ∇ in P_GM with $T^\nabla = 0$ (1-*integrable* G-*structure*).

Suppose now that N is a G-invariant complement of $\operatorname{Im}\delta$ in $\Lambda^2T^* \otimes T$. Then there exists a connection with Torsion in $\Gamma(N(M))$. If furthermore δ is injective, then this connection $\nabla^{0,N}$ is unique and is called *the canonical connection* of P_GM with respect to N.

Examples:

1. $G = SO(n)$ or $O(n)$.

$$\delta_{\mathfrak{so}(n)} : T^* \otimes \underbrace{\mathfrak{so}(n)}_{\cong \Lambda^2 T^*} \to \Lambda^2 T^* \otimes \underbrace{T}_{\cong T^*}$$

is an isomorphism. Therefore

- $\operatorname{Im}\delta_{\mathfrak{so}(n)} = \Lambda^2 T^* \otimes T$
$$\Rightarrow \Lambda^2 T^* \otimes T / \operatorname{Im}\delta_{\mathfrak{so}(n)} = 0$$
$\Rightarrow T_{P_{SO(n)}M} = 0$ and thus there exists a connection ∇ in $P_{SO(n)}M$ with $T^\nabla = 0$.
- $\operatorname{Ker}\delta_{\mathfrak{so}(n)} = 0$. Therefore ∇ is unique.

∇ is the Levi-Civita connection.

2. $G \subset SO(n)$ or $O(n)$.

Then $\mathfrak{g} \subset \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ Thus

$$\delta_{\mathfrak{g}} : T^* \otimes \mathfrak{g} \overset{i}{\hookrightarrow} T^* \otimes (\mathfrak{g} \oplus \mathfrak{g}^\perp) = T^* \otimes \mathfrak{so}(n) \overset{\delta_{\mathfrak{so}(n)}}{\longrightarrow}$$

$\Lambda^2 T^* \otimes T = \delta_{\mathfrak{so}(n)}(T^* \otimes \mathfrak{g}) \oplus \delta_{\mathfrak{so}(n)}(T^* \otimes \mathfrak{g}^\perp)$

So $\delta_{\mathfrak{g}}$ is injective, $\operatorname{Im} \delta_g = \delta_{\mathfrak{so}(n)}(T^* \otimes \mathfrak{g})$ and $\delta_{\mathfrak{so}(n)}(T^* \otimes \mathfrak{g}^\perp)$ is a G-invariant complement of $\operatorname{Im} \delta_g$ in $\Lambda^2 T^* \otimes T$. Thus there exists a unique connection ∇^0 with Torsion $T^{\nabla^0} \in \Gamma(\delta_{\mathfrak{so}(n)}(T^*M \otimes \mathfrak{g}^\perp(M)))$. Equivalently, ∇^0 is characterized by $\nabla^0 = \nabla + A_0$ with $A_0 \in \Gamma(T^*M \otimes \mathfrak{g}^\perp(M))$.

Because of the isomorphisms

$$\Lambda^2 T^* \otimes T / \operatorname{Im} \delta_{\mathfrak{g}} \cong \delta_{\mathfrak{so}(n)}(T^* \otimes \mathfrak{g}^\perp) \cong T^* \otimes \mathfrak{g}^\perp$$

we have $\quad T_{P_G M} \leftrightarrow T^{\nabla^0} \leftrightarrow A_0$

and the Gray-Hervella-type classification is usually done in terms of a decomposition $T^* \otimes \mathfrak{g}^\perp = W_1 \oplus \cdots \oplus W_k$ of $T^* \otimes \mathfrak{g}^\perp$ into irreducible G-invariant summands.

2 3-Sasakian manifolds

Def. (M,g) is a 3-*Sasakian manifold* if the cone $(\widehat{M} = M \times \mathbb{R}_+, \widehat{g} = r^2 g + \mathrm{d}r^2)$ is hyper-Kähler.

In this case there exist orthogonal

$$\widehat{I}, \widehat{J}, \widehat{K} \in \Gamma(End(T\widehat{M}))$$

which satisfy the quaternionic identities. Let

$$\xi_I = -\widehat{I}\partial_r|_{r=1}, \xi_J = -\widehat{J}\partial_r|_{r=1}, \xi_K = -\widehat{K}\partial_r|_{r=1},$$

$$V = \mathrm{span}\{\xi_I, \xi_J, \xi_K\},$$

$$I = \widehat{I}|_{V^\perp}, \quad J = \widehat{J}|_{V^\perp}, \quad K = \widehat{K}|_{V^\perp},$$

$$I|_V = 0, \quad J|_V = 0, \quad K|_V = 0.$$

Then $TM = V^\perp \oplus V$, V is trivialised by the orthonormal frame ξ_I, ξ_J, ξ_K, and I, J, K satisfy the quaternionic identities and are orthogonal on $V^\perp$. Thus we obtain an $Sp(n)$-structure on M, where the action of $Sp(n) \subset SO(4n+3)$ on $\mathbb{R}^{4n+3} = \mathbb{R}^{4n} \oplus \mathbb{R}^3$ is given by the standard representation on $\mathbb{R}^{4n}$ and the trivial one on $\mathbb{R}^3$.

3 $Sp(n)Sp(1)$-structures on $(4n+3)$-dimensional manifolds

If we consider an $Sp(n)$-structure as above, we have

$$T^* \otimes \mathfrak{sp}(n)^\perp = \underbrace{15\mathbb{R} \oplus \text{other summands}}_{\substack{57 \\ n\geq 3} \text{ or } \substack{54 \\ n=2} \text{ or } \substack{33 \\ n=1}}.$$

Since the dimension of the trivial representation is too big, we shall consider a more general G-structure.

Let $G := Sp(n)Sp(1) \subset SO(4n+3)$ acting on $\mathbb{R}^{4n+3} = \mathbb{R}^{4n} \oplus \mathbb{R}^3$ by the standard representation of $Sp(n)Sp(1)$ on $\mathbb{R}^{4n} \cong \mathbb{H}^n$ and through the projection $Sp(n)Sp(1) \to SO(3)$ on $\mathbb{R}^3$. (Then $Sp(n) \subset G$ acts on $\mathbb{R}^{4n+3}$ as above.)

We have

$$T^* \otimes \mathfrak{g}^\perp = \underbrace{2\mathbb{R} \oplus \text{other summands}}_{31 \text{ for } n>1 \text{ or } 18 \text{ for } n=1},$$

$$T^* \otimes \mathfrak{g} = \underbrace{\mathbb{R} \oplus \text{other summands}}_{9 \text{ for } n>1 \text{ or } 8 \text{ for } n=1}.$$

One basis of $2\mathbb{R} \subset T^* \otimes \mathfrak{g}^\perp$ is given by

$$A(P,Q) = \mathfrak{S}(g(IP,Q)\xi_I - \eta_I(Q)IP),$$
$$B(P,Q) = \mathfrak{S}(\eta_I(P)IQ - n\eta_J \wedge \eta_K(P,Q)\xi_I)$$

and $\mathbb{R} \subset T^* \otimes \mathfrak{g}$ is spanned by

$$C(P,Q) = \mathfrak{S}\eta_I(P)(IQ+2\eta_J(Q)\xi_K-2\eta_K(Q)\xi_J).$$

Here η_I, η_J, η_K are dual to ξ_I, ξ_J, ξ_K and $\mathfrak{S}$ denotes the cyclic sum with respect to I, J, K.

Let T_A, T_B, T_C be the corresponding torsions. Then all invariant complements of

$$\mathbb{R} = \mathrm{span}\{T_C\} = \mathrm{Im}\,\delta \cap \underbrace{\mathrm{span}\{T_A, T_B, T_C\}}_{3\mathbb{R}\subset\Lambda^2T^*\otimes T}$$

are of the form

$$N_{x,y} = \mathrm{span}\{T_A + xT_C, T_B + yT_C\}, x, y \in \mathbb{R}.$$

For the canonical connections $\nabla^{0,N_{x,y}}$ we have

$$T^{\nabla^{0,N_{x,y}}} = \lambda(T_A + xT_C) + \mu(T_B + yT_C),$$
$$\nabla^{0,N_{x,y}} = \nabla + \lambda(A + xC) + \mu(B + yC),$$

where in the first instance λ and μ are functions. Notice that they are the same for all x, y.

Thm 1 *If the the torsion of* $\nabla^0 = \nabla^{0,N_{0,0}}$ *is* $T^0 = \lambda T_A + \mu T_B$*, then* λ *and* μ *are constants and the curvature tensors of* ∇^0 *and* ∇ *satisfy*

$$R^0 = \widetilde{R}^0 + R_{\text{hyper}}, \quad R = \widetilde{R} + R_{\text{hyper}},$$

where $\widetilde{R}^0$ *and* $\widetilde{R}$ *are explicit* G*-invariant tensors (which depend on* λ, μ*) and* R_{hyper} *is a hyper-Kähler curvature tensor on* $V^\perp$*. In particular, Ric has two eigenvalues:*

$$Ric|_V = 2(n+2)(2\lambda^2 + 4\lambda\mu + (n+2)\mu^2),$$
$$Ric|_{V^\perp} = 2\lambda((4n+5)\lambda + 2(n+2)^2\mu).$$

Proof: $R^0 \in \Lambda^2 \otimes \mathfrak{g}$, $\nabla^0 T^0 \in 2T^* \otimes \mathbb{R}$ and $T^0(T^0(\cdot,\cdot),\cdot)$ is an explicit G-invariant tensor. Then decompose the spaces $\Lambda^2 \otimes \mathfrak{g}$ and $2T^*$ into G-irreducible components and use the Bianchi identity

$$b(R^0 - \nabla^0 T^0 - T^0(T^0(\cdot,\cdot),\cdot)) = 0$$

and Schur's lemma. □

General constructions:

Let (M, g) have a G-structure, so that the potential of ∇^0 is $\lambda A + \mu B$.

1. Then for $g_{c,d} = d^2(g|_{V^\perp} + c^2 g|_V)$ we obtain a G-structure, where the potential of $\nabla^{0,g_{c,d}}$ is $\lambda_{c,d} A^{g_{c,d}} + \mu_{c,d} B^{g_{c,d}}$ with

$$\lambda_{c,d} = \frac{c}{d}\lambda, \quad \mu_{c,d} = \frac{1}{cd}\left(\mu - \frac{2(c^2-1)}{n+2}\lambda\right).$$

2. If we change the sign of ξ_I, ξ_J, ξ_K, then we obtain a G-structure, where the signs of λ and μ are also changed.

Examples:

1. Let (M, g) be 3-Sasakian. Then
$$\lambda = -1, \quad \mu = \frac{1}{n+2} \quad \text{for } g,$$
$$\lambda = -\frac{c}{d}, \quad \mu = \frac{2-c^2}{(n+2)cd} \quad \text{for } g_{c,d}.$$
In all cases $\quad \lambda < 0, \quad 2\lambda + (n+2)\mu < 0.$

2. Let (M, g) be 3-Sasakian with signature $(3, 4n)$. Then for the metric $d^2(-g|_{V^\perp} + c^2 g|_V)$
$$\lambda = \frac{c}{d}, \quad \mu = -\frac{1+2c^2}{(n+2)cd}.$$
In all cases $\quad \lambda > 0, \quad 2\lambda + (n+2)\mu < 0.$

3. Let M' be hyper-Kähler, $M = M' \times SO(3)$ with the product metric. On M we have a G-structure with $\quad \lambda = 0, \quad \mu < 0$ (depending on the scaling of the metric on $SO(3)$) and we have $2\lambda + (n+2)\mu < 0$.

4. Let (M', g', I', J', K') be hyper-Kähler. Then
$$\mathrm{d}\Omega_{I'} = 0, \quad \mathrm{d}\Omega_{J'} = 0, \quad \mathrm{d}\Omega_{K'} = 0.$$
Hence locally there exist $\alpha_{I'}$, $\alpha_{J'}$, $\alpha_{K'}$ such that
$$\Omega_{I'} = \mathrm{d}\alpha_{I'}, \quad \Omega_{J'} = \mathrm{d}\alpha_{J'}, \quad \Omega_{K'} = \mathrm{d}\alpha_{K'}.$$
Let $M = M' \times \mathbb{R}^3$ and u, v, w be the coordinates on $\mathbb{R}^3$. Fix $\nu < 0$ and define
$$\xi_I = \partial_u, \quad \eta_I = \mathrm{d}u - \nu\alpha_{I'},$$
$$\xi_J = \partial_v, \quad \eta_J = \mathrm{d}v - \nu\alpha_{J'},$$
$$\xi_K = \partial_w, \quad \eta_K = \mathrm{d}w - \nu\alpha_{K'},$$
$$V = \operatorname{span}\{\xi_I, \xi_J, \xi_K\} = T\mathbb{R}^3,$$
$$V^\perp = \{X : \eta_I(X) = \eta_J(X) = \eta_K(X) = 0\}$$
(notice that $V^\perp \neq TM'$),
$$g = g' + \eta_I^2 + \eta_J^2 + \eta_K^2,$$
$$I|_V = 0, \quad J|_V = 0, \quad K|_V = 0,$$
$$IX' = hI'X', X' = hJ'X', KX' = hK'X'$$
for $X' \in TM'$. Thus we obtain a G-structure on M with $\quad \lambda = -\frac{\nu}{2} > 0, \quad \mu = \frac{\nu}{n+2} < 0$ and $2\lambda + (n+2)\mu = 0$.

5. We obtain further examples if we apply the second "general construction" on the above ones.
6. Let M' be hyper-Kähler. Then $M = M' \times \mathbb{R}^3$ with the product metric has a G-structure with $\quad \lambda = 0, \quad \mu = 0.$

Thm 2 *1. Every pair (λ, μ) appears exactly once in the above list of examples.*

2. A manifold with a G-structure of the considered type with torsion $T^{\nabla^0} = \lambda T_A + \mu T_B$ is locally equivalent to the corresponding example.

Rem. For each (λ, μ) there exists a unique connection with totally skew-symmetric torsion:

$$\nabla^a = \nabla + \lambda A + \mu B + (\lambda - \mu) C.$$

Consider a 3-Sasakian Wolf space, written in the form $H \cdot Sp(1)/L \cdot Sp(1)$. Then the second Einstein metric is one of the normal metrics and ∇^a is the corresponding canonical connection.