Riemannian 4-manifolds with ‘small’ holonomy

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Concern

surfaces with 1-dimensional Chern holonomy

Key interplay : *complex structures* $\leftrightsquigarrow$ *curvature*

Core results on (M^4, g, I) Hermitian:

$*$ Ricci I-invariant, or lcK $\Longrightarrow$ W^+ degenerate
$\Longleftarrow$ if compact

[Apostolov–Gauduchon 97]

$*$ Einstein $(W^+ \not\equiv 0) \Longrightarrow$ lcK, $\exists$ Hamiltonian Killing field

[Derdziński 83, Boyer 86, Nurowski 93, AG 97]

Our work:

- $Ric \in \Omega_I^{1,1} \iff W^+$ degenerate, yet M^4 may still not be lcK
- **local classification** $\rightsquigarrow$ explicit constructions!

(esp. *non-compact* surfaces)

often arising from Kähler surfaces of Calabi type see

Plan

- scenario: complex structures in dimension 4
- what small holonomy means
- the classification

Let M^4 be a real, oriented, smooth 4-manifold (a 'surface') with an almost Hermitian structure (g, I) (I is an OCS)

$$I^2 = -Id_{TM}, \qquad g(I\cdot, I\cdot) = g(\cdot, \cdot) > 0, \qquad \omega_I = g(I\cdot, \cdot).$$

- The bundle of real 2-forms decomposes as

$$\Omega^2 = \Omega^{1,1} \oplus \Omega^{\{2,0\}} = \mathbb{R}\,\omega_I \oplus \Omega_0^{1,1} \oplus K_M$$

- Paramount feature

$$\Omega^2 = \Omega^+ \oplus \Omega^-$$

so

$$\boldsymbol{\Omega^+ = \mathbb{R}\,\omega_I \oplus K_M, \qquad \Omega^- = \Omega_0^{1,1}}$$

Similarly for curvature: $R = s \oplus Ric_0 \oplus W^+ \oplus W^- \in Sym^2(\Lambda^2)$.

Orthogonal complex structures

A Kähler surface (M^4, g, J) is naturally oriented, say $\omega_J \in \Omega^-$.

Are there 'interesting' structures on Ω^+ ?

cf. hypercomplex, hypersymplectic, ...

[Hitchin 90, Salamon 91, Geiges-Gonzalo 95]

[Pontecorvo 97, Kamada 99, Bande-Kotschick 06...]

But what about *existence*? Well,

if W^+ is co-closed can define an OCS I such that $\omega_I \in \Omega^+$, plus

- I and J commute
- lcK $(d\theta = 0)$ $(d\omega_I = \theta \wedge \omega_I)$
- θ is preserved by $I \circ J$

Kähler surfaces with $\delta W^+ = 0$ are called *weakly self-dual*, and were defined and classified by **[Apostolov–Calderbank–Gauduchon 03]**

Canonical connection

Geometry of any almost Hermitian (M^{2n}, g, I) determined by (θ or)

$$\eta = \tfrac{1}{2}(\nabla I)I \in \Omega^1 \otimes \Omega^{\{2,0\}}$$

eg: (g, I) Hermitian $\iff \eta \in \Omega^{1,1} \otimes \Omega^1$

Consider

$$\overline{\nabla} \;=\; \nabla \;+\; \eta$$

- metric, Hermitian, with torsion $T(X, Y) = \eta_X Y - \eta_Y X$
- called 2nd *canonical* Hermitian connection
- $\overline{\nabla} = \nabla^{\text{Chern}}$ when M complex cf. **[Gauduchon 97]**

Corresponding curvature: $\overline{R} = W^- + \frac{s}{12} Id_{\Omega^-} + \frac{1}{2} Ric_0^{1,1} + \frac{1}{2}\,\overline{\gamma} \otimes \omega_I$

$$\overline{\gamma} = \rho^I + W^+\omega_I + \tfrac{1}{2} d^+\theta - \tfrac{s}{6}\omega_I$$ (essentially, first Chern form)

cf. $\overline{R}/R$ comparison of **[Cleyton-Swann 04]**

"Small" curvature

Holonomy algebra generated by $\overline{R}(X, Y) \in \Omega^{1,1}$

Interested in the case: M^4 with

$$\mathfrak{hol}(\overline{\nabla}) \;\subset\; \Omega_0^{1,1} \oplus \mathbb{R} \;\subset\; \Omega^- \oplus \Omega^+$$

of **dimension 1 at most**:

$$\overline{R} = \tfrac{1}{2}\,\overline{\gamma} \otimes (\mathbf{F_0} + \alpha\, \omega_{\mathbf{I}})$$

Three rather different situations: $\begin{cases} \overline{R} & \equiv & 0 \\ F_0 & \equiv & 0 \\ F_0 & \neq & 0 \end{cases}$

Chern-flat surfaces

Proposition

(M^4, g, I) almost Hermitian with $\overline{R} = 0 \implies g$ flat.

Better: if (σ_i) is a ∇-parallel ON basis of Ω^+,

$$\omega_I = \sigma_1 \cos\varphi \cos\psi + \sigma_2 \cos\varphi \sin\psi + \sigma_3 \sin\varphi$$

where $d\psi \wedge d\varphi = 0$ (actually $\psi = \psi(\varphi)$).

NB: even not compact

Corollary:
M^4 either Hermitian or almost Kähler, $\overline{R} = 0 \implies$ flat Kähler.

Compare to

M^n compact almost Kähler, $\overline{R} = 0 \implies$ flat Kähler **[Vezzoni-Di Scala 10]**

M^n compact Hermitian, holom. torsion + CHSC $\implies$ Kähler or flat **[Balas-Gauduchon 85]**

SDE 4-manifolds

If $F_0 = 0$:

Proposition

$$\overline{R} = \tfrac{1}{2}\overline{\gamma} \otimes \omega_I \quad \iff \quad \textit{Ricci-flat and self-dual.}$$

In particular:

$$g \text{ flat} \implies \dim \mathfrak{hol}(\overline{\nabla}) \leqslant 1$$

$$\text{compact} \implies \text{flat Kähler}$$

NB: Self-dual Einstein-Hermitian surfaces classified

[Apostolov–Gauduchon 02]

Example: a symplectic army

(M^4, g, I) Hermitian:

$$\overline{R} = -\tfrac{1}{4} d(I\theta) \otimes \omega_I \qquad \Longleftrightarrow \qquad \exists\ 5 \text{ symplectic forms: } \omega_i \wedge \omega_j = \pm\, vol(g)$$

Can always arrange for $span\{\omega_1, \omega_2, \omega_3\} = \Omega^-, \quad \omega_4,\ \omega_5 \in \Omega^+,$

latter 2 not Kähler if g not flat **[Armstrong 97]**

complete frame by $\omega_I = \omega_6 \in \Omega^+$ (non-closed)

Proposition

(M^4, g) non-flat with 5 ON symplectic forms $\Longrightarrow$

- *there exists a tri-holomorphic Killing field*
- *(M, g) is locally isometric to $\mathbb{R}^+ \times Nil^3$ with*

$$dt^2 + (\tfrac{2}{3}t)^{3/2}(\sigma_1^2 + \sigma_2^2) + (\tfrac{2}{3}t)^{-3/2}\sigma_3^2.$$

$\rightsquigarrow$ *quotient of KT mfd with diagonal Bianchi metric of class II, chm = 1*

Not complete, or (global) symmetry would force flatness **[Bielawski 99]**

Kähler-Hermitian surfaces

If $F_0 \neq 0$:

parametrise $F_0 = \omega_J$ using J with orientation opposite to I

Proposition

These statements are equivalent:

- $\mathfrak{hol}(\overline{\nabla})$ *is generated by* $F \in \Omega^{1,1}$ *with* $F_0 \neq 0$;
- $\overline{\nabla}$ *is not flat and there is a negative* **Kähler** J *such that* $\overline{\gamma} = \alpha\rho^J$ *(*ρ^J *Ricci form).*

Either implies $\overline{R} = \frac{\rho^J}{2} \otimes (\omega_J + \alpha\omega_I)$.

Call this a **Kähler-Hermitian surface $(\mathbf{M^4}, \mathbf{g}, \mathbf{J}, \mathbf{I})$**

To study a KH surface need to understand features of ('vertical' and 'horizontal') distributions

see

$$\mathcal{V} := \mathrm{Ker}(IJ - Id),\ \mathcal{H} := \mathcal{V}^\perp \quad \Longrightarrow \quad TM^4 = \mathcal{V} \oplus \mathcal{H}$$

Example

Hermitian line bundle over a Riemann surface **[Calabi 82]**

$$\mathbb{C}^\times \hookrightarrow (L, h) \longrightarrow (\Sigma, g_\Sigma, I_\Sigma, \omega_\Sigma)$$

$$TL^\times = \mathcal{H} \oplus \mathcal{V}$$

For some map f of r = norm of fibres $\mathcal{V}$,

$$\omega_\Sigma + dJ_\mathcal{V} d\, f(r) \text{ is Kähler on } M^4 = L^\times$$

Morally: $\boxed{J = I_\Sigma \oplus J_\mathcal{V} \quad \text{Kähler}}$

(M^4, g, J) is a Kähler surface **of Calabi type** *if*

$$I_{|\mathcal{V}} := -J, \quad I_{|\mathcal{H}} := J$$

satisfies $\theta \in \mathcal{H}$ and $d\theta = 0$ $\boxed{I = I_\Sigma \oplus -J_\mathcal{V} \quad \text{lcK}}$

Standard local form c/o **[ACG 03]**

some compact instances: $M^4 \xrightarrow{T^2} T^2$, $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow \mathbb{P}^1 \ldots$

back to concern

Dichotomy

(M^4, g, J, I) Kähler-Hermitian, with $\mathfrak{hol}(\overline{\nabla}) = \langle \alpha\omega_I + \omega_J \rangle$

Proposition (the KH balance)

1. *If* $\alpha \neq \pm 1$
   - W^+ *degenerate* $(\iff$ *Ric* $\in \Omega_I^{1,1})$
   - *compact* $\Longrightarrow$ *Calabi type (I lcK)*
2. *If* $\alpha = \pm 1$ $(\mathcal{V}$ *flat)*
   - $T^2 \hookrightarrow M \to \Sigma$
   - W^+ *degenerate* $\iff$ *I lcK*
   - *compact* $\Longrightarrow$ *I Kähler* *(M local product)*

Flippin' sign corresponds to reversing I on $\mathcal{H}, \mathcal{V}$

Normalisation

(M^4, g, J, I) is said **normal** if

$$(d \log |\theta|)_{\mathcal{V}} = V(|\theta|)\, \theta$$

for some smooth $V : \mathbb{R}^+ \to \mathbb{R}$

- Kähler of Calabi type $\Longrightarrow$ normal
- W^+ degenerate $\Longrightarrow$ normal

(apparently-obnoxious technical property that solves a lot of problems)

Proposition

A Kähler-Hermitian (M^4, g, J, I) is, locally, either

- *a torus bundle, or*
- *a deformed Calabi-type structure (g_0, J_0, I_0)* see*, or*
- *'normalisable' with*

$$V = -\frac{1}{4}\left(1 + \frac{2s^+}{|\theta|^2}\right) \quad \textit{and } \mathcal{V} \textit{ of CSC } s^+$$

Local structure

Normality sets up the construction

Kähler-Hermitian $\leftrightsquigarrow$ deformed Calabi-type

Theorem (gist)

A non-degenerate, normal (M^4, g, J, I) is locally obtained from

1. *a Hermitian line bundle $L \longrightarrow (\Sigma, g_\Sigma, J_\Sigma, \omega_\Sigma)$* *with* $c_1(L) = -[\omega_\Sigma]$
2. *a constant $s^+ \in \mathbb{R}$*
3. $\xi \in \Omega^{0,1}(\Sigma, L^m)$ *giving a Calabi-type structure (g_0, J_0, I_0)*

$$\partial_{I_\Sigma} \xi = 0 \quad \textit{and} \quad (1 - \tfrac{s^+}{2m}|\xi|^2)\omega_\Sigma \textit{ calibrates } J_\Sigma$$

Remarks

- J_Σ, ξ are lifted *horizontally*, I_Σ important only to choose ξ and fix ω_Σ.
- $J_\Sigma = I_\Sigma$, $\xi = 0$ yields Calabi-type surface (g_0, J_0, I_0).
- *Symp*(Σ, ω_Σ) acts on $\mathcal{M}$ ('rough' space of data) by connections preserving lifts to L
- When (M^4, g, J, I) is not lcK nor ASD:
  - Goldberg-Sachs ensures $[g]$ has no Einstein metric
  - $J_\Sigma = I_\Sigma \implies (d\theta)^+ = 0$ and *Ric* has double eigenvalue s^+ (!)

Correspondence

Theorem

(M^4, g, J, I) with $\mathfrak{hol}(\overline{\nabla}) = \langle \alpha\omega_I + \omega_J \rangle$, $\alpha \neq \pm 1$, are locally in 1-1 correspondence with

- *when $s^+ = 0$:* $(\Sigma, g_\Sigma, I_\Sigma)$ *with* $s_\Sigma = \frac{2\alpha m}{1-\alpha}$ *and* $\xi \in H^{0,1}(\Sigma, L^m)$

- *when $\alpha = -\frac{1}{3}$:*

  local solutions $u(x,y) \in \mathbb{R}^2$ to

  $$\Delta u = \frac{m}{2}(e^{-u} + \frac{s^+}{2m} e^{2u})$$

  Tzitzéica equation

  $\rightsquigarrow$ *Chimaera?*

Unknown - "The Chimaera of Arezzo", about 400-375 B.C. - Bronze. Object: H: 78.5 X L: 129 cm.
Museo Archeologico Nazionale, Florence, Italy

Ashes to ashes ?

The Tzitzéica equation has to do with

- Abelian vortex eqns
- hyperbolic affine spheres
- $SL(3,\mathbb{R})$ ADSYM eqns
- minimal Lagrangian surfaces in $\mathbb{C}\mathcal{H}^2$
- SL cones in $\mathbb{C}^3$

(Phoenix rising from its ashes)

appendix: holomorphic distributions

A distribution $\mathscr{D}$ on an *almost complex* surface (M^4, I) is holomorphic if

$$I\mathscr{D} = \mathscr{D} \quad \text{and} \quad (L_{\mathscr{D}} I)TM \subseteq \mathscr{D}$$

(so I integrable $\implies \mathscr{D}$ is locally spanned by $T_I^{1,0}M$)

Proposition

1. (M^4, g, J, I) *KH surface* $\implies \mathcal{V} = \mathrm{Ker}(IJ - Id)$ *totally geodesic, both I- and J-holomorphic.*
2. (M^4, g, J) *Kähler with a holomorphic* $\mathscr{D}$*, define the OCS*

$$I_{|\mathscr{D}} = -J, \quad I_{|\mathscr{D}^\perp} = J.$$

Then $\mathscr{D}$ *is I-holo,*

$\theta \in \mathscr{D}$,

I *integrable* $\iff \mathscr{D}$ *tot. geodesic (superminimal)*

cf. [Wood 92]

▸ KH surfs

Given a Hermitian line bundle $L \to (\Sigma, g_\Sigma)$ over a Riemann surface, there is (g_0, J_0) Kähler on $TL^\times = \mathcal{V} \oplus \mathcal{H}$ [Calabi 82]

➤ Reverse orientation on fibres $\mathcal{V} \rightsquigarrow$ get OCS (g_0, I_0)

Let $w : \Sigma \to \mathbb{D}$ be holomorphic, and $T \in End\ TL^\times$ such that

$$T_{|\mathcal{V}} = \begin{pmatrix} Re\, w & Im\, w \\ Im\, w & -Re\, w \end{pmatrix}, \qquad T_{|\mathcal{H}} = 0.$$

➤ Deform (g_0, J_0, I_0) (vertically, and canonically):

$$J_w = (1-T)J_0(1-T)^{-1}$$
$$I_w = (1-T)I_0(1-T)^{-1}$$
$$g_w(\cdot,\cdot) = g_0\big((1+T)(1-T)^{-1}\cdot,\cdot\big)$$

Note

$$\boxed{\omega_{J_w} = \omega_{J_0}\ ,\ \omega_{I_w} = \omega_{I_0}}$$

back

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