Special geometries and their Dirac operators

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1. Introduction

(M^n, g) compact Riemannian spin manifold

$\Rightarrow$ ΣM spinor bundle, $D^g : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ Dirac operator

- Schrödinger ('32): $(D^g)^2 = \Delta^g + \frac{1}{4}\,\mathrm{Scal}^g$
- Lichnerowicz ('63): $\mathrm{Scal}^g > 0 \Rightarrow \hat{\mathcal{A}}(M) = 0$ (index theorem)
- Friedrich ('80): $\mathrm{Scal}^g > 0 \Rightarrow (D^g)^2 > \frac{1}{4}\,\mathrm{Scal}^g_{\text{min}}$ – optimal lower bound?

Answer [Friedrich 1980]: the perturbed operator $D^g - f$ satisfies

$$(D^g - f)^2 = \Delta^f + \frac{1}{4}\,\mathrm{Scal}^g + (1-n)f^2$$

$\Rightarrow$

- $(D^g)^2 \geq \frac{1}{4}\,\frac{n}{n-1}\,\mathrm{Scal}^g_{\text{min}}$
- limiting spinors satisfy the Killing equation $\nabla^g_X \psi = \mu \cdot X \cdot \psi$.

$(M^n, g, \mathcal{R}, \nabla^c)$ compact Riemannian spin mf. equipped with

- $\mathcal{R}$ a non-integrable geometric structure
- $\nabla^c = \nabla^g + \frac{1}{2} T^c$ characteristic connection with parallel torsion

$$T^c \in \Lambda^3 T^* M, \nabla^c T^c = 0$$

Examples

- G/H naturally reductive $\Rightarrow$ $\nabla^c = \nabla^{\mathrm{can}} \Longleftrightarrow \nabla^c T^c = 0, \nabla^c R^c = 0$
- *Sasaki structures* in dim. $2k+1 \Rightarrow T^c = \eta \wedge \mathrm{d}\eta$ [Friedrich,Ivanov (2002)]
- *nearly Kähler structures* $(M^6, g, J) \Rightarrow T^c = -J((\nabla^g_{.} J)(.))$ [Gray,Kirichenko,Bismut,Friedrich,Ivanov]
- *nearly parallel G_2-structures* $(M^7, \varphi) \Rightarrow T^c = \frac{1}{6} a \cdot \varphi$ [Friedrich,Ivanov (2002)]

Interpretation

Strominger equations of type-II string theorie

(M^n, g, H, ψ) – $H \in \Lambda^3 T^* M^n$ B-field, ψ spinor

$$\nabla^g_X \psi + \frac{1}{4} X \lrcorner H \cdot \psi = 0, \;\; H \cdot \psi = \mu \cdot \psi$$

$$\delta(H) = 0, \;\; \delta(\mathrm{Ric}^g_{ij} - \frac{1}{4} H_{imn} H_{jmn}) = 0$$

idea: $\nabla = \nabla^g + \frac{1}{2} H \Rightarrow \nabla \psi = 0$

Ansatz: non-integrable geometries $(M^n, g, \mathcal{R}, \nabla^c)$.

$\nabla^{1/3}$ metric connection with torsion $1/3\ T^c \Rightarrow$

$$D^{1/3} = \mu \circ \nabla^{1/3} : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$$

elliptic symmetric differential operator of first order.

Facts

•(M,g) naturally reductive $\Rightarrow$ $D^{1/3}$ coincides with *Kostant's cubic Dirac operator* and satisfies a nice Parthasarathy type formula [Agricola 2003]

•(M,g) Hermitian $\Rightarrow$ $D^{1/3}$ can be identified with the *Dolbeault-Operator* [Bismut,Gauduchon]

•$(D^{1/3})^2$ satisfies the (universal) SL formula [Agricola, Friedrich 2003]

$$(D^{1/3})^2 = \Delta^T + \tfrac{1}{4}\,|T^c|^2 + \tfrac{1}{8}\,T^{c2} + \tfrac{1}{4}\,\mathrm{Scal}^g,$$

where $\Delta^T = \nabla^{c*}\nabla^c$ – optimal lower bound?

Consequence:

$$(D^{1/3})^2 \circ T^c = T^c \circ (D^{1/3})^2$$

$\Rightarrow$
estimate $(D^{1/3})^2_{|\Sigma_\mu}$ for all $\Sigma_\mu = \ker(T^c - \mu), \mu \in \mathrm{spec}_p T^c$.

Remark:

$D^{1/3}$ and T^c do not commute!

Theorem ((S-def.) SL Formula). $S \in \mathrm{End}(\Sigma M)$ a symmetric and ∇^c parallel endomorphism. Then:

$$\langle (D^{1/3}+S)^2\psi,\psi\rangle_{L^2} = ||\nabla^S\psi||^2_{L^2} - \tfrac{1}{4}\textstyle\sum_{i=1}^n ||(e_i\cdot S + Se_i)\psi||^2_{L^2} -$$

$$\tfrac{1}{4}||T^c\psi||^2_{L^2} + \tfrac{1}{8}|T^c|^2\cdot||\psi||^2_{L^2} + \tfrac{1}{4}\textstyle\int_M \mathrm{Scal}^g|\psi|^2 + ||S\psi||^2_{L^2} - \langle T^c\cdot S\psi,\psi\rangle_{L^2},$$

where

$$\nabla^S_X\psi := \nabla^c_X\psi - \frac{1}{2}(SX\cdot\psi + X\cdot S\psi).$$

Ansatz:

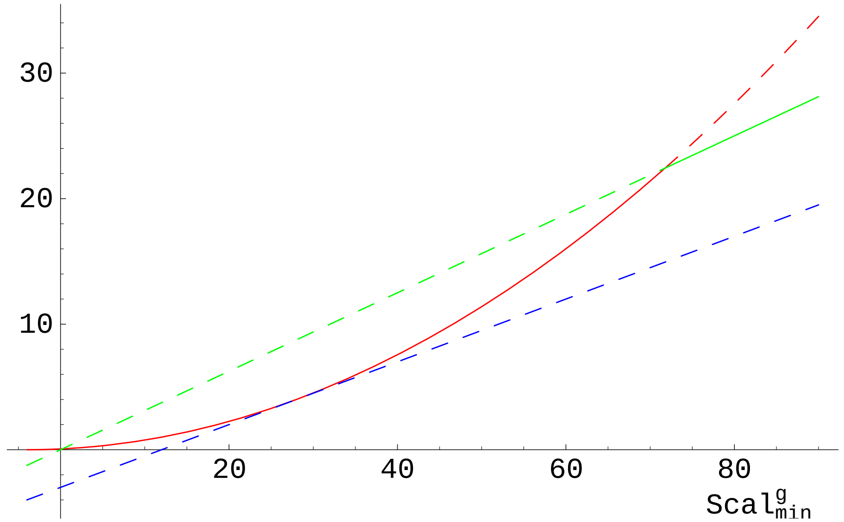
- Fix a bundle Σ_μ. We choose S as polynomials $P(T^c)$ in T^c such, that Σ_μ is $\nabla^{P(T^c)}$ parallel!

- The algebraic type of T^c is kwown for $\dim M \leq 7$ $\Rightarrow$ $\sum_{i=1}^n ||(e_i\cdot S + Se_i)\psi||^2_{L^2}$ can be controlled for a fixed geometric structure!

Theorem[AF–]. $(M^5, g, \xi, \eta, \phi)$ compact Sasaki manifold
$T^c = \eta \wedge d\eta,\ |T^c|^2 = 8,\ \Sigma M = \Sigma^1_{-4} \oplus \Sigma^2_0 \oplus \Sigma^1_4$ and $-4 < \mathrm{Scal}^g_{\mathrm{min}} \Rightarrow$

$$\lambda_{\mathrm{min}}((D^{1/3})^2) \geq \begin{cases} 1/16\ (1 + 1/4\ \mathrm{Scal}^g_{\mathrm{min}})^2; -4 < \mathrm{Scal}^g_{\mathrm{min}} \leq 4(9+4\sqrt{5}) \\ 5/16\ \mathrm{Scal}^g_{\mathrm{min}}; 4(9+4\sqrt{5}) \leq \mathrm{Scal}^g_{\mathrm{min}} \end{cases}$$

If
$\lambda_{\mathrm{min}}((D^{1/3})^2) < \frac{1}{16}(1 + \frac{1}{4}\mathrm{Scal}^g_{\mathrm{min}})^2$,
then necessarily
$\lambda_{\mathrm{min}}((D^{1/3})^2_{|\Sigma^2_0}) = \lambda_{\mathrm{min}}((D^{1/3})^2_{|\Sigma^1_{\pm 4}})$.



Remark

- quadratic dependence on the scalar curvature
- positive lower bound even for negative scalar curvature
- $\mathrm{Scal}_{\mathrm{min}} = 28 \Rightarrow$ spaces with ∇^c parallel spinors [Friedrich,Ivanov (2002)]

The limiting case

Definition. (M, g, ξ, η, ϕ) is called η-Einstein, if $\mathrm{Ric} = \lambda g + \mu \eta \otimes \eta$.

Theorem. 1) (M, g, ξ, η, ϕ) compact Sasaki mf. with $-4 < \mathrm{Scal}_{\text{min}} \leq 4(9 + 4\sqrt{5})$. If $\psi \in \Gamma(\Sigma^1_{\pm 4})$ is eigenspinor to the eigenvalue $\lambda = 1/16(1 + 1/4\mathrm{Scal}_{\text{min}})^2$, then (M, g, ξ, η, ϕ) is η-Einstein.

2) (M, g, ξ, η, ϕ) simply connected η-Einstein space with $-4 < \mathrm{Scal}^g_{\text{min}}$, then $\lambda = 1/16(1 + 1/4\, \mathrm{Scal}^g_{\text{min}})^2$ is eigenvalue of the operator $(D^{1/3})^2_{|\Sigma^1_{\pm 4}}$ and realizes the smallest eigenvalue for $\mathrm{Scal}^g_{\text{min}} \leq 4(9 + 4\sqrt{5})$.

Example. 5-dim. η-Einstein-Sasaki spaces can be obtained as total spaces of certain S^1-bundles over 4-dim Kähler-Einstein spaces.

Example. There are many non-regular examples. [Boyer,Galicki,Matzen 2006]

2. $D^{1/3}$ on 6-dim. almost Hermitian spaces

Spaces of strict type $W1$ – the *nearly Kähler* case

(M^6, g, J) compact (simply connected) *nearly Kähler* space, i.e.

$$(\nabla^g_X J)X = 0, \ X \in TM.$$

• (M^6, g, J) complete Einstein space of positive scalar curvature Scal^g with $\mathrm{Scal}^g = 15/2\,|T^c|^2$.

• $M^6 \neq S^6 \Rightarrow$ 2 ∇^c-parallel Killing spinors φ_1, φ_2.

[Friedrich,Grunewald,Ivanov]

• $\Sigma M = \Sigma^6_0 + \Sigma^1_{2|T^c|} + \Sigma^1_{-2|T^c|}$.

Proposition. φ_1 and φ_2 realize the universal lower bound for $(D^{1/3})^2_{|\Sigma_{\pm 2|T^c|}}$,

$$(D^{1/3})^2\varphi_i = 2/15\ \mathrm{Scal}^g \cdot \varphi_i, \quad i = 1, 2.$$

The bundle Σ_0

•Universal bound: $(D^{1/3})^2_{|\Sigma_0} \geq 4/15\,\mathrm{Scal}^g$ – never optimal!

•$\psi \in \Gamma(\Sigma_0) \Rightarrow$

$$\langle \psi, (D^{1/3})^2\psi\rangle_{L^2} = \langle \psi, (D^g)^2\psi\rangle_{L^2} \geq \inf_{\alpha\perp\varphi_1,\varphi_2} \langle \alpha, (D^g)^2\alpha\rangle_{L^2}$$

Theorem. (M^6, g, J) compact nearly Kähler space and not isometric to the sphere S^6 $\Rightarrow$

$$\lambda_1((D^{1/3})^2_{|\Sigma_0}) \geq \lambda_2((D^g)^2).$$

Questions:

•optimal lower bound for $\lambda_2((D^g)^2)$? – remains open

•nessecary conditions for $\lambda_1((D^{1/3})^2_{|\Sigma_0}) = \lambda_2((D^g)^2)$

We apply our method to the bundle Σ_0 and obtain:

Theorem 1. (M^6, g, J) compact nearly Kähler space and λ eigenvalue with eigenspinor $(D^{1/3})^2_{|\Sigma_0}\psi = \lambda\psi$. Then:

$$\lambda \geq \frac{1}{4}\frac{\langle D^{1/3}\psi, \psi\rangle_{L^2}}{||\psi||^2_{L^2}} + \frac{4}{15}\,\mathrm{Scal}^g.$$

Theorem 2. In Theorem 1 equality never holds !

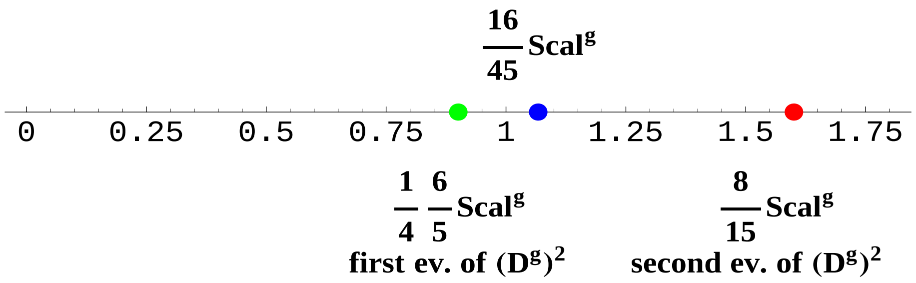
Consequence:

Theorem. (M^6, g, J) compact nearly Kähler space with $\lambda_1((D^{1/3})^2_{|\Sigma_0}) = \lambda_2((D^g)^2) \Rightarrow$

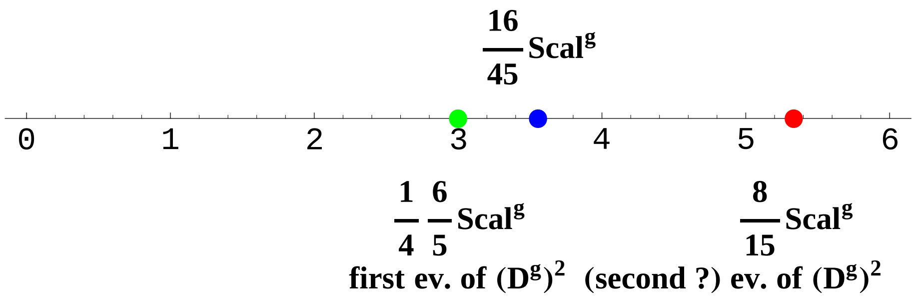
$$\frac{16}{45}\,\mathrm{Scal}^g < \lambda_2((D^g)^2).$$

$$\lambda_1((D^{1/3})^2_{|\Sigma_0}) = \lambda_2((D^g)^2) \Rightarrow \tfrac{16}{45}\,\mathrm{Scal}^g < \lambda_2((D^g)^2)$$

nearly Kähler structure on S^6, $\mathrm{Scal}^g = 3$



nearly Kähler structure on $S^3 \times S^3$, $\mathrm{Scal}^g = 10$



3. $D^{1/3}$ on 7-dim. non-integrable G_2-spaces

The case of a nearly parallel G_2-space

- (M^7, φ) nearly parallel $\Leftrightarrow d\varphi = -a * \varphi$.
- ∇^c exists and is determined by $T^c = -1/6\, a \cdot \varphi$.
- (M^7, φ) simply-conn. $\Rightarrow \exists$ (at least) one real Killing spinor.
- $T^c : \Sigma M \to \Sigma M \quad \Rightarrow \quad \Sigma M^7 = \Sigma^1_{-7/6\, a} \oplus \Sigma^7_{1/6\, a}$.

The canonical ∇^c parallel spinor $\psi^* \in \Gamma(\Sigma^1_{-7/6\, a})$ realizes the universal lower bound and the first eigenvalue in $\Sigma^1_{-7/6\, a}$,

$$(D^{1/3})^2 \psi^* = \frac{7}{54}\, \mathrm{Scal}^g \cdot \psi^* .$$

For an eigenvalue λ with eigenspinor $(D^{1/3})^2\psi = \lambda\psi$, $\psi \in \Gamma(\Sigma^7_{1/6\,a})$, we obtain

$$\lambda \geq \frac{1}{6}\frac{\langle D^{1/3}\psi,\psi\rangle^2_{L^2}}{||\psi||^4_{L^2}} + \frac{1}{36}a\frac{\langle D^{1/3}\psi,\psi\rangle_{L^2}}{||\psi||^2_{L^2}} + \frac{583}{2268}\,\mathrm{Scal}^g.$$

In the limiting case ψ satisfies the equation

$$(1) \qquad \nabla^c_X\psi - \frac{1}{2}(P(T^c)\cdot X + X\cdot P(T^c))\cdot\psi = 0,\ X \in TM,$$

where $P \in \mathbb{R}[T^c]$.

•The integrability conditions give more restrictions on P.

•An algebraic calculation shows that (1) is equivalent to the Killing equation !

Theorem. (M^7, φ) nearly parallel G_2-space and $(D^{1/3})^2_{|\Sigma^7_{1/6\,a}}\psi = \lambda\psi$. Then:

$$\lambda \geq \frac{1}{6}\frac{\langle D^{1/3}\psi, \psi\rangle^2_{L^2}}{||\psi||^4_{L^2}} + \frac{1}{36}a\frac{\langle D^{1/3}\psi, \psi\rangle_{L^2}}{||\psi||^2_{L^2}} + \frac{583}{2268}\mathrm{Scal}^g.$$

In the limiting case:

1) $\lambda = \frac{121}{378}\cdot \mathrm{Scal}^g$ and ψ is a second Killing spinor.

2) (after scaling) (M^7, φ) is Einstein-Sasaki.

3) (M^7, φ) nearly parallel and $\psi \in \Gamma(\Sigma^7_{1/6\,a})$ a second Killing spinor then the limiting case is realized.

Remark. If (M^7, φ) is *essentially* nearly parallel we can estimate $\lambda_1((D^{1/3})^2_{|\Sigma^7_{1/6}})$ in terms of $\lambda_2((D^g)^2)$ (as in the nearly Kähler case)!

Summary

•optimal eigenvalue estimates on Sasaki manifolds

•(M^6, g, J) nearly Kähler $\Rightarrow$

$$\lambda_1((D^{1/3})^2_{|\Sigma_0}) \geq \lambda_2((D^g)^2).$$

Next step: compute the spectrum of $D^{1/3}$ on the naturally reductive nearly Kähler space $\mathbf{CP}^3$

•optimal eigenvalue estimates on almost Hermitian manifolds of strict type $W3$ and $W4$ $\leftrightarrow$ Sasaki case

•non-integrable G_2-manifolds of type $\mathfrak{hol}^c = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2),\ \mathfrak{u}(2)$

Literature

–, *Dirac operators in geometries with torsion*, to appear in Ann. Global Anal. Geom.