Abelian complex structures and related geometries

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Plan

- Abelian complex structures.
- Relation to HKT geometry.
- Affine Lie algebras and abelian double products.
- Kähler Lie algebras with abelian complex structures.
- The first canonical connection.
- Flat complex connections with $(1, 1)$-torsion.

Abelian complex structures

A complex structure J on a real Lie algebra $\mathfrak{g}$ is called *abelian* when it satisfies:

$$[Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g}. \tag{1}$$

Equivalently, $\mathfrak{g}^{1,0}$ is an abelian subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

If G is a Lie group with Lie algebra $\mathfrak{g}$ these conditions imply the vanishing of the Nijenhuis tensor on the invariant almost complex manifold (G, J), that is, J is integrable on G.

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Relation to HKT geometry

- A *hyperhermitian* structure on a smooth manifold M is $(\{J_\alpha\}_{\alpha=1,2,3}, g)$, where
  1. $\{J_\alpha\}_{\alpha=1,2,3}$ are complex structures such that
  $$J_1 J_2 = -J_2 J_1 = J_3,$$
  2. g is a Riemannian metric which is Hermitian with respect to J_α, $\alpha = 1, 2, 3$.

- Given a hyperhermitian structure $(\{J_\alpha\}_{\alpha=1,2,3}, g)$ on M, g is called *hyper-Kähler with torsion* (HKT) if there exists a connection ∇ on M satisfying
  1. $\nabla g = 0, \quad \nabla J_\alpha = 0, \alpha = 1, 2, 3,$
  2. the torsion tensor $c(X, Y, Z) = g(X, T(Y, Z))$ is skew-symmetric.

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Relation to HKT geometry

This class of metrics has been introduced by P.S. Howe - G.Papadopoulos (1996).

- Dotti - Fino (2002): If G is a 2-step nilpotent Lie group with a left invariant HKT structure $(\{J_\alpha\}_{\alpha=1,2,3}, g)$, then the hypercomplex structure is abelian.
- B - Dotti - Verbitsky (2009): Let $(N, \{J_\alpha\}_{\alpha=1,2,3}, g)$ be an HKT nilmanifold such that $\{J_\alpha\}$ is left invariant. Then the hypercomplex structure $\{J_\alpha\}$ is abelian.

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- Let $(\mathcal{A}, \cdot)$ be a finite dimensional associative, commutative algebra. Set $\mathfrak{aff}(\mathcal{A}) := \mathcal{A} \oplus \mathcal{A}$ with Lie bracket:

$$[(a, a'), (b, b')] = (0, a \cdot b' - b \cdot a'), \qquad a, b, a', b' \in \mathcal{A},$$

In particular, when $\mathcal{A} = \mathbb{R}$ or $\mathcal{A} = \mathbb{C}$, we obtain the Lie algebra of the group of affine motions of either $\mathbb{R}$ or $\mathbb{C}$.

- Let J be the endomorphism of $\mathfrak{aff}(\mathcal{A})$ defined by

$$J(a, a') = (a', -a), \qquad a, a' \in \mathcal{A}.$$

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Properties of Lie algebras carrying abelian complex structures

Let J be an abelian complex structure on the Lie algebra $\mathfrak{g}$. Then:

(i) The center $\mathfrak{z}$ of $\mathfrak{g}$ is J-stable.

(ii) For any $x \in \mathfrak{g}$, $\mathrm{ad}_{Jx} = -\mathrm{ad}_x J$.

(iii) $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is abelian, equivalently, $\mathfrak{g}$ is 2-step solvable [Petravchuk, 1988].

(iv) $J\mathfrak{g}'$ is an abelian subalgebra.

(v) $\mathfrak{g}' \cap J\mathfrak{g}' \subseteq \mathfrak{z}(\mathfrak{g}'_J)$, where $\mathfrak{g}'_J := \mathfrak{g}' + J\mathfrak{g}'$.

Abelian double products

[Andrada-Salamon, 2005] Consider a finite dimensional real vector space $\mathcal{A}$ with two structures of commutative associative algebra, $(\mathcal{A}, \cdot)$ and $(\mathcal{A}, *)$, with the following compatibility conditions:

$$a * (b \cdot c) = b * (a \cdot c), \qquad a \cdot (b * c) = b \cdot (a * c), \qquad (2)$$

for every $a, b, c \in \mathcal{A}$.

Then, $\mathcal{A} \oplus \mathcal{A}$ with the bracket:

$$[(a, a'), (b, b')] = (-(a*b' - b*a'), a \cdot b' - b \cdot a'), \qquad a, b, a', b' \in \mathcal{A},$$

is a Lie algebra denoted by $(\mathcal{A}, \cdot) \bowtie (\mathcal{A}, *)$ and the endomorphism J defined by

$$J(a, a') = (-a', a), \qquad a, a' \in \mathcal{A}, \qquad (3)$$

is an abelian complex structure, called the *standard* complex structure on $(\mathcal{A}, \cdot) \bowtie (\mathcal{A}, *)$.

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More examples

We show next that there is a large family of Lie algebras with abelian complex structure which are not abelian double products.

Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{v}$ where $\mathfrak{a} = \text{span}\{f_1, f_2\}$ and $\mathfrak{v}$ is a $2n$-dimensional real vector space. We fix an endomorphism J of $\mathfrak{g}$ such that $J^2 = -I$, $Jf_1 = f_2$ and $\mathfrak{v}$ is J-stable. Given a linear isomorphism T of $\mathfrak{v}$ commuting with $J|_{\mathfrak{v}}$, we define a Lie bracket on $\mathfrak{g}$ such that $\mathfrak{a}$ is an abelian subalgebra, $\mathfrak{v}$ is an abelian ideal and the bracket between elements in $\mathfrak{a}$ and $\mathfrak{v}$ is given by:

$$[f_1, v] = TJ(v), \qquad [f_2, v] = T(v), \qquad \text{for every } v \in \mathfrak{v}.$$

It turns out that J is an abelian complex structure on $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is not an abelian double product, unless $n = 1$. In this case, $\mathfrak{g} = \mathfrak{aff}(\mathbb{C})$ with an abelian complex structure which is NOT the standard one.

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Theorem (Andrada -B - Dotti, 2011)

Let $\mathfrak{g}$ be a solvable Lie algebra with an abelian complex structure J such that $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = \mathfrak{u} + J\mathfrak{u}$. Then:

(i) *if $\mathfrak{u}$ is an abelian subalgebra of $\mathfrak{g}$ then $\mathfrak{g} = \mathfrak{a} \oplus J\mathfrak{a}$ is an abelian double product with $\mathfrak{a} \subset \mathfrak{u}$;*

(ii) *if $\mathfrak{u}$ is an abelian ideal of $\mathfrak{g}$ and, moreover, $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$, then $(\mathfrak{g}, J)$ is holomorphically isomorphic to $\mathfrak{aff}(\mathcal{A})$ for some commutative associative algebra $(\mathcal{A}, \cdot)$.*

Corollary

Let $\mathfrak{g}$ be a solvable Lie algebra with an abelian complex structure J. Then:

1. *$\mathfrak{g}'_J$ is an abelian double product and if $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$, then $(\mathfrak{g}'_J, J)$ is holomorphically isomorphic to $\mathfrak{aff}(\mathcal{A})$ for some commutative associative algebra $(\mathcal{A}, \cdot)$;*
2. *if $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}'$, then $\mathfrak{g} = \mathfrak{u} \oplus J\mathfrak{u}$ is an abelian double product for some subalgebra $\mathfrak{u} \subset \mathfrak{g}'$.*

Kähler Lie algebras with abelian complex structure

Let $(\mathfrak{g}, J, g)$ be a Kähler Lie algebra with J abelian. It can be shown that:

(i) $\mathfrak{z} = (\mathfrak{g}'_J)^\perp$.

(ii) $(\mathfrak{g}')^\perp$ is abelian.

(iii) $\mathrm{ad}_z|_{\mathfrak{g}'}$ is symmetric for all $z \in \mathfrak{g}$.

Theorem (Andrada - B -Dotti, 2011)

Let $(\mathfrak{g}, J, g)$ be a Kähler Lie algebra with J an abelian complex structure. Then $\mathfrak{g}$ is isomorphic to

$$\mathfrak{aff}(\mathbb{R}) \times \cdots \times \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}^{2s},$$

and this decomposition is orthogonal and J-stable.

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Corollary

Let G be a simply connected Lie group equipped with a left-invariant Kähler structure (J, g) such that J is abelian. If the commutator subgroup is n-dimensional and the center is $2s$-dimensional, then

$$G = H^2(-c_1) \times \cdots \times H^2(-c_n) \times \mathbb{R}^{2s},$$

where $c_i > 0$, $i = 1, \ldots, n$, and $H^2(-c_i)$ denotes the 2-dimensional hyperbolic space of constant curvature $-c_i$.

Corollary

Let $M = \Gamma \backslash G$ be a compact quotient with a left invariant Kähler structure (J, g) such that J is abelian. Then M is diffeomorphic to a torus.

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The first canonical Hermitian connection

Given a Hermitian Lie algebra $(\mathfrak{g}, J, g)$, consider the connection ∇^1 on $\mathfrak{g}$ defined by

$$g\left(\nabla^1_x y, z\right) = g\left(\nabla^g_x y, z\right) + \frac{1}{4}\left(d\omega(x, Jy, z) + d\omega(x, y, Jz)\right),$$

where ω is the Kähler form. This connection satisfies

$$\nabla^1 g = 0, \quad \nabla^1 J = 0, \quad T^1 \text{ is of type } (1,1).$$

∇^1 is known as the first canonical Hermitian connection associated to $(\mathfrak{g}, J, g)$ [Lichnerowicz, 1962].

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Another expression for ∇^1 [Agricola, 2005]:

$$\nabla^1_x y := \nabla^g{}_x y + \frac{1}{2}(\nabla^g{}_x J)\, Jy = \frac{1}{2}(\nabla^g{}_x y - J\nabla^g{}_x Jy),$$

for $x, y \in \mathfrak{g}$. We write the above equation with any affine connection ∇ and define

$$\overline{\nabla}_x y := \nabla_x y + \frac{1}{2}(\nabla_x J)\, Jy = \frac{1}{2}(\nabla_x y - J\nabla_x Jy), \qquad (4)$$

for $x, y \in \mathfrak{g}$.

$\overline{\nabla}$ satisfies:

- $\overline{\nabla} J = 0$
- if ∇ is torsion-free, then $\overline{T}(x, y) = \overline{T}(Jx, Jy)$, i.e. $\overline{T}$ is of type $(1, 1)$ with respect to J.

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Lemma

Let ∇ be a torsion-free connection and J a complex structure on $\mathfrak{g}$. Assume that $\overline{\nabla} = 0$, that is, $\nabla_x J = -J\nabla_x$ for every $x \in \mathfrak{g}$. Then J is abelian.

Theorem (Andrada - B - Dotti, 2011)

Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra such that its associated first canonical connection ∇^1 satisfies $\nabla^1_x y = 0$ for every $x, y \in \mathfrak{g}$, that is, ∇^1 coincides with the $(-)$-connection. Then $\mathfrak{g}$ is abelian.

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Corollary

Let $M = \Gamma\backslash G$ be a compact quotient of a simply connected Lie group G by a discrete subgroup Γ. If (J, g) is a left invariant Hermitian structure on M such that its first canonical connection ∇^1 coincides with the connection ∇^0, then M is diffeomorphic to a torus.

Flat first canonical connection

Lemma

Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra with J abelian. If the associated first canonical connection ∇^1 is flat, then $\mathfrak{z} \cap \mathfrak{g}' = \{0\}$.

Theorem

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Let ∇ be an affine connection on a manifold M with torsion tensor field T and J an almost complex structure on M. The Nijenhuis tensor of J can be calculated as follows:

$$
\begin{aligned}
N(X,Y) &= (\nabla_{JX}J)\,Y - (\nabla_{JY}J)\,X + (\nabla_X J)\,JY - (\nabla_Y J)\,JX \\
&\quad + T(X,Y) - T(JX,JY) + J\,(T(JX,Y) + T(X,JY)),
\end{aligned}
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for all X, Y vector fields on M.

Lemma

Let (M,J) be an almost complex manifold with a complex connection ∇. Then J is integrable if and only if the torsion T of ∇ satisfies:

$$T(X,Y) - T(JX,JY) + J\,(T(JX,Y) + T(X,JY)) = 0,$$

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Proposition

Let (M, J) be an almost complex manifold.

(i) *If ∇ is a complex connection on M whose torsion is of type $(1, 1)$ with respect to J, then J is integrable.*

(ii) *If J is integrable, then there exists a complex connection ∇ whose torsion is of type $(1, 1)$ with respect to J.*

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(ii) *If J is integrable, then there exists a complex connection ∇ whose torsion is of type $(1,1)$ with respect to J.*

Let M be an n-dimensional connected manifold and ∇ an affine connection on M with trivial holonomy. Then the space $\mathcal{P}^{\nabla}$ of parallel vector fields on M is an n-dimensional real vector space.

$$T(X,Y) = -[X,Y], \qquad \text{for all } X, Y \in \mathcal{P}^{\nabla}.$$

Well known result:

Lemma

The space $\mathcal{P}^{\nabla}$ of parallel vector fields is a Lie subalgebra of $\mathfrak{X}(M)$ if and only if the torsion T of ∇ is parallel.

Complex connections with trivial holonomy

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Well known result:

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The next result gives equivalent conditions for an affine connection with trivial holonomy on an almost complex manifold to be complex.

Lemma

Let M, $\dim M = 2n$, be a connected manifold with an almost complex structure J. Assume that there exists an affine connection ∇ on M with trivial holonomy. Then the following conditions are equivalent:

(i) $\nabla J = 0$;

(ii) *the space $\mathcal{P}^\nabla$ of parallel vector fields is J-stable;*

(iii) *there exist parallel vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of M.*

Flat complex connections with $(1,1)$-torsion

Proposition

Let M be a connected $2n$-dimensional manifold with an almost complex structure J. Then the following conditions are equivalent:

(i) *there exist vector fields $X_1, \dots, X_n, JX_1, \dots, JX_n$, linearly independent at every point of M, such that*

$$[X_k, X_l] = [JX_k, JX_l], \qquad [JX_k, X_l] = -[X_k, JX_l], \qquad k < l;$$

(ii) *there exist n commuting vector fields $Z_1, \dots, Z_n$ which are linearly independent sections of $T^{1,0}M$ at every point of M;*

(iii) *there exist n linearly independent $(1,0)$-forms $\alpha_1, \dots, \alpha_n$ on M such that $d\alpha_i$ is a section of $\Lambda^{1,1}M$ for every i;*

(iv) *there exists a complex connection ∇ on M with trivial holonomy whose torsion tensor field T is of type $(1,1)$.*

Moreover, any of the above conditions implies that J is integrable.

An affine connection ∇ on a connected almost complex manifold (M, J) is called an *abelian* connection if it satisfies condition (iv) of the previous Proposition.

Corollary

Let (M, J) be a connected complex manifold and ∇ an affine connection with trivial holonomy. Then ∇ is an abelian connection on (M, J) if and only if the space $\mathcal{P}^\nabla$ of parallel vector fields is J-stable and J satisfies

$$[JX, JY] = [X, Y] \quad \textit{for any} \quad X, Y \in \mathcal{P}^\nabla.$$

Complete abelian connections with parallel torsion

Theorem

Let ∇ be an abelian connection on a connected complex manifold (M, J) such that ∇ is complete and the torsion tensor field T is parallel. Then (M, J, ∇) is equivalent to $(\Gamma \backslash G, J_0, \nabla^0)$, where G is a simply connected Lie group equipped with a left invariant abelian complex structure and $\Gamma \subset G$ is a discrete subgroup.