5. Relations between $\mathcal{RBM}$ and $\mathcal{GHW}$

Let us start to consider the relations between these two classes of flat manifolds. We start with an easy observation

$$\mathcal{RBM}(n) \cap \mathcal{GHW}(n) = \{M(A) \mid rank_{\mathbb{Z}_2} A = n-1\} =$$

$$= \{M(A) \mid a_{1,2}a_{2,3}...a_{n-1,n} = 1\}.$$

These manifolds are classified in [3, Example 3.2] and for $n \geq 2$

$$\#(\mathcal{RBM}(n) \cap \mathcal{GHW}(n)) = 2^{(n-2)(n-3)/2}. \qquad (10)$$

There exists the classification, see [15] and [3], of diffeomorphism classes of $\mathcal{GHW}$ and $\mathcal{RBM}$ manifolds in low dimensions. For dim ≤ 6 we have the following table.

dim	number of GHW manifolds		number of RBM manifolds		number of $GHW \cap RBM$ manifolds
	total	oriented	total	oriented	total
1	0	0	1	1	0
2	1	0	2	1	0
3	3	1	4	2	1
4	12	0	12	3	2
5	123	2	54	8	8
6	2536	0	472	29	64

Proposition 1 $\Gamma_n \in \mathcal{GHW} \cap \mathcal{RBM}$.

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6. Existence of Spin and $\mathrm{Spin}^{\mathbb{C}}$ structures on real Bott manifolds

A closed oriented differential manifold N has a $\mathrm{Spin}-$ structure if and only if the second Stiefel-Whitney class $w_2(N) = 0$. In the case of an oriented real Bott manifold $M(A)$ we have the formula for w_2.
Recall, see [10], that for the Bott matrix A

$$H^*(M(A); \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_2, ..., x_n]/(x_j^2 = x_j \Sigma_{i=1}^n a_{i,j} x_i \mid j = 1, 2, ..., n) \qquad (11)$$

as graded rings. Moreover, from [11, (3.1) on page 3] the k-th Stiefel-Whitney class

$$w_k(M(A)) = (B(p))^* \sigma_k(y_1, y_2, ..., y_n) \in H^k(M(A); \mathbb{Z}_2), \qquad (12)$$

where σ_k is the k-th elementary symmetric function,

$$p : \pi_1(M(A)) \to G \subset O(n)$$

a holonomy representation, $B(p)$ is a map induced by p on the classification spaces and $y_i = w_1(L_{i-1})$. Hence,

$$w_2(M(A)) = \Sigma_{1 \leq i < j \leq n} y_i y_j \in H^2(M(A); \mathbb{Z}_2). \qquad (13)$$

There exists a general condition, see [4, Theorem 3.3], for the calculation of the second Stiefel-Whitney for flat manifolds with $(\mathbb{Z}_2)^k$ holonomy of diagonal type but we prefer the above explicit formula (13). Its advantage follows from the knowledge of the cohomology ring (11) of real Bott manifolds.

An equivalent condition for the existence of a Spin structure is as follows. An oriented flat manifold M^n (a Bieberbach group $\pi_1(M^n) = \Gamma$) has a Spin structure if and only if there exists a homomorfism $\epsilon : \Gamma \to \mathrm{Spin}(n)$ such that $\lambda_n \epsilon = p$. Here $\lambda_n : \mathrm{Spin}(n) \to SO(n)$ is the covering map, see [6]. We have a similar condition, under assumption $H^2(M^n, \mathbb{R}) = 0$, for the existence of $\mathrm{Spin}^{\mathbb{C}}$ structure, [6, Theorem 1]. In this case M^n (a Bieberbach group Γ) has a $\mathrm{Spin}^{\mathbb{C}}$ structure if an only if there exists a homomorphism

$$\bar{\epsilon} : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n) \qquad (14)$$

such that $\bar{\lambda_n}\bar{\epsilon} = p$. $\bar{\lambda_n} : \mathrm{Spin}^{\mathbb{C}}(n) \to SO(n)$ is the homomorphism induced by λ_n, see [6]. We have the following easy observation. If there existe $H \subset \Gamma$, a subgroup of finite index, such that the finite covering $\tilde{M}^n$ with $\pi_1(\tilde{M}^n) = H$ has no Spin ($\mathrm{Spin}^{\mathbb{C}}$) structure, then M^n has also no such structure.

Theorem 1 *Let A be a matrix of an orientable real Bott manifold $M(A)$ of dimension n.*

1. *Let $l \in \mathbb{N}$ be an odd number. If there exist $1 \leq i < j \leq n$ and rows $A_{i,*}, A_{j,*}$ such that*

$$\#\{m \mid a_{i,m} = a_{j,m} = 1\} = l \qquad (15)$$

and

$$a_{i,j} = 0, \qquad (16)$$

then $M(A)$ has no Spin structure. Moreover, if

$$\#\{J \subset \{1, 2, ..., n\} \mid \#J = 2, \Sigma_{j \in J} A_{*,j} = 0\} = 0, \qquad (17)$$

then $M(A)$ has no $\mathrm{Spin}^{\mathbb{C}}$ structure.

2. *If there exist $1 \leq i < j \leq n$ and rows*

$$A_{i,*} = (0, ..., 0, a_{i,i_1}, ..., a_{i,i_{2k}}, 0, ..., 0),$$
$$A_{j,*} = (0, ..., 0, a_{j,i_{2k+1}}, ..., a_{j,i_{2k+2l}}, 0, ..., 0)$$

such that $a_{i,i_1} = a_{i,i_2} = ... = a_{i,i_{2k}} = 1, a_{i,m} = 0$ for $m \notin \{i_1, i_2, ..., i_{2k}\}$ $a_{j,i_{2k+1}} = a_{j,i_{2k+2}} = ... = a_{j,i_{2k+2l}} = 1, a_{j,r} = 0$ for $r \notin \{i_{2k+1}, i_{2k+2}, ..., i_{2k+2l}\}$ and l, k odd then $M(A)$ has no Spin structure.

Example 1 From *[14]* we have the list of all 5-dimensional oriented real Bott manifolds. There are 7 such manifolds without the torus. Here are their matrices:

$$A_4 = \begin{bmatrix} 0\,1\,0\,1\,0 \\ 0\,0\,1\,0\,1 \\ 0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix}, A_{23} = \begin{bmatrix} 0\,1\,1\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix},$$

$$A_{29} = \begin{bmatrix} 0\,1\,1\,1\,1 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix}, A_{37} = \begin{bmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0 \\ 0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix},$$

$$A_{40} = \begin{bmatrix} 0\,0\,1\,0\,1 \\ 0\,0\,1\,1\,0 \\ 0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix}, A_{48} = \begin{bmatrix} 0\,0\,1\,0\,1 \\ 0\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix},$$

$$A_{49} = \begin{bmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix}.$$

From the first part of Theorem 1 above, for $i = 1, j = 2$ the manifold $M(A_4)$ has no $\mathrm{Spin}^{\mathbb{C}}$ structure. For the same reasons (for $i = 1, j = 2$) manifolds $M(A_{40})$ and $M(A_{48})$ have no Spin structures. The manifold $M(A_{23})$ has no a Spin structure, because it satisfies for $i = 1, j = 3$ the second part of the *Theorem 1*. Since any flat oriented manifold with $\mathbb{Z}_2$ holonomy has Spin strucure, *[9, Theorem 3.1]* manifolds $M(A_{29}), M(A_{49})$ have it. Our last example, the manifold $M(A_{37})$ has Spin structure and we leave it as an exercise.

In all these cases it is possible to calculate the w_2 with the help of (**??**), (13) and (11). In fact, $w_2(M(A_4)) = (x_2)^2 + x_1x_3, w_2(M(A_{23})) = x_1x_3, w_2(M(A_{40})) = w_2(M(A_{48})) = x_1x_2$. In all other cases $w_2 = 0$.

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