HOLOMORPHIC REDUCTIONS OF PSEUDOCONVEX HOMOGENEOUS MANIFOLDS

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Classical Levi problem : characterize domains of holomorphy in $\mathbb{C}^n$

Domain is already holomorphically separable ; in order for it to be Stein one has to "control" the geometry as one "goes to infinity" in the domain in order to get holomorphic convexity.

For today's talk a complex manifold is pseudoconvex if it admits a continuous plurisubharmonic (psh) exhaustion function.

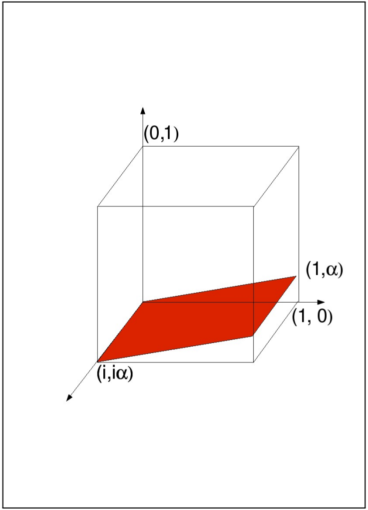
A complex manifold admitting a smooth strictly plurisubharmonic exhaustion function is Stein. [Grauert, Ann.of Math. **68** (1958)]

An interesting survey : Y.-T. Siu, Pseudoconvexity and the problem of Levi, Bull. Amer. Math. Soc. **84** (1978).

Constructing a Cousin Group

- A connected complex Lie group G with no non-constant holomorphic functions is called a Cousin group, a toroidal group, or an HC-group
- Such a G lies in the kernel of $\mathrm{Ad} : G \to GL(n, \mathbb{C})$ and this kernel is central in G, i.e., G is Abelian.
- In the Abelian setting : exp : $\mathfrak{g} \to G$ is a surjective homomorphism $\Longrightarrow G = \mathbb{C}^n/\Gamma_{n+k}, 1 \leq k \leq n$.
- $\{(1,0),(0,1),(i,i\alpha)\} \subset \mathbb{C}^2$ lin. indep./$\mathbb{R}$; $\alpha \in [0,1] \cap \mathbb{R} \setminus \mathbb{Q}$.
- Set $\Gamma := \langle (1,0),(0,1),(i,i\alpha)\rangle_{\mathbb{Z}}$ and $V := \langle \Gamma \rangle_{\mathbb{R}}$. Note that $C := \mathbb{C}^2/\Gamma = K \times \mathbb{R}$, where $K = V/\Gamma = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Clearly, K is the maximal compact subgroup of C.

The complex geometry



Recall $V := \langle (1,0),(0,1),(i,i\alpha)\rangle_{\mathbb{R}}$.
The figure shows a cube that is a fundamental domain for this lattice in the real 3-dimensional space $V \subset \mathbb{C}^2$ and a portion of the maximal complex subspace $\mathfrak{m} = \langle (1,\alpha)\rangle_{\mathbb{C}} = V \cap iV$ of V.

The orbit of $M := \exp \mathfrak{m}$ is dense in the quotient of V by the lattice.
Reason : the orbit of $\langle (1,\alpha)\rangle_{\mathbb{R}}$ is dense in $\langle (1,0),(0,1)\rangle_{\mathbb{R}}/\langle (1,0),(0,1)\rangle_{\mathbb{Z}}$.
(This is a skew line on the torus $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R}^2/\langle (1,0),(0,1)\rangle_{\mathbb{Z}}$.)

$\mathscr{O}(C) \simeq \mathbb{C}$

Suppose $f \in \mathcal{O}(C)$. Define $\sigma : \mathbb{C} \to \mathfrak{m} := V \cap iV, z \mapsto z \cdot (1, \alpha)$. Consider the composite holomorphic map

$$\mathbb{C} \xrightarrow{\sigma} \mathfrak{m} \xrightarrow{\exp} M \xrightarrow{\alpha} M/(M \cap \Gamma) \xrightarrow{f} \mathbb{C}.$$

Since $M/(M \cap \Gamma)$ lies in K and $f(K)$ is compact, the holomorphic function $f \circ \alpha \circ \exp \circ \sigma$ is a bounded entire function, and thus is constant. But the holomorphic maps σ, exp, α are not constant. Therefore $f|_{M/(M \cap \Gamma)}$ is constant.

Since $M/(M \cap \Gamma)$ is dense in K, it follows that $f|_K$ is constant. But K has real codimension one in C and so f is constant.

Remark : A non-compact Cousin group provides an example of a pseudoconvex homogeneous space of a reductive complex Lie group that is not holomorphically convex ! Subtlety : we change groups $\mathbb{C}^2/\langle(1,0),(0,1)\rangle_{\mathbb{Z}} \simeq \mathbb{C}^* \times \mathbb{C}^*$; so $C \simeq \mathbb{C}^* \times \mathbb{C}^*/\langle(i, i\alpha)\rangle_{\mathbb{Z}}$.

Holomorphic Reductions of Homogeneous Manifolds

For any connected complex manifold X define $x_1 \sim x_2 \iff f(x_1) = f(x_2)\ \forall f \in \mathscr{O}(X)$. This gives an equivalence relation. Does $X/\sim$ have a complex structure ? Is $\pi : X \to X/\sim$ holomorphic ?

Let G be a connected complex Lie group with H a closed complex subgroup that is not necessarily connected. Set $X := G/H$.

For $X = G/H$ there is a Lie theoretic homogeneous fibration $\pi : G/H \to G/J, gH \mapsto gJ$, called the holomorphic reduction of X, where $J := \{\, g \in G \mid f(gH) = f(eH) \quad \forall f \in \mathscr{O}(G/H) \,\}$.

By definition J is a closed complex subgroup of G containing H, G/J is holomorphically separable and $\mathscr{O}(G/H) \simeq \pi^* \mathscr{O}(G/J)$.

Optimal : X holomorphically convex $\iff$ G/J Stein and J/H compact. This is the Remmert Reduction.

Next Best : G/J Stein and $\mathscr{O}(J/H) \simeq \mathbb{C}$.

Holomorphic Reductions - some results

G complex Abelian Lie group : then $G = \mathbb{C}^k \times (\mathbb{C}^*)^p \times C$, where C is a Cousin group.

For complex Lie groups : the base of the holomorphic reduction is Stein and its fiber is a Cousin group, see [Morimoto (1964)]

G nilpotent :

1. G/J is Stein and $\mathscr{O}(J/H) \simeq \mathbb{C}$; [G–Huckleberry (1978)]
2. J/H is a Cousin group tower ; Akhiezer/K. Oeljeklaus (1980's)
3. Every nilmanifold is pseudoconvex [Huckleberry (2011)]

G solvable :

1. G/J is Stein, see [Huckleberry–E. Oeljeklaus (1986)]
2. fiber J/H can itself be Stein ; Coeuré-Loeb example $G/H \to G/J$ with $G/J \simeq \mathbb{C}^*$ and $J/H \simeq \mathbb{C}^* \times \mathbb{C}^*$
3. provides a homogeneous counterexample to the Serre problem : with fiber and base Stein, total space not !

Reductive Groups

Consider $G = K^{\mathbb{C}} = S \times Z$ a complex reductive Lie group.
G carries a (unique) structure of an algebraic group.

Remark : One has $GL(n, \mathbb{C}) = SL(n, \mathbb{C}) \cdot \mathbb{C}^*$ with finite intersection $\{ \alpha I_n \mid \alpha^n = 1 \}$. A finite covering is a direct product.

Here S is a connected complex semisimple Lie group and $Z \cong (\mathbb{C}^*)^k$ is the center of G. Note that $G' = S$.

Important Observations :

G/J is Stein iff J is reductive : Matsushima/Onishchik (1960)

For G reductive one has $\overline{H} \subset J$, where $\overline{H}$ denotes the Zariski closure of H in G : [Barth–Otte (1973)]
In particular, the isotropy subgroup J of the holomorphic reduction of G/H is algebraic.

Motivating Example

Example : Set $\Gamma = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\} \subset SL(2, \mathbb{C}) =: S$ and $J = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}$.

Note that $J = \overline{\Gamma}$ is the Zariski closure of Γ in S.

Then $S/\Gamma \xrightarrow{\mathbb{C}^*} S/J = \mathbb{C}^2 - \{(0,0)\}$ is the holomorphic reduction of G/Γ ; follows, since $\mathbb{C}^2 - \{(0,0)\}$ is holomorphically separable and J is the smallest algebraic subgroup of S containing Γ.

Note that S/J is not Stein.

One has $\mathscr{O}(J/\Gamma) \not\simeq \mathbb{C}$. Here $J/\Gamma = \mathbb{C}^*$ is even Stein itself ; see [Barth–Otte (1973)].

Theorem (G-Miebach-K. Oeljeklaus ; 2012)

Suppose G/H is pseudoconvex and let $G/H \to G/J$ be its holomorphic reduction.

1. *For G reductive :*

   a) G/J is Stein and $\mathscr{O}(J/H) \simeq \mathbb{C}$.
   b) G/H also Kähler $\Longrightarrow$ the fiber $J/H = \overline{H}/H \times J/\overline{H}$, where $\overline{H}/H$ is a Cousin group and $J/\overline{H}$ is a flag manifold.

2. *For G solvable : J/H is a Cousin group tower ; in particular, $\mathscr{O}(J/H) \simeq \mathbb{C}$.*

Remark : 1) b) extends Matsushima '57 and Borel–Remmert '62 : every compact homogeneous Kähler manifold is a product $T \times Q$, where T is a compact complex torus and Q is a flag manifold

Theorem (G-Miebach-K. Oeljeklaus ; 2012)

Suppose $p : D \to X$ is a pseudoconvex domain spread over $X = G/H$ such that $eH \in p(D)$. If D is not Stein, then there exists a connected complex Lie subgroup $\widehat{H}$ of G containing H^0 with $\dim \widehat{H} > \dim H$ and a foliation $\mathscr{F}$ of D such that

1. *every leaf of $\mathscr{F}$ is a relatively compact complex manifold immersed in D*
2. *every inner integral curve in D passing through a point $x \in D$ lies in the leaf F_x containing x*
3. *the leaves are homogeneous under a covering group of $\widehat{H}$ and the restriction of p to a leaf is a finite covering map onto a leaf in X*

Remark : Generalizes Kim-Levenberg-Yamaguchi (2011) : they consider pseudoconvex, relatively compact domains $D \subset G/H$ with smooth boundary – not useful for non–compact G/H !

Steps in the proof

1.) Let $p : D \to X := G/H$ with $x_0 \in D$ and $p(x_0) = eH$. Note that $T_{x_0}(D)$ is generated by holo vector fields $\tilde{\xi}_X$ for $\xi \in \mathfrak{g}$.

2.) Finding the "queen bee" :
$\widehat{\mathfrak{h}} := \{\xi \in \mathfrak{g} \mid \tilde{\xi}_X \varphi(x_0) = 0 \ \forall \varphi$ continuous psh functions on D $\}$

3.) Show that $\widehat{\mathfrak{h}}$ is a complex Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$.

4.) Hirschowitz [1975] : D not Stein $\Longrightarrow$ D contains an inner integral curve, i.e, a relatively compact integral curve of a holomorphic vector field coming from $\mathfrak{g} \Longrightarrow \dim \widehat{\mathfrak{h}} > \dim \mathfrak{h}$.

5.) F_{x_0} is an immersed complex manifold in D that contains all integral curves through the point x_0 – this is a leaf of the foliation.

6.) Move F_{x_0} by the local action of G to get a foliation $\mathscr{F}$ of D.

Characterization of Holomorphic Convexity

Theorem (GMO ; 2012)

Suppose $p : D \to X$ is a pseudoconvex domain spread over $X = G/H$. The complex group $H\widehat{H}$ is closed in G iff D is holomorphically convex. Then the Remmert reduction of D is a holomorphic fiber bundle $D \to D_0$ induced by the bundle $G/H \to G/H\widehat{H}$.

The idea is to define $D_0 := D/\mathscr{F}$ and show that this works!

By a result of H. Holmann the leaf space $D/\mathscr{F}$ has a canonical complex structure whenever this leaf space is Hausdorff.

One needs existence of open saturated neighborhoods inside every open neighborhood of a leaf that contain the leaf – this follows easily from compactness.

Projective orbits

Theorem (GMO ; 2012)

Suppose G is a complex linear group and H is a closed complex subgroup of G, i.e., G/H is an orbit in some projective space $\mathbb{CP}^N$. If G/H is pseudoconvex, then G/H is holomorphically convex. Hence G/J is Stein and J/H is a homogeneous rational manifold.

1.) Theorem of Chevalley '51 : $\mathfrak{g}' = \overline{\mathfrak{g}}'$ and so $G' = \overline{G}'$ and thus the G'–action on $\mathbb{CP}^N$ is algebraic. So G' has closed orbits.

2.) Additional important fact : $G/HG' \hookrightarrow \overline{G}/\overline{HG}' \simeq \mathbb{C}^p \times (\mathbb{C}^*)^q$, so $\widehat{H} \subset G'$ and one can reduce to the case $G = G'$.

3.) In this setting H is algebraic and $\widehat{H}$ is also algebraic.

4.) Then $\widehat{H}$-orbits are closed. So the $\widehat{H}$-orbits are homogeneous rational manifolds with Stein quotient $G/H\widehat{H}$ by previous slide.

Main Theorem in semi-simple case

RTP : G semisimple, G/H pseudoconvex $\Longrightarrow G/H$ holomorphically convex ; i.e., J/H is compact and G/J is Stein.

$$\begin{array}{ccc} X = G/H & \xrightarrow{\quad \pi \quad} & G/J \\ & \searrow \qquad \nearrow & \\ & G/\overline{H} & \end{array}$$

The holomorphic reductions of G/H and $G/\overline{H}$ are both G/J !

For G semi-simple, every right H-invariant plurisubharmonic function on G is also right $\overline{H}$-invariant $\Longrightarrow G/\overline{H}$ is pseudoconvex and $\overline{H}/H$ is compact ; see [Berteloot (1987)] and [Berteloot–K. Oeljeklaus (1988) Now in an algebraic setting – use previous result.

Example : Set $S := SL(3, \mathbb{C}) \supset T := \mathbb{C}^* \times \mathbb{C}^* \supset H := \mathbb{Z}$ with T/H a non-compact Cousin group. Hol. red. : $S/H \to S/T$. Then S/H is not pseudoconvex, because it is not holomorphically convex.

Extension of Kiselman Minimum Principle

Let $u : \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{R}$ be an $\mathbb{R}^q$–invariant psh function. Then the function $\widehat{u}(w) := \min_{z \in \mathbb{C}^q} u(w, z)$ is psh on $\mathbb{C}^p$.

Lemma (GMO ; 2012)

Suppose X is pseudoconvex and $X \to Y$ is a holomorphic fiber bundle with fiber a Cousin group. Then Y is pseudoconvex.

On a local trivialization $W \times \mathbb{C}^n/\Gamma_{n+k}$ let $u : W \times \mathbb{C}^n/\Gamma_{n+k} \to \mathbb{R}$ be a psh function. Pull back $u(w, .)$ to a Γ_{n+k}–invariant function v on $\mathbb{C}^n$ for $w \in W$. Let M be the Lie subgroup of $\mathbb{C}^n/\Gamma_{n+k}$ with algebra $\mathfrak{m} := \langle \Gamma_{n+k} \rangle_{\mathbb{R}} \cap i \langle \Gamma_{n+k} \rangle_{\mathbb{R}}$. Its orbit is dense in the maximal compact subgroup $\langle \Gamma_{n+k} \rangle_{\mathbb{R}}/\Gamma_{n+k}$. So v is $\langle \Gamma_{n+k} \rangle_{\mathbb{R}}$–invariant and pushes down to a function on $\mathbb{C}^n/\mathfrak{m}$ that is $\langle \Gamma_{n+k} \rangle_{\mathbb{R}}/\mathfrak{m}$–invariant . Apply Kiselman, noting $\langle \Gamma_{n+k} \rangle_{\mathbb{R}}/\mathfrak{m}$ is a real form of $\mathbb{C}^n/\mathfrak{m}$.

Remark : An analogue of the Lemma also holds for $(\mathbb{C}^*)^k$–principal bundles. Choose an $(S^1)^k$-invariant psh exhaustion of X, etc.

Main Theorem in reductive case

Step 1 : Show G/J is Stein.

Let N be those connected components of the normalizer of H in G that meet H. Note that N/H is a connected complex Lie group.

Let $N/H \to N/I$ be its holomorphic reduction ; I/H is a Cousin group by Morimoto's result.

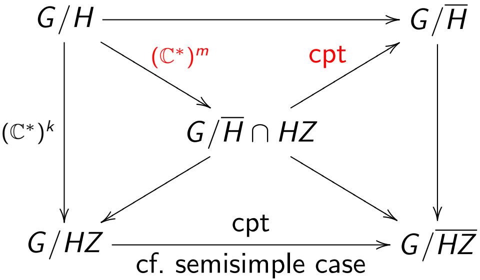
Apply Kiselman's minimum principle for Cousin fibrations in order to push down the psh exhaustion to G/I, i.e., previous Lemma

Apply induction if $\dim G/I < \dim G/H$; same holo reductions !

Otherwise, $\dim G/I = \dim G/H \Longrightarrow I/H$ is Stein. Note $Z \subset N$.

The bundle $G/H \to G/HZ$ is a principal $(\mathbb{C}^*)^k$–bundle $\Longrightarrow G/HZ$ pseudoconvex $\Longrightarrow G/HZ = S/S \cap H$ holomorphically convex

Not done ! Don't know how $\mathscr{O}(G/H)$ and $\mathscr{O}(G/HZ)$ are related.



Consider $\overline{H}/H \to \overline{H}/\overline{H} \cap HZ$. Torus $\overline{H} \cap Z$ transitive on fiber $\Longrightarrow (\mathbb{C}^*)^m$–principal bundle $\Longrightarrow G/\overline{H} \cap HZ$ pseudoconvex

$\overline{HZ}/HZ = \overline{H}Z/HZ$ compact $\Longrightarrow \overline{H}/\overline{H} \cap HZ = \overline{HZ}/HZ$ compact

Thus $G/\overline{H}$ is pseudoconvex and hence holomorphically convex and G/H and $G/\overline{H}$ have the same holomorphic reduction.

The base G/J of the holomorphic reduction of G/H is Stein

Remarks on the fiber

Step 2 : Show $\mathscr{O}(J/H) \simeq \mathbb{C}$ with $G/H \to G/J$ the holomorphic reduction of a pseudoconvex G/H and G is reductive.

For G reductive G/J is Stein iff J is reductive
Matsushima and Onishchik, both 1960

Let $J/H \to J/I$ be the holomorphic reduction ; note that J/H inherits pseudoconvexity and J is reductive

Step 1 implies J/I is Stein and thus I is reductive

Apply Matsushima–Onishchik to conclude that G/I is Stein and thus $I = J$; in other words $\mathscr{O}(J/H) \simeq \mathbb{C}$ in the first place !

The Kähler setting

Use characterization of Kähler reductive homogeneous manifolds.

Theorem (G-Miebach-K. Oeljeklaus ; Math. Ann. 349 (2011))

Suppose G is a complex reductive Lie group and H is a closed complex subgroup of G. Then G/H is Kähler if and only if $S \cap H$ is algebraic and SH is closed in G

$N_G(S \cap H)$ algebraic $\Longrightarrow \overline{H} \subset N_G(S \cap H) \Longrightarrow \overline{H}/S \cap H$ group

$\overline{H}/S \cap H$ Abelian ; since $\overline{H}/S \cap H = \overline{H/S \cap H}$ and $H' \subset S \cap H$. Thus $\overline{H}/H$ is an Abelian group without any $\mathbb{C}$ factor.

G/H pseudoconvex $\Longrightarrow G/\overline{H}$ pseudoconvex ; Kiselman minimality type Lemma for Cousin bundles and principal $(\mathbb{C}^*)^k$–bundles

Then $G/\overline{H}$ pseudoconvex $\Longrightarrow G/\overline{H}$ holomorphically convex. Finally $\mathscr{O}(G/H) \simeq \mathbb{C} \Longrightarrow G/\overline{H}$ compact, thus a flag manifold.

Continuation of proof in Kähler case

Claim : there are no $\mathbb{C}^*$'s in $\overline{H}/H$

Lemma : A pseudoconvex holomorphic $(\mathbb{C}^*)^k$-principal bundle over a flag manifold is trivial.

Idea underlying the proof : unless the bundle is trivial, construct a closed embedding of a finite quotient of $\mathbb{C}^2 - \{(0,0\}$ in the bundle space - this quotient inherits the pseudoconvexity, a contradiction.

Finally, show the triviality of the fibration $G/H \to G/\overline{H}$

The algebraic variety $G' \cap \overline{H}/G' \cap H$ is closed subgroup of Cousin group $\overline{H}/H$; recall $G' = S$ and $S \cap H$ is algebraic.

$\Longrightarrow [G' \cap \overline{H} : G' \cap H] < \infty$. But $G' \cap \overline{H}$ parabolic implies it is connected ; i.e., the bundle has trivial structure group.

An example that is not Kähler

Let $\Gamma \subset SL(2, \mathbb{C})$ be a cocompact discrete subgroup such that Γ/Γ' contains an element of infinite order. [Millson, Ann. of Math. 1976]

Let $\varphi : \Gamma \to \mathbb{C}^*$ be a homomorphism with dense image in S^1.

Let Γ_G be the graph of φ in $G := SL(2, \mathbb{C}) \times \mathbb{C}^*$; set $X := G/\Gamma_G$.

CLAIM : X is pseudoconvex. Choose ρ to be an S^1-invariant strictly plurisubharmonic exhaustion function on $\mathbb{C}^*$.

The function $G \to \mathbb{R}^{\geq 0}, (s, z) \mapsto \rho(z)$ is Γ_G-invariant and psh ; so defines a psh function on X. Since the closure of every S-orbit is a compact real hypersurface in X, this function is an exhaustion on X and so X is pseudoconvex.

Note that $\mathscr{O}(X) \simeq \mathbb{C}$ and the group $\widehat{\Gamma}_G = S$ has no locally closed orbits in X. It follows that X is not Kähler and also not holomorphically convex, because it is not compact.