KÄHLER STRUCTURES ON HERMITIAN SURFACES

SIMON G. CHIOSSI
Politecnico di Torino

http://calvino.polito.it/~chiossi

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CONTENTS

SCENARIO

4-dimensional manifolds with symmetry,
be this Einstein, selfdual or Kähler geometry

Let (M^4, g) be a smooth Riemannian four-manifold

- If $\omega_1, \ldots, \omega_6$ are ON symplectic forms $\Longrightarrow$ g is flat

  "the convergence of six figures in a flat space has a comforting geometry" **McEwan (1998)**

- Pairs: **Salamon (1991)**, **Bande–Kotschick (2006)**

- Triples: **Geiges–Gonzalo Pérez (1995), (2009)**

Might as well consider **five ON symplectic forms**

$$\mathit{span}\{\omega_1, \omega_2, \omega_3\} = \Lambda^2_- M^4, \quad \{\omega_4, \omega_5\} \subset \Lambda^2_+ M^4$$

Consequence: M is hyperKähler.
Expected: constraints on curvature.

I want to study

Hermitian surfaces with 1-dimensional Chern holonomy.

Core results on (M^4, g, I) Hermitian:

Einstein $\Longrightarrow W^+$ degenerate (ie eigenvalues not distinct)

(aka *'Riemannian Goldberg-Sachs'* **Nurowski, 1993**)

lcK $\Longrightarrow W^+$ degenerate ($\Longleftrightarrow$ when compact)

Ricci I-invariant $\Longrightarrow W^+$ degenerate

Apostolov–Gauduchon (1997)

This presentation:

$*$ despite Ricci is I-invariant, M may still not be lcK.

$**$ But if this happens: **complete local classification**, in most cases deformations of Kähler surfaces of Calabi-type.

$* * *$ NEW Hermitian, non-lcK examples whose Ricci admits exactly one constant eigenvalue equal 0.

Plus, there is a positive complex structure, unlike examples of **Apostolov–Armstrong–Drăghici (2002)**

TWO-FORMS

Let M^4 be a real, oriented, smooth 4-manifold with an almost Hermitian structure (g, I)

$$I^2 = -Id_{TM}, \qquad g(I\cdot, I\cdot) = g(\cdot,\cdot) > 0, \qquad \omega_I = g(I\cdot, \cdot).$$

- The bundle of real 2-forms decomposes as

$$\Lambda^2 = \overbrace{\mathbb{R}\omega_I \oplus \lambda_0^{1,1}}^{\lambda^{1,1}} \oplus \overbrace{\lambda^{2,0} \oplus \lambda^{0,2}}^{\lambda^2}$$

where $\lambda^{1,1}$ are the I-invariant two-forms

$\lambda_0^{1,1} = \mathrm{Ker}(\omega_I \wedge \cdot)$ are the primitive $(1,1)$-forms

$\lambda^2 \cong K_{(M,I)}$ is the canonical line bundle

- Paramount feature of dim 4 is *'self-duality'*

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

originating from $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, so

$$\boxed{\boldsymbol{\Lambda^+ = \mathbb{R}\,\omega_I \oplus \lambda^2}, \qquad \boldsymbol{\Lambda^- = \lambda_0^{1,1}}}$$

SOME DIFFERENTIAL AND ALGEBRAIC CONSEQUENCES OF THE EINSTEIN FIELD EQUATIONS*

BY

K. W. LAMSON

Introduction. If a set of four directions in a Riemannian four-dimensional space, V_4, is orthogonal, then the ds^2 can be expressed in terms of their sixteen parameters, $h_k{}^\alpha(x_0, x_1, x_2, x_3)$, as in Einstein's recent papers.

The first purpose of this paper is to set up sixteen invariant linear first-order partial differential equations in these parameters (§2). The solutions of these equations include all solutions for empty space of the Einstein field equations of 1917. There is a restriction which excludes some special cases. In addition to the $h_k{}^\alpha$, these sixteen equations contain four linear combinations of the components of the curvature tensor. These four combinations are to be taken as independent variables, x^k.

Since only alternating tensors appear it is convenient to use Cartan's notation† for symbolic differential forms and for their derivatives and products. Covariant differentiation in the sense of the absolute differential calculus is not used, except in §4.

The components of the curvature tensor may be taken as coefficients in the equation of a quadratic line complex in a three-dimensional projective space, P_3‡§. The directions $h_k{}^\alpha$ correspond to the vertices of a tetrahedron in P_3. Bivector and simple bivector correspond to linear complex and special complex. Where there is no danger of misunderstanding, the language of V_4 will be used interchangeably with that of P_3. The second purpose of this paper is the application of some of the theory of quadratic complexes to the study of the curvatures in V_4.¶

The lines which lie in a plane in P_3 and which belong to the quadratic complex are tangent to a conic. The envelope of planes for which this conic

* Presented to the Society, December 27, 1923, and September 12, 1930; received by the editors February 20, 1930.

† Goursat, *Leçons sur le Problème de Pfaff*, chapters I and III.

‡ D. J. Struik, *On sets of principal directions in a Riemannian manifold in four dimensions*, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 7 (1928), p. 193.

§ E. Kretschmann, Annalen der Physik, vol. 53 (1917), p. 592.

¶ Jessop, *Treatise on the Line Complex*. See also Hudson, *Kummer's Quartic Surface*, and Zindler's article in the Encyclopedia, IIIC8.

709

K.W.Lamson, TAMS (1930):
contains the SD equations !

CANONICAL CONNECTION

Via the Levi-Civita connection ∇ defined by g build the **intrinsic torsion** of (g, I)

$$\eta = \frac{1}{2}(\nabla I)I \in \Lambda^1 \otimes \lambda^2.$$

The components of η determine the type and features of the almost Hermitian structure, like

$$\begin{aligned}(M^4, g, I) \text{ Hermitian } &\iff (M^4, I) \text{ complex surface}\\ &\iff \eta \in \lambda^{1,1} \otimes \Lambda^1\end{aligned}$$

Lee form: unique $\theta \in \Lambda^1$ such that $d\omega_I = \theta \wedge \omega_I$

The second **canonical** Hermitian **connection** of (g, I)

Gauduchon (1984)

$$\boxed{\nabla^C = \nabla + \eta}$$

is metric, Hermitian, with torsion

$$T(X, Y) = \eta_X Y - \eta_Y X$$

Notation: $\eta_X = \frac{1}{2}(\nabla_X I)I \in \lambda^2, \qquad \nabla^C_X Y = \nabla_X Y + (\eta_X Y)^\sharp$

CURVATURE

The Riemannian curvature decomposes

$$R = \left(\begin{array}{c|c} W^+ + \frac{1}{12}s & \frac{1}{2}\,Ric_0 \\ \hline \frac{1}{2}\,Ric_0^{\mathrm{T}} & W^- + \frac{1}{12}s \end{array}\right),$$

$$W^\pm = \tfrac{1}{2}(W \pm \star W), \qquad Ric = tr\,R,$$
$$\kappa = 3\langle W^+\omega_I, \omega_I\rangle \qquad s = tr\,Ric\ .$$

I will call **Chern curvature**

$$R^C(X,Y) = \nabla^C_{[X,Y]} - [\nabla^C_X, \nabla^C_Y] \in \lambda^{1,1}.$$

Comparison formula

$$R^C(X,Y) = R(X,Y) - d^C\eta(X,Y) + [\eta_X, \eta_Y] - \eta_{T(X,Y)}$$

Lemma $R^C = W^- + \frac{s}{12}Id_{\Lambda^-} + \frac{1}{2}\,Ric_0^- + \frac{1}{2}\gamma_1^C \otimes \omega_I$

where $\gamma_1^C(X,Y) \propto \langle R^C(X,Y), \omega_I\rangle$ is the first Chern form.

HOLONOMY

Consider when $\mathfrak{hol}^C \subseteq \mathfrak{u}(2)$ is **one-dimensional**, let F be a generator

$$\boxed{R^C = \gamma \otimes F}$$

As ∇^C is Hermitian

$$F \;=\; F_0 \;+\; \alpha\,\omega_I \;\in\; \mathfrak{su}(2) \;\oplus\; \mathbb{R},$$

where $\alpha \neq 0$ is constant, and F_0 is either identically, or never, zero.

If $F_0 \neq 0$, can parametrise $F_0 = g(J\cdot,\cdot) = \omega_J$ by means of an OCS J with orientation **opposite** to that of I.

Proposition. The following statements are equivalent:

- **i)** $\mathfrak{hol}^C$ is 1-dimensional, generated by F in $\lambda^{1,1}$ with $F_0 \neq 0$;
- **ii)** ∇^C is not flat and there is a '*negative*' Kähler J such that $\gamma_1^C = \alpha\rho_J$, where ρ_J is the Ricci form.

Either assumption implies $\quad R^C = \dfrac{\rho_J}{2} \otimes (\omega_J + \alpha\omega_I).$

Proposition. When $F_0 = 0$

$$R^C = \tfrac{1}{2}\gamma_1^C \otimes \omega_I \iff Ric = 0 \text{ and } W^- = 0.$$

In particular: g flat $\implies \dim \mathfrak{hol}^C \leqslant 1$.

Corollary. Let (M^4, g, I) be compact Hermitian, with $R^C = \frac{1}{2}\gamma_1^C \otimes \omega_I$. Then (g, I) is a flat Kähler structure (and these are classified).

cf. **Apostolov–Gauduchon (2002)** classifies SD Einstein-Hermitian surfaces

CHERN-FLAT SURFACES:

Proposition. Let (M^4, g, I) be almost Hermitian with $R^C = 0$. Then g is flat.

Compare to

M^{n} Hermitian with holomorphic torsion and constant holo sectional curvature $\implies$ either Kähler or flat.

Balas (1985)

M^{n} *compact* almost Kähler, Chern-flat $\implies$ flat Kähler.

Di Scala–Vezzoni (arXiv 2008)

5-FRAMES

A **closed 5-frame** of two-forms on M^4 (just smooth!) is a system $\omega_1, \ldots, \omega_5$ of symplectic forms satisfying

$$\omega_i \wedge \omega_j = \pm\delta_{ij}\, \omega_1 \wedge \omega_1$$

at each point. This induces an orientation on M.

As

$$\frac{GL(4,\mathbb{R})}{CO(4)} \cong \frac{SL(4,\mathbb{C})}{SO(4)} \cong \frac{SO_0(3,3)}{SO(3) \times SO(3)},$$

Atiyah–Hitchin–Singer (1978), **Salamon (1982)**

in every conformal class there is a unique $g > 0$ such that

$$\mathit{span}\{\omega_1, \omega_2, \omega_3\} = \Lambda^-, \qquad \{\omega_4, \omega_5\} \subset \Lambda^+.$$

Complete to a basis with $\omega_I = g(I\cdot, \cdot) \in \Lambda^+$ (non-closed)

Proposition.
Let M^4 possess a closed 5-frame. Then (g, I) is Hermitian and $R^C = \frac{1}{2}\gamma_1^C \otimes \omega_I$.

(M^4, g, I) Hermitian: pointwise there is a closed 5-frame if and only if $R^C = -\frac{1}{4}d(I\theta) \otimes \omega_I$.

EXAMPLE: 'Gibbons-Hawking Ansatz'

Let (M^4, g, I, J, K) be hyperKähler with tri-holomorphic Killing field X.

There are local coordinates (x, y, z, u) such that $X = \frac{\partial}{\partial u}$, $X \lrcorner\, \omega_I = dx,\ X \lrcorner\, \omega_I = dy,\ X \lrcorner\, \omega_K = dz.$

$\|X\|^{-2} = U(x, y, z)$ is harmonic for the flat metric on $\mathbb{R}^3$.

Let Θ satisfy the monopole equation $d\Theta = \star_{\mathbb{R}^3} dU$.

Then the metric reads

$$g = U(dx^2 + dy^2 + dz^2) + U^{-1}(du + \Theta)^2$$

and Λ^- is trivialised.

Vice versa, $U(x, y, z) > 0$ harmonic and $d\Theta = \star dU \Longrightarrow$ g hyperKähler, locally.

Gibbons-Hawking (1978), see also **Gauduchon-Tod (1998)**

Importance:

- Key method yielding (strictly) almost Kähler, Einstein metrics in dimension 4.
- First example of this kin by **Nurowski-Przanowski (1999)**

• No coincidence this construction admits a closed 5-frame, for

Theorem.

1. An M^4 with a closed 5-frame is determined via the Gibbons-Hawking Ansatz by choosing $U(y) = ay + b$.

2. (M, g) is isometric to a quotient of a diagonal Bianchi metric of class II on $S^1 \times \mathcal{H}^3$.

Proof relies on:

Let (M^4, g, I) be Hermitian, with $Ric = 0 = W^-$. Then

i) $W^+\omega_I = \frac{\kappa}{6}\omega_I$

ii) $(\kappa^{\frac{2}{3}}g, I)$ is Kähler

iii) $X = I grad\,(\kappa^{-\frac{1}{3}})$ is holomorphic Killing

iv) $d^+X^\flat = -\frac{1}{12}\kappa^{\frac{2}{3}}\omega_I$.

i) $\Longleftrightarrow$ ii) on compact, not necessarily Einstein, Hermitian surfaces

Apostolov–Gauduchon (1997), **Boyer (1986)**, **Derdziński (1983)**, **Nurowski, 1993**

The metric cannot be complete, otherwise X tri-holomorphic would force flatness **Bielawski (1999)**

NB: $\mathcal{H}^3$ is the real three-dimensional Heisenberg group

KÄHLER-HERMITIAN SURFACES

A Hermitian surface (M^4, g, I) admitting a Kähler structure (g, J) with the <u>*opposite*</u> orientation will be referred to as a **Kähler-Hermitian surface (M, g, J, I)**.

(lest we forget, from Proposition on p. 9)

The endomorphisms I and J commute and $J \neq \pm I$.

The rank-two orthogonal distributions

$$\boxed{\mathscr{D}^{\pm} = \mathrm{Ker}(IJ \mp Id), \qquad TM = \mathscr{D}^+ \oplus \mathscr{D}^-}$$

are invariant under J and I, and preserved by ∇^C.

Proposition. The distribution $\mathscr{D}^+$ is totally geodesic, and holomorphic for both I and J.

> *A distribution $\mathcal{V}$ on (M^4, I) complex is said* holomorphic *if I preserves it ($I\mathcal{V} \subseteq \mathcal{V}$), and it is locally spanned by holomorphic vectors (ie $(L_{\mathcal{V}} I)X \in \mathcal{V}$ for any X, by Frobenius)*

Conversely,

Proposition. Let (M, g, J) be a Kähler surface equipped with a holomorphic distribution E, and call $F = E^{\perp}$. Define the OCS

$$I_{|E} = -J, \quad I_{|F} = J.$$

Then **(i)** E is I-holomorphic

(ii) $(IJ)\theta = \theta$

(iii) I integrable $\iff$ E totally geodesic.

The notion of holomorphic distribution on a Kähler surface has yet another geometrical interpretation

A foliation with leaf-tangent-distribution $\mathcal{V}$ *is* conformal *if*

$$(L_{\mathcal{V}} g)(X, Y) = k(\mathcal{V}) g(X, Y) \qquad X, Y \in \mathcal{V}^{\perp}$$

and homothetic *if* $dk = 0$.

On M^4 *Kähler: complex conformal* $\iff$ *holomorphic*

Important in theory of harmonic morphisms

Although few examples are known on Kähler surfaces,

Theorem. Let (M, g, J) be a Kähler surface equipped with a complex homothetic foliation $\mathscr{F}$.

If $\mathscr{F}$ is totally geodesic, M arises as a holomorphic line bundle over a Riemann surface (*à la* Calabi)

$$\mathbb{C} \hookrightarrow M \to \Sigma.$$

REVIEW OF CALABI'S CONSTRUCTION

$(\Sigma, \omega_\Sigma, I_\Sigma)$ Riemann surface

$\mathscr{L}$ complex line bundle with $c_1(\mathscr{L}) = -[\omega_\Sigma]$

Pick a Hermitian metric h on the fibres and a Hermitian connection with curvature $-\omega_\Sigma$

The distribution $\mathcal{V}$ tangent to the fibres has a complex structure $I_\mathcal{V}$ induced by h, so

$$T\mathscr{L} \cong T\Sigma \oplus \mathcal{V}$$

(pullbacks omitted) secures an integrable complex structure

$$I = I_\Sigma + I_\mathcal{V}.$$

If f is a function of the fibres' norm,

$$\omega = \omega_\Sigma + dIdf$$

defines an I-compatible Kähler metric on $\mathscr{L}\backslash\{0\}$.

Calabi, 1982

Lemma. When $(JI)\theta = \theta$

$$W^+ \text{ is degenerate} \iff Ric \text{ is } I\text{-invariant}$$

in contrast to Hermitian surfaces (no J), where only $\Longleftarrow$ holds.

Hence I am interested in

$$\boxed{(JI)\theta = \theta}$$

and call (M, g, J, I) a **Kähler surface of Calabi-type with respect to *I*** if

$$(JI)\theta = \theta \quad \text{and} \quad d\theta = 0.$$

Examples

1. Calabi's Kähler metrics on $\mathbb{C} \hookrightarrow \mathscr{L} \to \Sigma$
2. some weakly SD Kähler surfaces (W^- harmonic)

   Apostolov–Calderbank–Gauduchon (2003)

where 'Calabi-type' means there's a Hamiltonian Killing X with (g, I) conformally Kähler, and $I = J$ on $span\{X, JX\}$, $I = -J$ on $span\{X, JX\}^\perp$

CLASSIFICATION I (generic case)

Theorem.
(M^4, g, J, I) is connected Kähler-Hermitian, with $\mathfrak{hol}^C$ spanned by $\alpha\omega_I + \omega_J$, $\alpha \neq \pm 1$, if and only if

- $M = \overline{M_+ \cup M_-}$, for some $M_\pm$ open, disjoint subsets
- $(IJ)\theta = \pm\theta$ on $M_\pm$,
- constraints (on α, θ, s, ρ)
- W^+ degenerate everywhere on M

NB: The 'minus' corresponds to flipping the sign of I.

Corollary.
A compact Kähler-Hermitian surface with $\mathfrak{hol}^C = \langle \alpha\omega_I + \omega_J \rangle$ is of Calabi-type for I.

Theorem.
(M^4, g, J, I) Kähler-Hermitian, with $(IJ)\theta = \theta$. On any connected component of $\{d\theta \wedge \theta \neq 0\}$

- the scalar curvature s^+ of the leaves is constant (if not 0, then α is fixed).
- $d^+\theta = 0 \Longrightarrow$ there is an **explicit 'normal form'**

CLASSIFICATION II $(\alpha = \pm 1)$

Theorem.
A Kähler-Hermitian (M^4, g, J, I) with $\mathfrak{hol}^C = \langle \pm\omega_I + \omega_J \rangle$ is locally a torus-bundle over a Riemann surface

$$T^2 \hookrightarrow M \to \Sigma \ ,$$

with $d^+\theta \neq 0$ where $\theta \neq 0$.

If M compact, (g, I) is Kähler.

How '$\Longleftarrow$' goes, in two words:
Curvatures $\quad d\theta_k = \frac{f_k}{2}\omega_\Sigma \qquad (f_1^2 + f_2^2 = 1)$
$\mathscr{D}^+ = span\{\theta_1^\sharp, \theta_2^\sharp\}$ the torus-action distribution
$\mathscr{D}^-$ the horizontal distribution

Build integrable OCSs

$$I\theta_1 = -\theta_2, \qquad I|_{\mathscr{D}^-} = I_\Sigma$$
$$J\theta_1 = \theta_2, \quad J|_{\mathscr{D}^-} = I_\Sigma.$$

for the Riemannian metric $g = \theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2 + t g_\Sigma$ ($t > 0$ a map)

Then
$\omega_J = -\theta_1 \wedge \theta_2 + t\omega_\Sigma$ is closed, forcing (g, J) Kähler
$\theta = t^{-1}(-f_2\theta_1 + f_1\theta_2)$ belongs in $\mathscr{D}^+$
$\theta_k^\sharp$ are I-holo, and $\eta_{\theta_k} = 0$, so $\nabla^C\theta_k = 0$.

NB: $d\theta \wedge \theta = 0 \iff f_1, f_2$ are constant.

CLASSIFICATION III

cases left: $d\theta \wedge \theta = 0$ or $d\theta \neq 0$.

Theorem.
In both cases the Kähler-Hermitian surface is a suitable *'deformation'* of a Kähler surface of Calabi-type, where $\theta \neq 0$.

This means

$$\mathbb{C} \hookrightarrow \mathscr{L} \to \Sigma$$

inherits coords from the Hamiltonian Killing X field tangent to S^1-action on $M = \mathscr{L}$.

Use $\gamma \in \Lambda^{0,1}(\Sigma, L^{-1})$ to modify Calabi's construction for my purposes...

If not lcK, Goldberg-Sachs guarantees there's no Einstein metric in the conformal class.

Yet, *Ric* has one constant double eigenvalue!

All cases dealt with.

The local structure of each is fully described.

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