Einstein manifolds with skew torsion

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1. Abstract

We present a notion of Einstein manifolds with skew torsion on Riemannian manifolds for all dimensions, initiated by the doctoral work of the second author in dimension four.

2. Overview

- Torsion, and in particular skew torsion, has been a topic of interest to both mathematicians and physicists in recent decades.
- The first attempts to introduce torsion in general relativity go back to the 1920's with the work of É. Cartan. More recently, torsion makes its appearance in string theory, where the basic model for type II consists of a Riemannian manifold, a connection with skew torsion, a spinorial field and a dilaton function.
- From the mathematical point of view, skew torsion has played a significant role in the work of Bismut and his local index theorem for non-Kähler manifolds. Skew torsion is also an important feature in generalized geometry, where there are two natural connections with skew torsion that come from the exterior derivative of the B-field.
- Torsion is also ubiquitous in the theory of non-integrable geometries. The idea is to choose a G-structure so that the G-connection with torsion admits desired parallel objects, in particular spinors, interpreted as supersymmetry transformations. As a first step in this investigation, T. Friedrich and S. Ivanov proved that many non-integrable geometric structures admit a unique invariant connection with parallel totally anti-symmetric torsion, thus being a natural replacement for the Levi-Civita connection.

3. Metric connections with skew torsion

Definition, existence and uniqueness

- Let (M,g) be a Riemannian manifold. Suppose that ∇ is a connection on TM and let T be its (1,2) torsion tensor. If we contract T with the metric we get a (0,3) tensor which we still call the torsion of ∇. If T is a three-form then we say that ∇ is a connection with skew-symmetric torsion.
- Given any three-form H on M then there exists a unique metric connection with skew torsion H which is defined explicitly by

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}H(X,Y,Z)$$

where ∇^g is the Levi-Civita connection.

4. Motivation and definition

The obvious way to define the Einstein equations with skew torsion is simply to set $\mathrm{Ric}^\nabla = \lambda g$ for some function λ. This, however, presents two immediate problems. The first is that the function λ might not be constant, and also that Ric^∇ might not be symmetric since

$$\mathrm{Ric}^\nabla(X,Y) = \mathrm{Ric}^g(X,Y) - \frac{1}{4}g(H(e_i,X),H(e_i,Y)) - \frac{1}{2}d^*H(X,Y),$$

where $\{e_i\}$ is an orthonormal frame of the tangent bundle; and requiring H to be co-closed is, a priori, too restrictive.
The standard Einstein equations of Riemannian geometry can be obtain by a variational argument. They are the critical points of the Hilbert functional

$$g \longmapsto \int_M (s^g - 2\Lambda)\ \mathrm{dvol}_g,$$

where Λ is the cosmological constant.
So one way of obtaining Einstein equations with skew torsion is to look for the critical points of the following functional

$$(g,H) \longmapsto \int_M \left(s^\nabla - 2\Lambda\right)\ \mathrm{dvol}_g = \int_M \left(s^g - \frac{3}{2}\|H\|^2 - 2\Lambda\right)\ \mathrm{dvol}_g.$$

Proposition 1: *The critical points of the functional*

$$\mathcal{L}(g,H) = \int_M \left(s^\nabla - 2\Lambda\right)\ \mathrm{dvol}_g$$

are given by pairs (g,H) such that the Ricci tensor Ric^∇ is symmetric and satisfies the equation

$$-\mathrm{Ric}^\nabla + \frac{1}{2}s^\nabla g - \Lambda g = 0.$$

In the context of non-integrable geometries, requiring that H is ∇ parallel is a natural condition and this will then imply that H is co-closed. Also, under the assumption that $\nabla H = 0$, the ∇-curvature tensor simplifies to

$$R^\nabla(X,Y,Z,W) = R^g(X,Y,Z,W) + \tfrac{1}{4}g(H(X,Y),H(Z,W)) + \tfrac{1}{8}dH(X,Y,Z,W)$$

Following the classical theory of decomposition of algebraic curvature tensors, if $\odot$ denotes the Kulkarni-Nomizu product, we have the following:

Proposition 2: *Under the action of the orthogonal group, the ∇-curvature tensor decomposes as*

$$R^\nabla = W^\nabla + \frac{1}{n-2}(Z^\nabla \odot g) + \frac{s^\nabla}{n(n-1)}g \odot g + \frac{1}{12}dH$$

So it is natural to set:

Definition: *Let (M,g,H) be a Riemannian manifold equipped with a three-from H. We say that (M,g,H) is Einstein with skew torsion H if the trace free part of the Ricci-tensor*

$$Z^\nabla = \mathrm{Ric}^\nabla - \frac{s^\nabla}{n}g,$$

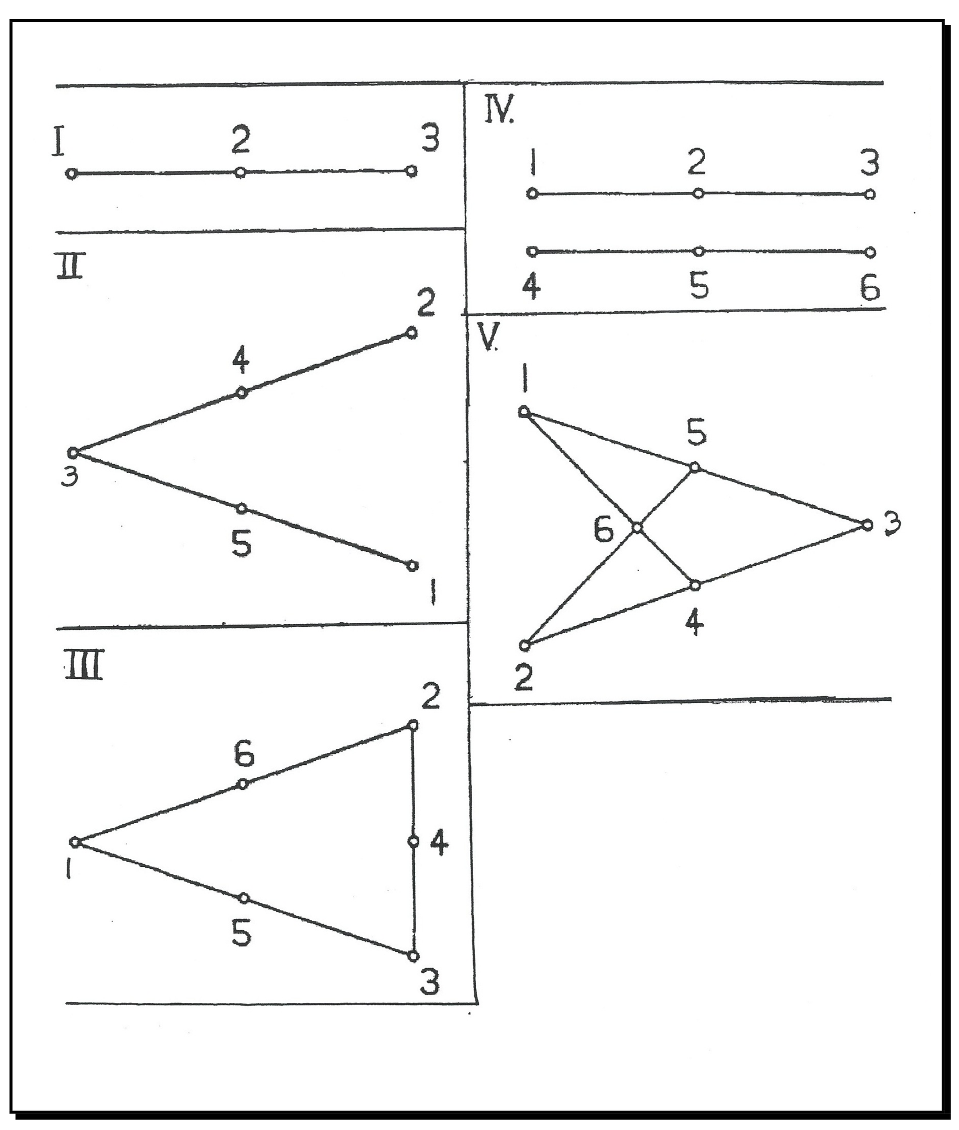
where ∇ is the metric connection with skew torsion H, vanishes.

We also have the following remarkable property:

Theorem 1: If (M,g,H) is Einstein with parallel skew torsion H then the scalar curvature s^∇ is constant.

5. ∇^g-Einstein and ∇-Einstein

An interesting question is what is the relation between the standard Einstein condition and the Einstein condition with skew torsion. By looking at the formulas for the Ricci tensors, we see that if (M,g) is Einstein then (M,g,H) being Einstein depends only on the algebraic type of the three-form H. Up to dimension 6, we have a classification provided by Schouten which can be easily illustrated by the following graphs.



It can be checked by direct computation that type I. works in dimension 3, and types IV. and V. work in dimension 6. Types II. and III. never work, so in particular an Einstein metric never gives an Einstein metric with skew torsion in dimensions 4 and 5.

Example: Consider the 3-dimensional sphere S^3 and take g to be the round metric. Then (S^3,g) is Einstein with $s^g = 6$. Consider a global frame $\{e_1,e_2,e_3\}$ of TS^3, f any non-constant smooth function and define the three-form H by

$$H = 2fe^1 \wedge e^2 \wedge e^3.$$

Then the connection given by $\nabla = \nabla^g + \frac{1}{2}H$ is Einstein with skew torsion and

$$s^\nabla = s^g - \frac{3}{2}\|H\|^2 = 6 - 6f(x)^2$$

which is clearly not a constant.

6. Special geometries

Lie groups

Classical examples of manifolds where skew torsion arises naturally are those of Lie groups equipped with a bi-invariant inner product on the corresponding Lie algebra. They provide many examples of Einstein manifolds with skew torsion.

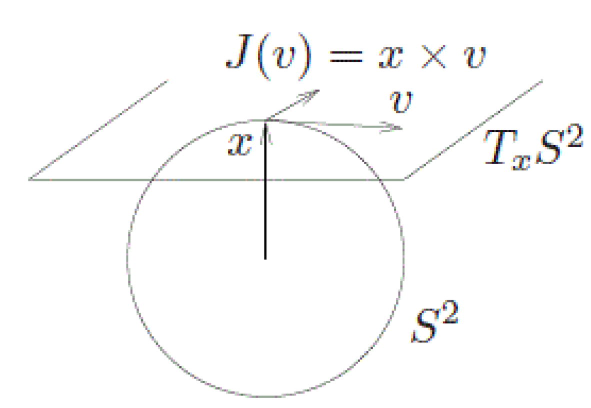
Theorem 2: Let G be a Lie group equipped with a bi-invariant metric. Consider the 1-parameter family of connections with skew torsion

$$\nabla^t_X Y = t[X,Y].$$

Then for $t=0$ and $t=1$, ∇^t is Einstein (it is in fact flat) and for $t \neq 0$ and $t \neq 1$, ∇^t is Einstein if and only if ∇^g is Einstein.

Nearly Kähler in dimension 6

$S^6 \subset \mathbb{R}^7$ has an almost complex structure J inherited from the "cross product" on $\mathbb{R}^7$. J is not integrable and it is still an open problem whether or not S^6 admits a complex structure. Here is the analogous picture for the 2-sphere in $\mathbb{R}^3$:



Even though J is not integrable, $\nabla^g J \neq 0$, we have that J is an example of a nearly Kähler structure: $\nabla^g_X J(X) = 0$ (g is the round metric).
For any nearly Kähler manifold, the Gray connection

$$\overline{\nabla}_X Y + \frac{1}{2}(\nabla^g_X J)JY$$

is a connection with totally skew symmetric torsion. In dimension 6, all nearly Kähler manifolds are Einstein and of constant type so it can be easily proved that the Gray connection is Einstein with skew torsion (as observed by T. Friedrich and S. Ivanov '02).

G_2 T-manifolds in dimension 7

A G_2 structure can be seen as a triple (M^7,g,ω) where ω is a 3-form of general type at any point. A G_2 T-manifold - G_2 manifold with (skew) torsion - is such that there exists a one-form θ such that $d*\omega = \theta \wedge *\omega$. In this case, there exists a unique connection with totally skew symmetric torsion which preserves both the metric g and the three-form ω:

$$\nabla = \nabla^g + \frac{1}{2}\left(-*d\omega - \frac{1}{6}g(d\omega, *\omega)\omega + *(\theta \wedge \omega)\right).$$

Furthermore, its torsion is proportional to ω and thus it is ∇-parallel. (T. Friedrich and S. Ivanov '02)
Let (M,g,ω) be a G_2 T-manifold which is also nearly parallel, that is, $d\omega = \lambda * \omega$, for some $\lambda \in \mathbb{R}$. Then the M^7 is Einstein and ∇ is Einstein with skew torsion.

Einstein-Sasaki in odd dimensions

Let $(M^{2k+1},g,\varphi,\xi,\eta)$ be a Sasaki manifold. Then it admits a connection ∇ with skew torsion H such that

$$\nabla g = \nabla \eta = \nabla \varphi = 0$$

given by the torsion $H = \eta \wedge d\eta$ and also we have that H is ∇-parallel.
In this particular instance, it is not too difficult to check that a Sasaki-Einstein metric never gives that ∇ is Einstein with skew torsion.
However, if we consider the Tanno deformation $g_t = tg + (t^2-t)\eta\otimes\eta$, $t>0$, of g in the direction of ξ, then $(M^{2k+1},g_t,\xi_t,\varphi,\eta_t)$, where $\xi_t = \frac{1}{t}\xi$ and $\eta_t = t\eta$, is also a Sasaki manifold and we can prove:

Theorem 3: Let (M,g,ξ,φ,η) be an Einstein-Sasaki manifold. Considering the 1-parameter family $(M,g_t,\xi_t,\varphi,\eta_t)$ described above, there exists a $t>0$ such that the connection

$$\nabla^t = \nabla^{g_t} + \frac{1}{2}\eta_t \wedge d\eta_t$$

is ∇^t-Ricci flat (but not flat).

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