Recent Progress in Homogeneous Einstein Metrics on Generalized Flag Manifolds

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Based on joint works with I. Chrysikos (Brno) and Y. Sakane (Osaka)

The problem

(M, g) Riemannian manifold is Einstein if $Ric(g) = c \cdot g$.

Consider G-invariant metrics on a homogeneous space $(M = G/K, g)$.

<u>Problem:</u> Find G-invariant Einstein metrics on $M = G/K$ and classify them (if they are not unique).

- $c > 0$ G/K is compact
- $c = 0$ G/K is Ricci flat $\Rightarrow$ flat
- $c < 0$ G/K is not compact

Review of selected research

- There exist compact homogeneous spaces G/K with no G-invariant Einstein metric (e.g. $SU(4)/SU(2)$) (Wang–Ziller 1986)
- (Böhm–Kerr 2006) Classified all compact, simply connected homogeneous spaces of dimension ≤ 11 admitting a G-invariant Einstein metric.
- (Nikonorov–Rodionov 2003, 2004) Found all G-invariant Einstein metrics on compact simply connected homogeneous spaces of dimension ≤ 7 (except $SU(2) \times SU(2)$).
- (Böhm–Wang–Ziller 2004) Variational approach
- (Böhm 2004) Simplicial complexes
- (Graev 2006) Newton Polytopes

- (A.A.–Nikonorov 2009) Introduced a construction for obtaining homogeneous Einstein metrics by restricting the isometry group. New examples of Einstein metrics on real and quaternionic Stiefel manifolds $SO(n)/SO(l)$, $Sp(n)/Sp(l)$.

Conjecture (Böhm–Wang–Ziller): Let $M = G/K$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.
If $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus m_s$, $\mathfrak{m}_i$ irreducible, non equivalent $Ad(K)$-modules, then the number of G-invariant Einstein metrics on M is finite.

Generalized flag manifolds

$M = G/K = G/C(T) \cong Ad(G)w$, $w \in \mathfrak{g}$.

$G =$ compact semisimple Lie group

$T =$ a torus in G

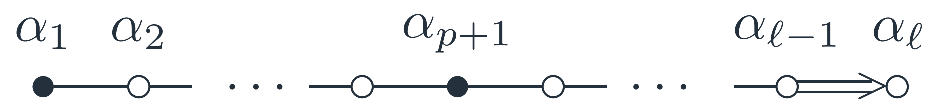
$Ad : G \to Aut(\mathfrak{g})$ (adjoint representation)

e.g. $SU(n)/S(U(n_1) \times \cdots U(n_s))$, $(n = \sum n_i)$

$SU(n)/S(U(1) \times \cdots U(1)) = SU(n)/T_{\max}$

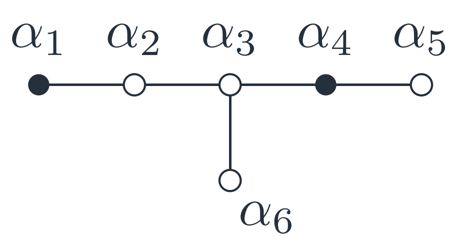
- Flag manifolds admit a finite number of G-invariant complex structures and for each of those there exists a compatible Kähler-Einstein metric.
- They exhaust all compact simply connected homogeneous manifolds.
- They are classified by the painted Dynking diagrams.
- There is an infinite series for each classical simple Lie group and a finite number of non isomorphic flag manifolds for each exceptional Lie group.

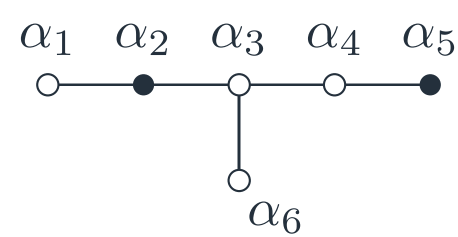
Examples of painted Dynkin diagrams



$$\Pi \backslash \Pi_0 = \Pi_{\mathfrak{n}} = \{\alpha_1, \alpha_{p+1} : 2 \leq p \leq \ell - 1\}$$

$$M = SO(2\ell+1)/(U(1) \times U(p) \times SO(2(\ell - p - 1) + 1))$$
$(2 \leq p \leq \ell - 1)$.





$\Pi \backslash \Pi_0 = \{\alpha_1, \alpha_4\}$ or $\Pi \backslash \Pi_0 = \{\alpha_2, \alpha_5\}$

Both define the flag manifold $E_6/(SU(4) \times SU(2) \times U(1) \times U(1))$.

Previous results about Einstein metrics on flag manifolds 1

- Wang-Ziller (1986) $M = G/T_{\max}$ admits the standard metric $g_B = -$ Killing form as Einstein if and only if $G \in \{SU(n), SO(2n), E_6, E_7, E_8\}$.
- Kimura (1990) Classification of flag manifolds with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus m_3$ and Einstein metrics for the classical cases.
- A.A. (1993) Lie theoretic expression for the Ricci tensor and Einstein metrics for certain flag mfds with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ and for $G_2/U(2)$.
- Sakane (1999) $G/T_{\max}$, $G \in \{SU(2n+1), SO(2n+1), SO(2n), Sp(n)\}$.
- Dos Santos-Negreiros (2006) $SU(2n)/T_{\max}$, $SU(2n+1)/T_{\max}$ $(n \geq 6)$.

Previous results about Einstein metrics on flag manifolds 2

- A.A.-Chrysikos (2011) Classification for $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$
- A.A.-Chrysikos (2010) Classification for $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$
- A.A.-Chrysikos-Sakane (2010, 2011) Completed the description of invariant Einstein metrics for certain classical cases
- Chrysikos-Anastassiou (2011) Redescovered the invariant Einstein metrics for flag mfds with 2 and 3 isotropy summands, as singularites at infinity of a dynamical system via the Ricci flow.
- A.A.-Chrysikos-Sakane (2012) $G_2/T_{\max} = G_2/U(1) \times U(1)$

$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$

It admits a unique Kähler-Einstein metric and two non-Kähler Einstein metrics (up to isometry). This is an example of a flag mfd of an exceptional Lie group which admits a non-Kähler, not normal homogeneous Einstein metric.

Previous results about Einstein metrics on flag manifolds 1

When the number of isotropy summands increases then the construction of the Einstein equation (actually the Ricci tensor) becomes more difficult and the solutions difficult to be obtained.

Ricci tensor of a compact homogeneous space

G a compact semisimple Lie group, K a connected closed subgroup.
The Killing form of $\mathfrak{g}$ is negative definite, so we can define an $Ad(G)$-invariant inner product B on $\mathfrak{g}$ given by $B = -$ Killing form of $\mathfrak{g}$.
Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$ with respect to B so that $[\,\mathfrak{k},\,\mathfrak{m}\,] \subset \mathfrak{m}$ and $\mathfrak{m} \cong T_o(G/K)$.
We assume that $\mathfrak{m}$ admits a decomposition into mutually non equivalent irreducible Ad(K)-modules as follows:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q. \tag{1}$$

Then any G-invariant metric on G/K can be expressed as

$$\langle\ ,\ \rangle = x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}, \tag{2}$$

for positive real numbers $(x_1, \ldots, x_q) \in \mathbb{R}^q_+$.

The Ricci tensor r of a G-invariant Riemannian metric on G/K is of the same form as (2), that is

$$r = r_1 x_1 B|_{\mathfrak{m}_1} + \cdots + r_q x_q B|_{\mathfrak{m}_q}.$$

The Ricci compenents r_i can be obtained as follows:

Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e. $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$.
We put $A^\gamma_{\alpha\beta} = B\left([e_\alpha, e_\beta], e_\gamma\right)$ so that $[e_\alpha, e_\beta] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$ and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$ (Wang-Ziller).

Then the positive numbers $\begin{bmatrix} k \\ ij \end{bmatrix}$ are independent of the B-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and $\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}$.

Let $d_k = \dim \mathfrak{m}_k$. Then we have the following:

Proposition 0.1 (Park–Sakane) *The components $r_1, \dots, r_q$ of the Ricci tensor r of the metric $\langle\ ,\ \rangle$ of the form* (2) *on G/K are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1, \cdots, q), \tag{3}$$

where the sum is taken over $i, j = 1, \dots, q$.

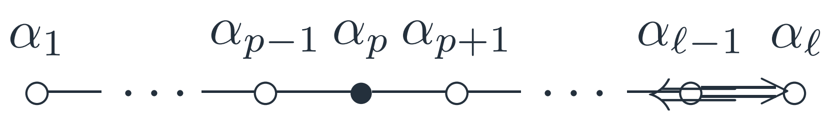
Since by assumption the submodules $\mathfrak{m}_i, \mathfrak{m}_j$ in the decomposition (1) are matualy non equivalent for any $i \neq j$, it will be $r(\mathfrak{m}_i, \mathfrak{m}_j) = 0$ whenever $i \neq j$.

Thus, the G-invariant Einstein metrics on $M = G/K$ are exactly the positive real solutions $g = (x_1, \ldots, x_q) \in \mathbb{R}^q_+$ of the polynomial system $\{r_1 = \lambda,\ r_2 = \lambda,\ \ldots,\ r_q = \lambda\}$, where $\lambda \in \mathbb{R}_+$ is the Einstein constant.

A simple example

Flag manifolds of B_ℓ with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Consider the flag manifolds $M = G/K = SO(2\ell+1)/(U(p) \times SO(2(\ell-p)+1))$ with $\ell \geq 2$, and $2 \leq p \leq \ell$.

This space is defined by the painted Dynkin diagram



G-invariant metrics: $\langle\ ,\ \rangle = x_1 B|_{\mathfrak{m}_1} + x_2 B|_{\mathfrak{m}_2}$.

Ricci components:

$$r_1 = \frac{1}{2x_1} - \frac{x_2}{2d_1 x_1^2}\begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

$$r_2 = \frac{1}{2x_2} - \frac{1}{2d_2 x_2}\begin{bmatrix} 1 \\ 21 \end{bmatrix} + \frac{x_2}{4d_2 x_1^2}\begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

Only $\begin{bmatrix} 1 \\ 21 \end{bmatrix} \neq 0.$

So $\langle\ ,\ \rangle$ is Einstein if and only if $r_1 = r_2$.

To avoid finding $\begin{bmatrix} 1 \\ 21 \end{bmatrix}$ from the definition we use tha fact that

$$\langle\ ,\ \rangle = 1 \cdot B|_{\mathfrak{m}_1} + 2 \cdot B|_{\mathfrak{m}_2}$$

is a Kähler-Einstein metric.

Thus $\begin{bmatrix} 1 \\ 21 \end{bmatrix} = \dfrac{d_1 d_2}{d_1 + 4d_2}.$

We normalize the equations $r_1 = r_2$ by setting $x_1 = 1$ and obtain a quadratic equation for x_2 with solutions $x_2 = 2$ and $x_2 = \frac{4d_2}{d_1+2d_2}$.

Thus a non Kähler Einstein metric is

$$x_1 = 2, \qquad x_2 = \frac{4d_2}{d_1 + 2d_2}.$$

Riemannian submersions

Let G be a compact semisimple Lie group and K, L two closed subgroups of G with $K \subset L$. Then there is a natural fibration $L/K \to G/K \xrightarrow{\pi} G/L$ with fiber L/K.

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$ with respect to B, and $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{l}$. Then we have the decompositions $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$, $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{q}$ so
$\mathfrak{g} = \mathfrak{k} \oplus \underbrace{\mathfrak{q} \oplus \mathfrak{p}}_{\mathfrak{m}}$.

An $Ad_G(L)$-invariant scalar product on $\mathfrak{p}$ defines a G-invariant metric $\check{g}$ on G/L, and an $Ad_L(K)$-invariant scalar product on $\mathfrak{q}$ defines an L-invariant metric $\hat{g}$ on L/K.
The orthogonal direct sum for these scalar products on $\mathfrak{q} \oplus \mathfrak{p}$ defines a G-invariant metric $g = \check{g} + \hat{g}$ on G/K, called *submersion metric*.

It is known that the map π is a Riemannian submersion from $(G/K, g)$ to $(G/L, \check{g})$ with totally geodesic fibers isometric to $(L/K, \hat{g})$.

$\mathfrak{q} =$ vertical subspace of $\mathfrak{m}$
$\mathfrak{p} =$ horizontal subspace of $\mathfrak{m}$.

O'Neill had introduced two tensors A, T. Since fibers are totally geodesic $T = 0$. Also,

$$A_X Y = \frac{1}{2}[X,\, Y]_{\mathfrak{q}} \quad \text{for } \; X, Y \in \mathfrak{p}.$$

Let r, $\check{r}$ be the Ricci tensors of the metrics g, $\check{g}$ respectively. Then we have (e.g. Besse)

$$r(X, Y) = \check{r}(X, Y) - 2g(A_X, A_Y) \quad \text{for } \; X, Y \in \mathfrak{p}.$$

(Note that there is a corresponding expression $r(U, V)$ for vertical vectors, but it does not contribute additional information in our approach.)

Let

$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_\ell$ into non equivalent irreducible $Ad(L)-$ modules,

$\mathfrak{q} = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_s$ into irreducible $Ad(K)-$ modules.

To compute the values $\begin{bmatrix} k \\ ij \end{bmatrix}$ for G/K, we use information from the Riemannian submersion $\pi : (G/K,\, g) \to (G/L,\, \check{g})$ with totally geodesic fibers isometric to $(L/K,\, \hat{g})$.

We consider G-invariant metrics on G/K defined by a Riemannian submersion $\pi : (G/K,\, g) \to (G/L,\, \check{g})$ given by

$$g = \underbrace{y_1 B|_{\mathfrak{p}_1} + \cdots + y_\ell B|_{\mathfrak{p}_\ell}}_{\check{g}} + \underbrace{z_1 B|_{\mathfrak{q}_1} + \cdots + z_s B|_{\mathfrak{q}_s}}_{\hat{g}} \tag{4}$$

for positive real numbers $y_1, \cdots, y_\ell, z_1, \cdots, z_s$.

Since $K \subset L$ we decompose each irreducible component $\mathfrak{p}_j$ into irreducible Ad(K)-modules:

$$\mathfrak{p}_j = \mathfrak{m}_{j,1} \oplus \cdots \oplus \mathfrak{m}_{j,\,k_j},$$

where the $Ad(K)$-modules $\mathfrak{m}_{j,t}$ $(j = 1, \ldots, \ell,\ t = 1, \ldots, k_j)$ are mutually non equivalent and are chosen to be (up to reordering) submodules from the decomposition (2).

Then the submersion metric (4) can be written as

$$g = y_1 \sum_{t=1}^{k_1} B|_{\mathfrak{m}_{1,t}} + \cdots + y_\ell \sum_{t=1}^{k_\ell} B|_{\mathfrak{m}_{\ell,t}} + z_1 B|_{\mathfrak{q}_1} + \cdots + z_s B|_{\mathfrak{q}_s} \quad (5)$$

and this is a special case of the G-invariant metric (2).

Lemma 0.2 *Let* $d_{j,t} = \dim \mathfrak{m}_{j,t}$. *The components* $r_{(j,\,t)}$ $(j = 1, \ldots, \ell,\ t = 1, \ldots, k_j)$ *of the Ricci tensor* r *for the metric* (5) *on* G/K *are given by*

$$r_{(j,\,t)} = \check{r}_j - \frac{1}{2d_{j,\,t}} \sum_{i=1}^{s} \sum_{j',\,t'} \frac{z_i}{y_j y_{j'}} \begin{bmatrix} & i & \\ (j,t) & & (j',t') \end{bmatrix}, \qquad (6)$$

where $\check{r}_j$ *are the components of Ricci tensor* $\check{r}$ *for the metric* $\check{g}$ *on* G/L.

Notice that when metric (4) is viewed as a metric (2) then the horizontal part of $r_{(j,\,t)}$ equals to $\check{r}_j$ ($j = 1, \ldots, \ell$), i.e. it is independent of t.

Let $M = G/K$ be a flag manifold with five isotropy summands $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$.

It follows that $[\mathfrak{m}_1, \mathfrak{m}_2] = \mathfrak{m}_3$, $[\mathfrak{m}_2, \mathfrak{m}_2] = \mathfrak{m}_4$, $[\mathfrak{m}_2, \mathfrak{m}_3] = \mathfrak{m}_5$ and $[\mathfrak{m}_1, \mathfrak{m}_4] = \mathfrak{m}_5$.
Thus the non zero structure constant are

$$\begin{bmatrix} 3 \\ 12 \end{bmatrix}, \begin{bmatrix} 4 \\ 22 \end{bmatrix}, \begin{bmatrix} 5 \\ 23 \end{bmatrix}, \begin{bmatrix} 5 \\ 14 \end{bmatrix}.$$

A G-invariant metric g on G/K is given by

$$g = x_1 B|_{\mathfrak{m}_1} + x_2 B|_{\mathfrak{m}_2} + x_3 B|_{\mathfrak{m}_3} + x_4 B|_{\mathfrak{m}_4} + x_5 B|_{\mathfrak{m}_5} \qquad (7)$$

where x_j $(j = 1, \ldots, 5)$ are positive numbers.
Put $d_i = \dim \mathfrak{m}_i$ for $i = 1, \cdots, 5$.

Then the components r_i $(i = 1, \ldots, 5)$ of the Ricci tensor for a G-invariant Riemannian metric (7) on G/K are given as follows:

$$
\begin{aligned}
r_1 &= \frac{1}{2x_1} + \frac{1}{2d_1}\begin{bmatrix}3\\12\end{bmatrix}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + \frac{1}{2d_1}\begin{bmatrix}5\\14\end{bmatrix}\left(\frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4} - \frac{x_4}{x_1x_5}\right),\\
r_2 &= \frac{1}{2x_2} + \frac{1}{2d_2}\begin{bmatrix}3\\12\end{bmatrix}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\right) - \frac{1}{2d_2}\begin{bmatrix}4\\22\end{bmatrix}\frac{x_4}{{x_2}^2} + \frac{1}{2d_2}\begin{bmatrix}5\\23\end{bmatrix}\left(\frac{x_2}{x_3x_5} - \frac{x_5}{x_2x_3} - \right.\\
r_3 &= \frac{1}{2x_3} + \frac{1}{2d_3}\begin{bmatrix}3\\12\end{bmatrix}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\right) + \frac{1}{2d_3}\begin{bmatrix}5\\23\end{bmatrix}\left(\frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5}\right),\\
r_4 &= \frac{1}{2x_4} + \frac{1}{2d_4}\begin{bmatrix}5\\14\end{bmatrix}\left(\frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5}\right) + \frac{1}{4d_4}\begin{bmatrix}4\\22\end{bmatrix}\left(-\frac{2}{x_4} + \frac{x_4}{{x_2}^2}\right),\\
r_5 &= \frac{1}{2x_5} + \frac{1}{2d_5}\begin{bmatrix}5\\23\end{bmatrix}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right) + \frac{1}{2d_5}\begin{bmatrix}5\\14\end{bmatrix}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right).
\end{aligned}
\tag{8}
$$

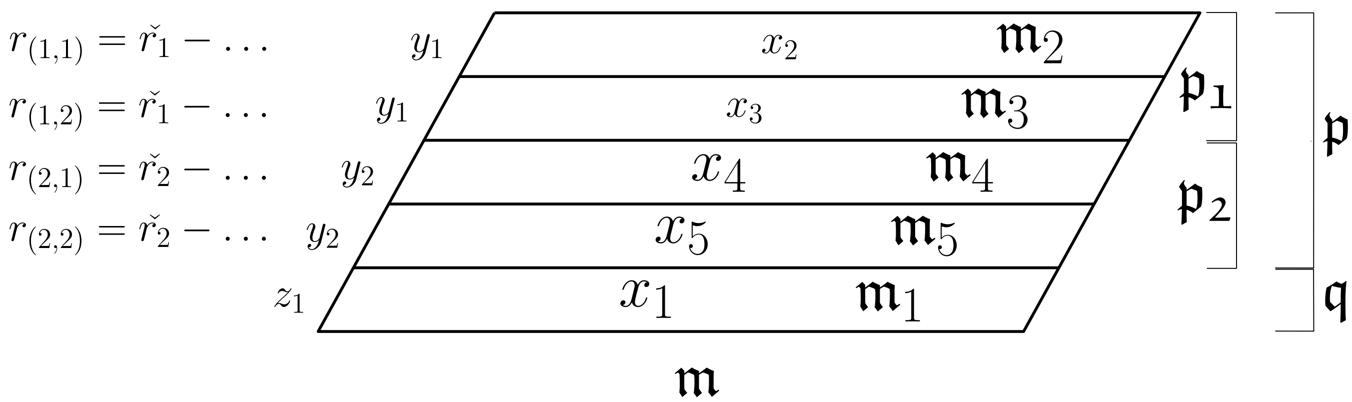
Let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to the Lie subgroup K. We consider a subspace $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{m}_1$ of $\mathfrak{g}$. Then $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$ and we have a natural fibration $\pi : G/K \to G/L$ with fiber L/K.

We decompose $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ and $\mathfrak{q} = \mathfrak{q}_1$, where $\mathfrak{p}_1 = \mathfrak{m}_2 \oplus \mathfrak{m}_3 \equiv \mathfrak{m}_{1,1} \oplus \mathfrak{m}_{1,2}$, $\mathfrak{p}_2 = \mathfrak{m}_4 \oplus \mathfrak{m}_5 \equiv \mathfrak{m}_{2,1} \oplus \mathfrak{m}_{2,2}$ and $\mathfrak{q}_1 = \mathfrak{m}_1$.

That is

$$\begin{aligned} \mathfrak{m} = \mathfrak{q} \oplus \mathfrak{p} = \mathfrak{q}_1 \oplus (\mathfrak{p}_1 + \mathfrak{p}_2) = \\ \mathfrak{m}_1 \oplus (\mathfrak{m}_2 \oplus \mathfrak{m}_3 + \mathfrak{m}_4 \oplus \mathfrak{m}_5) \equiv \\ \mathfrak{m}_1 \oplus (\mathfrak{m}_{1,1} \oplus \mathfrak{m}_{1,2} + \mathfrak{m}_{2,1} \oplus \mathfrak{m}_{2,2}) \end{aligned} \tag{9}$$

An example and a picture



$$
\begin{aligned}
\mathfrak{m} &= \mathfrak{q} \oplus \mathfrak{p} = \mathfrak{q}_1 \oplus (\mathfrak{p}_1 + \mathfrak{p}_2) = \mathfrak{m}_1 \oplus (\mathfrak{m}_2 \oplus \mathfrak{m}_3 + \mathfrak{m}_4 \oplus \mathfrak{m}_5) \\
&\equiv \mathfrak{m}_1 \oplus (\mathfrak{m}_{1,1} \oplus \mathfrak{m}_{1,2} + \mathfrak{m}_{2,1} \oplus \mathfrak{m}_{2,2})
\end{aligned} \tag{10}
$$

Then $r_{(1,1)} = r_2, r_{(1,2)} = r_3, r_{(2,1)} = r_4, r_{(2,2)} = r_5$

An example and a picture

We consider a G-invariant metric on G/K defined by a Riemannian submersion $\pi : (G/K,\, g) \to (G/L,\, \check{g})$ given by

$$g = y_1 B|_{\mathfrak{p}_1} + y_2 B|_{\mathfrak{p}_2} + z_1 B|_{\mathfrak{q}_1} \qquad (11)$$

and the metric $\check{g}$ on G/L

$$\check{g} = y_1 B|_{\mathfrak{p}_1} + y_2 B|_{\mathfrak{p}_2}$$

for positive real numbers y_1, y_2, z_1.

Note that, when we write the metric (11) as in the form (7), we have

$$g = y_1 B|_{\mathfrak{m}_2} + y_1 B|_{\mathfrak{m}_3} + y_2 B|_{\mathfrak{m}_4} + y_2 B|_{\mathfrak{m}_5} + z_1 B|_{\mathfrak{m}_1}\,. \qquad (12)$$

From (8) we obtain the components r_i of the Ricci tensor for the metric (12) on G/K as follows:

$$r_3 = \frac{1}{2y_1} - \frac{1}{2d_3}\begin{bmatrix}3\\12\end{bmatrix}\frac{z_1}{{y_1}^2} - \frac{1}{2d_3}\begin{bmatrix}5\\23\end{bmatrix}\frac{y_2}{{y_1}^2},$$

$$r_4 = \frac{1}{2y_2} - \frac{1}{2d_4}\begin{bmatrix}5\\14\end{bmatrix}\frac{z_1}{{y_2}^2} + \frac{1}{4d_4}\begin{bmatrix}4\\22\end{bmatrix}\left(\frac{y_2}{{y_1}^2} - \frac{2}{y_2}\right).$$

We put $\tilde{d}_1 = \dim \mathfrak{p}_1$ and $\tilde{d}_2 = \dim \mathfrak{p}_2$. Then $\tilde{d}_1 = d_2 + d_3$ and $\tilde{d}_2 = d_4 + d_5$.

By using the earlier very simple example $\mathfrak{m} = \mathfrak{m}_1 \bigoplus \mathfrak{m}_2$ the components $\check{r}_1, \check{r}_2$ of the Ricci tensor $\check{r}$ of a G-invariant metric $\check{g} = y_1 B|_{\mathfrak{p}_1} + y_2 B|_{\mathfrak{p}_2}$ are given by

$$\begin{cases} \check{r}_1 &= \dfrac{1}{2y_1} - \dfrac{y_2}{2\,\tilde{d}_1\,{y_1}^2}\left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right] \\ \check{r}_2 &= \dfrac{1}{2y_2} - \dfrac{1}{2\,\tilde{d}_2\,y_2}\left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right] + \dfrac{y_2}{4\,\tilde{d}_2\,{y_1}^2}\left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right], \end{cases} \tag{13}$$

where $\left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right] = \dfrac{\tilde{d}_1\tilde{d}_2}{\tilde{d}_1 + 4\tilde{d}_2}$.

Note that, in the notation of Lemma 0.2, we have that $r_{(1,1)} = r_2$, $r_{(1,2)} = r_3$, $r_{(2,1)} = r_4$ and $r_{(2,2)} = r_5$.

From Lemma 0.2 we see that the horizontal part of $r_{(1,2)}(= r_3)$ equals to $\check{r}_1$ and the horizontal part of $r_{(2,1)}(= r_4)$ equals to $\check{r}_2$, and thus we get

$$\frac{1}{2y_1} - \begin{bmatrix} 5 \\ 23 \end{bmatrix} \frac{1}{2d_3} \frac{y_2}{y_1^2} = \frac{1}{2y_1} - \frac{y_2}{2\tilde{d}_1 y_1^2} \left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right],$$

$$\frac{1}{2y_2} + \begin{bmatrix} 4 \\ 22 \end{bmatrix} \frac{1}{4d_4} (\frac{y_2}{y_1^2} - \frac{2}{y_2}) = \frac{1}{2y_2} - \frac{1}{2\tilde{d}_2 y_2} \left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right].$$

Therefore,

$$\begin{bmatrix} 5 \\ 23 \end{bmatrix} = d_3 \frac{1}{\tilde{d}_1} \left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right] = \frac{d_3(d_4 + d_5)}{(d_2 + d_3) + 4(d_4 + d_5)},$$

$$\begin{bmatrix} 4 \\ 22 \end{bmatrix} = d_4 \frac{1}{\tilde{d}_2} \left[\begin{bmatrix} 2 \\ 11 \end{bmatrix}\right] = \frac{d_4(d_2 + d_3)}{(d_2 + d_3) + 4(d_4 + d_5)}.$$

The two other triplets

$$\begin{bmatrix} 3 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

will be computed later on by taking into account the existence of Kähler-Einstein metric.

We remark that if we are going to study Einstein metrics on flag manifolds with six or more summands, then we need to apply the above method to two fibrations.

So the problem now is the following:

Classify all flag manifolds with five isotropy summands, use the above method to construct the Einstein equation and then study its solutions.

Classification of flag manifolds with five isotropy summands

Let G be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h}$ a maximal abelian subalgebra of $\mathfrak{g}$ with $\dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} = l = rkG$. There is a root space decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha\in\Delta} \mathfrak{g}_{\alpha}^{\mathbb{C}}$.
Let Π = $\{\alpha_1, \ldots, \alpha_l\}$ be a system of simple roots Δ. We denote by $\{\Lambda_1, \ldots, \Lambda_l\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π, that is $\dfrac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$ for any $1 \leq i, j \leq l$.

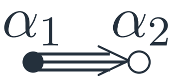
Let Π_0 be a subset of Π and set $\Pi_{\mathfrak{m}} = \Pi \backslash \Pi_0$ = $\{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$, where $1 \leq i_1 < \cdots < i_r \leq l$. We put $\Delta_0 = \Delta \cap \{\Pi_0\}_{\mathbb{Z}} = \{\beta \in \Delta : \beta = \sum_{\alpha_i\in\Pi_0} k_i\alpha_i,\ k_i \in \mathbb{Z}\}$, where $\{\Pi_0\}_{\mathbb{Z}}$ denotes the set of roots generated by Π_0 with integer coefficients (this is a the subspace of $\mathfrak{h}_0$). Then Δ_0 is a root subsystem of Δ.

Definition 0.3 *The roots of the set $\Delta_{\mathfrak{m}} = \Delta \backslash \Delta_0$ are called complemetary roots.*

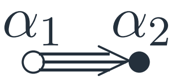
Painted Dynkin diagrams

Definition 0.4 *Let* $\Gamma(\Pi)$ *be the Dynkin diagram of* Π*. By painting black in* $\Gamma(\Pi)$ *the simple roots* $\alpha_i \in \Pi_{\mathfrak{m}} = \Pi \backslash \Pi_0$ *we obtain the painted Dynkin diagram* $\Gamma(\Pi_{\mathfrak{m}})$ *of* M*.*

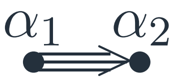
Example



$G_2(\alpha_2)$



$G_2(\alpha_1)$



G_2/T

These correspond to the flag manifolds

$$G_2/U(2) \quad \text{with } \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

$$G_2/U(2) \quad \text{with } \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3, \text{ and}$$

$$G_2/(U(1) \times U(1)) \text{ with } \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6.$$

Let

$$\mathfrak{t} = \mathfrak{z} \cap \mathfrak{h}_0 = \Big\{ H \in \mathfrak{h}_0 : (H, \Pi_0) = 0 \Big\}, \ \ (\mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}).$$

We consider the restriction map $\kappa : \mathfrak{h}_0^* \to \mathfrak{t}^*$, $\alpha \mapsto \alpha|_{\mathfrak{t}}$ and set $\Delta_{\mathfrak{t}} = \kappa(\Delta)$, $\kappa(\Delta_0) = 0$.

Definition 0.5 *The elements of $\Delta_{\mathfrak{t}}$ are called $\mathfrak{t}$-roots.*

Let $\mathfrak{m}^{\mathbb{C}} = T_o(G/K)^{\mathbb{C}}$ be the complexification of $\mathfrak{m}$. Then it is $\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \Delta_{\mathfrak{m}}} \mathfrak{g}_\alpha^{\mathbb{C}}$.

Proposition 0.6 (Alexkseevsky-Perelomov) *There exists a 1-1 correspondence between $\mathfrak{t}$-roots ξ and irreducible submodules $\mathfrak{m}_\xi$ of the $Ad_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$ given by*

$$\Delta_{\mathfrak{t}} \ni \xi \mapsto \mathfrak{m}_\xi = \sum_{\{\alpha \in \Delta_{\mathfrak{m}} : \kappa(\alpha) = \xi\}} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

If we denote by $\Delta_{\mathfrak{t}}^{+}$ the set of all positive $\mathfrak{t}$-roots (this is the restricton of the root system Δ^{+} under the map κ), then

$$\mathfrak{m} = \sum_{\xi\in\Delta_{\mathfrak{t}}^{+}} (\mathfrak{m}_{\xi} + \mathfrak{m}_{-\xi})^{\tau} = \sum_{\xi_i\in\Delta_{\mathfrak{t}}^{+}=\{\xi_1,\dots,\xi_q\}} \mathfrak{m}_i \tag{14}$$

as $Ad_G(K)$-modules. Also,

$$\dim_{\mathbb{R}} \mathfrak{m}_i = 2\cdot \left|\{\alpha\in\Delta_{\mathfrak{m}}^{+} : \kappa(\alpha) = \xi_i\}\right|,$$

G-invariant Riemannian metrics g on G/K can be expressed as

$$g = \sum_{\xi\in\Delta_{\mathfrak{t}}^{+}} x_{\xi} B|_{(\mathfrak{m}_{\xi}+\mathfrak{m}_{-\xi})^{\tau}} = \sum_{i=1}^{q} x_{\xi_i} B|_{(\mathfrak{m}_{\xi_i}+\mathfrak{m}_{-\xi_i})^{\tau}} \tag{15}$$

for positive real numbers x_{ξ}, x_{ξ_i}. Thus G-invariant Riemannian metrics on $M = G/K$ are parametrized by q real positive parameters.

In computing the Ricci tensor for a generalized flag manifold $M = G/K$ by using Riemannian submersions we will also use the well known fact that M admits a finite number of G-invariant Kähler–Einstein metrics.

Let $\delta_{\mathfrak{m}} = \dfrac{1}{2}\displaystyle\sum_{\alpha\in\Delta^+_{\mathfrak{m}}} \alpha \in \sqrt{-1}\mathfrak{h}$ (Koszul form).

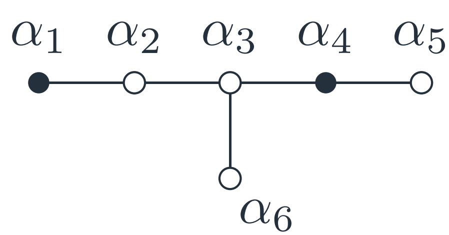
Then $2\delta_{\mathfrak{m}} = k_{\alpha_{i_1}}\Lambda_{\alpha_{i_1}} + \cdots + k_{\alpha_{i_r}}\Lambda_{\alpha_{i_r}}$, where $k_\alpha = \dfrac{2(2\delta_{\mathfrak{m}},\alpha)}{(\alpha,\alpha)}$ for $\alpha \in \Pi \setminus \Pi_0$.

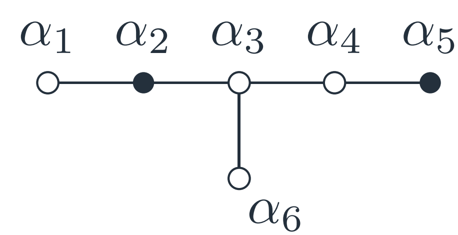
Proposition 0.7 *The G-invariant metric $g_{2\delta_{\mathfrak{m}}}$ on G/K corresponding to $2\delta_{\mathfrak{m}}$ is a Kähler-Einstein metric which is given by*

$$g_{2\delta_{\mathfrak{m}}} = \sum_{\xi\in\Delta^+_{\mathfrak{t}}} (2\delta_{\mathfrak{m}},\xi)B|_{(\mathfrak{m}_\xi+\mathfrak{m}_{-\xi})^\tau}$$

Case of E_6: (Type A)

Recall that the highest root $\tilde{\alpha}$ of E_6 is given by $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$. There are two pairs (Π, Π_0) of Type A, which determine flag manifolds with five isotropy summands, namely the choices $\Pi \backslash \Pi_0 = \{\alpha_1, \alpha_4\}$ and $\Pi \backslash \Pi_0 = \{\alpha_2, \alpha_5\}$. They correspond to the painted Dynkin diagrams





which both define the flag manifold $E_6/SU(4) \times SU(2) \times U(1)^2$. However, there is an outer automorphism of E_6 which makes these painted Dynkin diagrams equivalent (e.g. Bordeman et al). Thus we will not distinguish these two pairs (Π, Π_0) and we will work with the first one.

Let $\mathfrak{n}$ be the B-orthogonal complement of the isotropy subalgebra $\mathfrak{k}$ in $\mathfrak{e}_6$. The root system of the semisimple part of the isotropy subgroup K is given by $\Delta_0^+ = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_2+\alpha_3, \alpha_3+\alpha_6, \alpha_2+\alpha_3+\alpha_6\}$, thus

$$\Delta_{\mathfrak{n}}^+ = \left\{ \begin{array}{ll} \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6 & \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6 \\ \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+\alpha_6 & \alpha_1+\alpha_2+\alpha_3+\alpha_6 \\ \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6 & \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 \\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_6 & \alpha_3+\alpha_4+\alpha_5 \\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6 & \alpha_3+\alpha_4+\alpha_6 \\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6 & \alpha_3+\alpha_4 \\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_6 & \alpha_4+\alpha_5 \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6 & \alpha_4 \\ \alpha_1 & \alpha_3+\alpha_4+\alpha_5+\alpha_6 \end{array} \right. \tag{16}$$

Let $\alpha = \sum_{k=1}^{6} c_k\alpha_k \in \Delta_{\mathfrak{n}}^+$. Since $\Pi_{\mathfrak{n}} = \{\alpha_1, \alpha_4\}$, it follows that that $\kappa(\alpha) = c_1\overline{\alpha}_1 + c_4\overline{\alpha}_4$, where the numbers c_1, c_4 are such that $0 \leq c_1, c_4 \leq 2$.

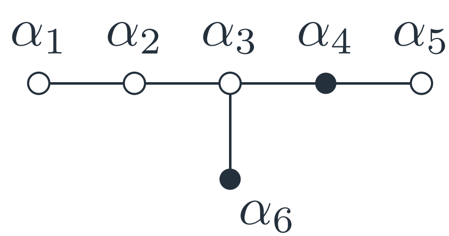
So, by using (16), we easily conclude that the positive $\mathfrak{t}$-roots are given by $\Delta(\mathfrak{n})^+_{\mathfrak{t}} = \{\overline{\alpha}_1,\ \overline{\alpha}_4,\ \overline{\alpha}_1 + \overline{\alpha}_4,\ 2\overline{\alpha}_4,\ \overline{\alpha}_1 + 2\overline{\alpha}_4\}$, and thus $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathfrak{n}_4 \oplus \mathfrak{n}_5$.

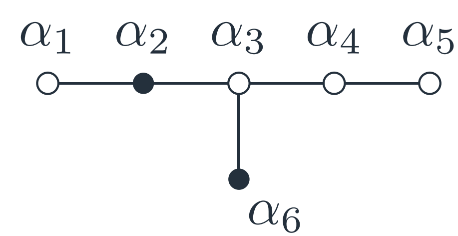
Also, we easily conclude that

$$\left.\begin{array}{lcl}
\dim_{\mathbb{R}} \mathfrak{n}_1 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{n}} : \kappa(\alpha) = \overline{\alpha}_1\}| = 2 \cdot 4 = 8, \\
\dim_{\mathbb{R}} \mathfrak{n}_2 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{n}} : \kappa(\alpha) = \overline{\alpha}_4\}| = 2 \cdot 12 = 24, \\
\dim_{\mathbb{R}} \mathfrak{n}_3 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{n}} : \kappa(\alpha) = \overline{\alpha}_1 + \overline{\alpha}_4\}| = 2 \cdot 8 = 16, \\
\dim_{\mathbb{R}} \mathfrak{n}_4 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{n}} : \kappa(\alpha) = 2\overline{\alpha}_4\}| = 2 \cdot 1 = 2, \\
\dim_{\mathbb{R}} \mathfrak{n}_5 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{n}} : \kappa(\alpha) = \overline{\alpha}_1 + 2\overline{\alpha}_4\}| = 2 \cdot 4 = 8.
\end{array}\right\} \tag{17}$$

Case of E_6: (Type B)

The flag manifold $E_6/SU(4)\times SU(2)\times U(1)^2$ is also defined by two pairs (Π,Π_0) of Type B, given by $\Pi\backslash\Pi_0=\{\alpha_4,\alpha_6\}$ and $\Pi\backslash\Pi_0=\{\alpha_2,\alpha_6\}$. They correspond to the painted Dynkin diagrams





Note that there is also an outer automorphism of E_6 which makes these painted Dynkin diagrams equivalent, and thus we can work with the first pair (Π,Π_0) only. By similar method we obtain that the positive $\mathfrak{t}$-roots are $\Delta(\mathfrak{m})^+_{\mathfrak{t}}=\{\overline{\alpha}_6,\ \overline{\alpha}_4,\ \overline{\alpha}_6+\overline{\alpha}_4,\ \overline{\alpha}_6+2\overline{\alpha}_4,\ 2\overline{\alpha}_6+2\overline{\alpha}_4\}$ and thus we obtain the decomposition $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4\oplus\mathfrak{m}_5$. where the dimensions of the submodules $\mathfrak{m}_i$ are given as follows:

Example B

$$\left.\begin{array}{rcl} \dim_{\mathbb{R}} \mathfrak{m}_1 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{m}} : \kappa(\alpha) = \overline{\alpha}_6\}| = 2 \cdot 4 = 8, \\ \dim_{\mathbb{R}} \mathfrak{m}_2 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{m}} : \kappa(\alpha) = \overline{\alpha}_4\}| = 2 \cdot 8 = 16, \\ \dim_{\mathbb{R}} \mathfrak{m}_3 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{m}} : \kappa(\alpha) = \overline{\alpha}_6 + \overline{\alpha}_4\}| = 2 \cdot 12 = 24, \\ \dim_{\mathbb{R}} \mathfrak{m}_4 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{m}} : \kappa(\alpha) = \overline{\alpha}_6 + 2\overline{\alpha}_4\}| = 2 \cdot 4 = 8, \\ \dim_{\mathbb{R}} \mathfrak{m}_5 & = & 2 \cdot |\{\alpha \in \Delta^+_{\mathfrak{m}} : \kappa(\alpha) = 2\overline{\alpha}_6 + 2\overline{\alpha}_4\}| = 2 \cdot 1 = 2. \end{array}\right\} \quad (18)$$

However We can show that these flag manifolds G/K of Type A and B are isometric as real manifolds, by an isometry arising from the action of the Weyl group of G.
Thus we study only flag manifolds of Type A.

Let $\Pi \backslash \Pi_0 = \Pi_{\mathfrak{n}} = \{\alpha_1, \alpha_4\}$. It is $2\delta_{\mathfrak{n}} = 5\Lambda_{\alpha_1} + 7\Lambda_{\alpha_4}$, Thus the Kähler Einstein metric $g_{2\delta_{\mathfrak{m}}}$ on G/K is given by

$$g_{2\delta_{\mathfrak{n}}} = 5B|_{\mathfrak{n}_1} + 7B|_{\mathfrak{n}_2} + 12B|_{\mathfrak{n}_3} + 14B|_{\mathfrak{n}_4} + 19B|_{\mathfrak{n}_5}.$$

Also, here $G = E_6$, $K = U(1) \times U(1) \times SU(2) \times SU(4)$,
$L = U(5) \times SU(2)$ and we have
$d_1 = 8, d_2 = 24, d_3 = 16, d_4 = 2, d_5 = 8$.

Thus by applying the expressions found earlier we obtain that

$$\begin{bmatrix} 5 \\ 23 \end{bmatrix} = 2, \quad \begin{bmatrix} 4 \\ 22 \end{bmatrix} = 1.$$

Since the Kähler Einstein metric $g_{2\delta_{\mathfrak{m}}}$ on G/K is given as above, we substitute the values $x_1 = 5$, $x_2 = 7$, $x_3 = 12$, $x_4 = 14$, $x_5 = 19$ into (8).

Consider the components r_2, r_3, r_4 and r_5 of the Ricci tensor for these values.

Then, from $r_2 - r_3 = 0$ and $r_4 - r_5 = 0$, we obtain that

$$\begin{bmatrix} 3 \\ 12 \end{bmatrix} = 2, \quad \begin{bmatrix} 5 \\ 14 \end{bmatrix} = \frac{1}{3}.$$

The classification of flag manifolds with five isotropy summands

Flag manifolds G/K of a simple Lie group G whose isotropy representation decomposes into a sum of five irreducible summands can be obtained by the following possible Dynkin diagrams:

(a) Paint black one simple root of Dynkin mark 5, that is

$$\Pi \backslash \Pi_0 = \{\alpha_p : \mathrm{Mrk}(\alpha_{\mathrm{p}}) = 5\}.$$

This case corresponds only to the flag manifold
$E_8/(U(1) \times SU(4) \times SU(5))$.
It was studied by Chrysikos-Sakane in a recent work in which they classified all flag manifolds M with $b_2(M) = 1$.
This space admits five non-Kähler Einstein metrics and a unique Kähler-Einstein metric.

(b) Paint black two simple roots, one of Dynkin mark 1 and one of Dynkin mark 2, that is

$$\Pi\backslash\Pi_0 = \{\alpha_i, \alpha_j : \mathrm{Mrk}(\alpha_\mathrm{i}) = 1,\ \mathrm{Mrk}(\alpha_\mathrm{j}) = 2\}. \qquad \textbf{Type A}$$

(c) Paint black two simple roots, both of Dynkin mark 2, that is

$$\Pi\backslash\Pi_0 = \{\alpha_i, \alpha_j : \mathrm{Mrk}(\alpha_\mathrm{i}) = \mathrm{Mrk}(\alpha_\mathrm{j}) = 2\}. \qquad \textbf{Type B}$$

For both cases $b_2(M) = 2$.

It can be shown that

Type A $\Rightarrow \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4,$ or
$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5.$

Type B $\Rightarrow \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5,$ or
$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6.$

The following table gives the pairs (Π, Π_0) of Type A and B, which determine flag manifolds G/K with $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_5$.

G Classical	$B_\ell = SO(2\ell+1)$	$D_\ell = SO(2\ell)$
Type A	$\Pi \backslash \Pi_0 = \{\alpha_1, \alpha_{p+1} : 2 \leq p \leq \ell - 1\}$	$\Pi \backslash \Pi_0 = \{\alpha_1, \alpha_{p+1} : 2 \leq p \leq \ell - 3\}$
Type B	$\Pi \backslash \Pi_0 = \{\alpha_p, \alpha_{p+1} : 2 \leq p \leq \ell - 1\}$	$\Pi \backslash \Pi_0 = \{\alpha_p, \alpha_{p+1} : 2 \leq p \leq \ell - 3\}$
G Exceptional	E_6	E_7
Type A	$\Pi \backslash \Pi_0 = \{\alpha_1, \alpha_4\}$	$\Pi \backslash \Pi_0 = \{\alpha_1, \alpha_7\}$
Type A	$\Pi \backslash \Pi_0 = \{\alpha_2, \alpha_5\}$	
Type B	$\Pi \backslash \Pi_0 = \{\alpha_4, \alpha_6\}$	$\Pi \backslash \Pi_0 = \{\alpha_6, \alpha_7\}$
Type B	$\Pi \backslash \Pi_0 = \{\alpha_2, \alpha_6\}$	

Since corresponding flag manifolds of Types A and B are isometric, it suffices to study only the following non isometric flag manifolds:

Generalized flag manifolds with five isotropy summands and $b_2(M) = 2$

$M = G/K$ classical	$M = G/K$ exceptional
$SO(2\ell+1)/U(1) \times U(p) \times SO(2(\ell-p-1)+1)$	$E_6/SU(4) \times SU(2) \times U(1)^2$
$SO(2\ell)/U(1) \times U(p) \times SO(2(\ell-p-1))$	$E_7/SU(6) \times U(1)^2$

MAIN THEOREM. (1) Let $M_1 = G_1/K_1$ be one of the flag manifolds $E_6/(SU(4) \times SU(2) \times U(1) \times U(1))$ or $E_7/(U(1) \times U(6))$.
Then M_1 admits exactly seven G_1-invariant Einstein metrics up to isometry.
There are two Kähler-Einstein metrics and five non Kähler metrics (up to scalar).

(2) Let $M_2 = G_2/K_2$ be one of the flag manifolds $SO(2\ell+1)/(U(1) \times U(p) \times SO(2(\ell-p-1)+1))$ or $SO(2\ell)/(U(1) \times U(p) \times SO(2(\ell-p-1)))$.
Then M_2 admits at least two G_2-invariant non Kähler-Einstein metrics up to isometry.

Solutions of algebraic systems of equations

For flag manifolds with five isotropy summands the Einstein equation reduces to an algebraic system of four equations with four unknowns.

These systems are difficult to be solved, especially in the cases where the coefficents depend on parameters (this happens for the flag manifolds of a classical Lie group). In this cases we only prove existence of a certain number of solutions.
For flag manifolds of an exceptional Lie group it is possible to obtain numerical solutions, however there is one case ($E_8/U(1)\times SU(2)\times SU(3)\times SU(5)$ with six isotropy summands and $b_2(M)=1$) where we can not obtain solutions (even numerical!).

To obtain numerical solutions or prove existence of solution for parametric systems of equations we use methods of Gröbner bases.

Case of E_6

Consider $M = E_6/SU(4)\times SU(2)\times U(1)^2$.
The components r_i ($i = 1, \cdots, 5$) of the Ricci tensor for a G-invariant Riemannian metric (7) on G/K are given as follows:

$$
\begin{aligned}
r_1 &= \frac{1}{2x_1} + \frac{1}{8}\Big(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\Big) + \frac{1}{48}\Big(\frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4} - \frac{x_4}{x_1x_5}\Big),\\
r_2 &= \frac{1}{2x_2} + \frac{1}{24}\Big(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\Big) - \frac{1}{48}\frac{x_4}{{x_2}^2} + \frac{1}{24}\Big(\frac{x_2}{x_3x_5} - \frac{x_5}{x_2x_3} - \\
r_3 &= \frac{1}{2x_3} + \frac{1}{16}\Big(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\Big) + \frac{1}{16}\Big(\frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5}\Big),\\
r_4 &= \frac{1}{2x_4} + \frac{1}{12}\Big(\frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5}\Big) + \frac{1}{8}\Big(-\frac{2}{x_4} + \frac{x_4}{{x_2}^2}\Big),\\
r_5 &= \frac{1}{2x_5} + \frac{1}{8}\Big(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\Big) + \frac{1}{48}\Big(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\Big).
\end{aligned}
\tag{19}
$$

We consider the system of equations:

$$r_1 = r_5, \;\; r_2 = r_3, \;\; r_3 = r_4, \;\; r_4 = r_5. \tag{20}$$

From $r_1 - r_5 = 0$, we see that

$$(x_1 - x_5)\left(x_1 x_2 x_3 + 3x_1 x_4 x_5 + 3{x_2}^2 x_4 - 12 x_2 x_3 x_4 + x_2 x_3 x_5 + 3{x_3}^2 x_4\right) = \tag{21}$$

Case of $x_5 = x_1$. We obtain four non isometric Einstein metrics (non Kähler) (Not presented here).

Case of $x_5 \neq x_1$. We normalize our equations by setting $x_1 = 1$. We see that the system of polynomial equations (20) reduces to the following system of polynomial equations:

We normalize our equations by setting $x_1 = 1$. We see that the system of polynomial equations (20) reduces to the following system of polynomial equations:

$$
\begin{aligned}
p_1 &= -8{x_2}^3 x_4 x_5 - 2{x_2}^3 x_4 - {x_2}^2 x_3 {x_4}^2 + 24{x_2}^2 x_3 x_4 x_5 - {x_2}^2 x_3 {x_5}^2 + {x_2}^2 \\
&\quad +2x_2 {x_3}^2 x_4 - 24 x_2 x_3 x_4 x_5 + 2x_2 x_4 {x_5}^2 + 8x_2 x_4 x_5 + x_3 {x_4}^2 x_5 = 0, \\
p_2 &= 5{x_2}^3 x_5 + 5{x_2}^3 - 24{x_2}^2 x_5 - 5x_2 {x_3}^2 x_5 - 5x_2 {x_3}^2 + 24 x_2 x_3 x_5 + x_2 x_5 \\
p_3 &= -3{x_2}^3 x_4 x_5 - 3{x_2}^3 x_4 - 4{x_2}^2 x_3 {x_4}^2 + 4{x_2}^2 x_3 {x_5}^2 - 12{x_2}^2 x_3 x_5 + 4x_2 \\
&\quad +3x_2 {x_3}^2 x_4 x_5 + 3x_2 {x_3}^2 x_4 - 3x_2 x_4 {x_5}^2 - 3x_2 x_4 x_5 - 6x_3 {x_4}^2 x_5 = 0, \\
p_4 &= 3{x_2}^2 x_4 - 12 x_2 x_3 x_4 + x_2 x_3 x_5 + x_2 x_3 + 3{x_3}^2 x_4 + 3x_4 x_5 = 0.
\end{aligned}
\tag{22}
$$

To find non zero solutions of equations (22), we consider a polynomial ring $R_2 = \mathbb{Q}[y, x_2, x_3, x_4, x_5]$ and an ideal I_2 generated by

$$\{p_1,\ p_2,\ p_3,\ p_4,\ y x_2 x_3 x_4 x_5 - 1\}.$$

We take a lexicographic order $>$ with $y > x_2 > x_5 > x_3 > x_4$ for a monomial ordering on R_2. Then a Gröbner basis for the ideal I_2 contains a polynomial

$$(5x_4 - 22)(5x_4 - 14)(17x_4 - 22)(19x_4 - 14)q_1,$$

where

$$\begin{aligned} q_1 = {} & 2568494494835430820312 5x_4^{24} - 312330714783423219879187500x_4^{23} \\ & -14789576030598686784365775000x_4^{22} + 169312435225853499159893370000x_4^{21} \\ & +16683190004942830656862088400 00x_4^{20} - 8641784992792389994443258331200x_4^{19} \\ & -10861158787935440551542665216640x_4^{18} + 87429206357937887857587009061632x_4^{17} \\ & -32949087665531461793795791137024x_4^{16} - 302754123930816030608716028461056x_4^{15} \\ & +377294987073145336781487843082240x_4^{14} + 229461889322385205525089296121856x_4^{13} \\ & -745488535262100375331097397100544x_4^{12} + 464674752074856427879419685109760x_4^{11} \\ & +120588308696762557788249740279808x_4^{10} - 332437403867399257596854179725312x_4^{9} \\ & +206232698781395558570755696361472x_4^{8} - 60625111325239908567111130152960x_4^{7} \\ & +5786387485742898687693985677312x_4^{6} + 1618103684685636757652930297856x_4^{5} \\ & -597859726821790689492624998400x_4^{4} + 84059799581674625557541683200x_4^{3} \\ & -2979131989754489205686272000x_4^{2} - 18429108055331433349120000 00x_4 \\ & +3336221218939338752000000 00. \end{aligned} \tag{23}$$

For the case when $(5x_4-22)(5x_4-14)(17x_4-22)(19x_4-14)=0$, we consider ideals I_3, I_4, I_5, I_6 of the polynomial ring $R_2=\mathbb{Q}[y,x_2,x_3,x_4,x_5]$ generated by

$$\{p_1,\ p_2,\ p_3,\ p_4,\ y,\ x_2x_3x_4x_5-1, 5x_4-22\},\quad \{p_1,\ p_2,\ p_3,\ p_4,\ y,\ x_2x_3x_4x_5-1, 5x_4-14\},$$
$$\{p_1,\ p_2,\ p_3,\ p_4,\ y,\ x_2x_3x_4x_5-1, 17x_4-22\},\quad \{p_1,\ p_2,\ p_3,\ p_4,\ y,\ x_2x_3x_4x_5-1, 17x_4-14\}$$

respectively.
We take a lexicographic order $>$ with $y>x_2>x_5>x_3>x_4$ for a monomial ordering on R_2. Then Gröbner bases for the ideals I_3, I_4, I_5, I_6 contain polynomials

$$\{5x_4-22, 5x_3-6, 5x_5-17, 5x_2-11\},\quad \{5x_4-14, 5x_3-12, 5x_5-19, 5x_2-7\},$$
$$\{17x_4-22, 17x_3-6, 17x_5-5, 17x_2-11\},\quad \{19x_4-14, 19x_3-12, 19x_5-5, 19x_2-7\}.$$

respectively. Thus we obtain the following solutions of equations (22):

$$1)\ x_1 = 1, x_2 = \frac{11}{5}, x_3 = \frac{6}{5}, x_4 = \frac{22}{5}, x_5 = \frac{17}{5}, \qquad 2)\ x_1 = 1, x_2 = \frac{7}{5}, x_3 = \frac{12}{5}, x_4 = \frac{14}{5}, x_5 =$$

$$3)\ x_1 = 1, x_2 = \frac{11}{17}, x_3 = \frac{6}{17}, x_4 = \frac{22}{17}, x_5 = \frac{5}{17}, \qquad 4)\ x_1 = 1, x_2 = \frac{7}{19}, x_3 = \frac{12}{19}, x_4 = \frac{14}{19}, x_5 = \qquad (24)$$

We normalize these solutions as follows:

1) $x_1 = 5, x_2 = 11, x_3 = 6, x_4 = 22, x_5 = 17,$ 2) $x_1 = 5, x_2 = 7, x_3 = 12, x_4 = 14, x_5 = 19,$
3) $x_1 = 17, x_2 = 11, x_3 = 6, x_4 = 22, x_5 = 5,$ 4) $x_1 = 19, x_2 = 7, x_3 = 12, x_4 = 14, x_5 = 5.$

and we get Kähler Einstein metrics for these values of x_i's. Note that the metrics corresponding to the cases 1) and 3) are isometric and the cases 2) and 4) are isometric.
For the case when $q_1 = 0$ and
$(5x_4 - 22)(5x_4 - 14)(17x_4 - 22)(19x_4 - 14) \neq 0$, we consider a ideal I_7 of the polynomial ring $R_2 = \mathbb{Q}[y, x_2, x_3, x_4, x_5]$ generated by

$$\{p_1,\ p_2,\ p_3,\ p_4,\ y(5x_4{-}22)(5x_4{-}14)(17x_4{-}22)(19x_4{-}14)x_2x_3x_4x_5{-}1\}.$$

We take the same lexicographic order $>$ with $y > x_2 > x_5 > x_3 > x_4$ for a monomial ordering on R_2. Then a Gröbner basis for the ideal I_7 contains the polynomial q_1 and polynomials of the form

$$b_2x_2 + v_2(x_4), \quad b_3x_3 + v_3(x_4), \quad b_5x_5 + v_5(x_4) \qquad (25)$$

where b_2, b_3, b_5 are positive integers and $v_2(x_4), v_3(x_4), v_5(x_4)$ are polynomials of degree 23 with integer coefficients.
By solving the equation $q_1 = 0$ for x_4 numerically, we obtain exactly 6 positive solutions, 8 negative solutions and 10 non-real solutions. The 6 positive solutions are approximately given by

$$1)\ x_4 \approx 1.157018562397866, \quad 2)\ x_4 \approx 2.075646788197390, \quad 3)\ x_4 \approx 2.145057741729789,$$
$$4)\ x_4 \approx 2.163849575049888, \quad 5)\ x_4 \approx 12.97930323340096, \quad 6)\ x_4 \approx 12207.19468694106.$$

We substitute the values for x_4 into the equations $b_2x_2 + v_2(x_4) = 0,\ b_3x_3 + v_3(x_4) = 0,\ b_5x_5 + v_5(x_4) = 0$. Then we obtain the following values approximately:

$$1)\ x_4 \approx 1.15702, x_2 \approx 0.641194, x_3 \approx 0.566074, x_5 \approx 0.557426,$$
$$2)\ x_4 \approx 2.07565, x_2 \approx 1.15028, x_3 \approx 1.01551, x_5 \approx 1.79396,$$
$$3)\ x_4 \approx 2.14506, x_2 \approx 8.87367, x_3 \approx 33.3409, x_5 \approx -1.12628,$$
$$4)\ x_4 \approx 2.16385, x_2 \approx 27.3523, x_3 \approx 7.26471, x_5 \approx -1.16127,$$
$$5)\ x_4 \approx 12.9793, x_2 \approx 1.3699, x_3 \approx 5.42602, x_5 \approx -1.49194,$$
$$6)\ x_4 \approx 12207.2, x_2 \approx 18.0447, x_3 \approx 1.46532, x_5 \approx -221.833.$$

Thus we see that only cases 1) and 2) correspond to Einstein metrics. We substitute these values for $\{x_1, x_2, x_3, x_4, x_5\}$ into (19) and get

$$1)\ r_1 = r_2 = r_3 = r_4 = r_5 \approx 0.31855, \qquad 2)\ r_1 = r_2 = r_3 = r_4 = r_5 \approx 0.571467. \tag{26}$$

Thus we obtain two Einstein metrics with Einstein constant 1:

$$\begin{array}{ll} 1) & x_1 \approx 0.31855, x_2 \approx 0.366421, x_3 \approx 0.323492, x_4 \approx 0.661198, x_5 \approx 0.571467, \\ 2) & x_1 \approx 0.571467, x_2 \approx 0.366421, x_3 \approx 0.323492, x_4 \approx 0.661198, x_5 \approx 0.31855. \end{array} \tag{27}$$

Now we see that these two metrics are isometric.

Case of E_6

Theorem 0.8 *The flag manifold $E_6/(SU(4)\times SU(2)\times U(1)\times U(1))$ admits exactly seven E_6-invariant Einstein metrics up to isometry. There are two Kähler-Einstein metrics (up to scalar) given by*

$$\{x_1=5, x_2=7, x_3=12, x_4=14, x_5=19\}, \quad \{x_1=5, x_2=11, x_3=6, x_4=22, x_5=17\}.$$

The other five are non-Kähler. These metrics are given approximately by

$$\{x_1\approx 0.571467, x_2\approx 0.366421, x_3\approx 0.323492, x_4\approx 0.661198, x_5\approx 0.31855\}, \tag{28}$$

$$\{x_1\approx 0.49572094, x_2\approx 0.39385688, x_3\approx 0.30158949, x_4\approx 0.093299706, x_5\approx 0.49572094\}, \tag{29}$$

$$\{x_1\approx 0.29495775, x_2\approx 0.40303263, x_3\approx 0.48143674, x_4\approx 0.10093004, x_5\approx 0.29495775\}, \tag{30}$$

$$\{x_1\approx 0.47024404, x_2\approx 0.35268279, x_3\approx 0.31380214, x_4\approx 0.62760315, x_5\approx 0.47024404\}, \tag{31}$$

$$\{x_1\approx 0.26465483, x_2\approx 0.42092053, x_3\approx 0.43231982, x_4\approx 0.42390247, x_5\approx 0.26465483\}, \tag{32}$$