Constructions of Sound-Alike Manifolds

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Definition

Let M be a compact Riemannian manifold, with or without boundary. The Laplace-Beltrami operator Δ on $C^\infty(M)$ is given by

$$\Delta(f) = -div(grad(f)).$$

Consider eigenvalue problem:

$$\Delta(f) = \lambda f.$$

(If $\partial(M) \neq \emptyset$, impose Dirichlet, Neumann or mixed boundary conditions.)

$$Spec(M) : 0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

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Inverse Spectral Problem

$$\mathrm{spec}(M) \overset{?}{\rightsquigarrow} \text{geometry of } M$$

Outline

- Geometry contained in the spectrum
- Isospectral manifolds with a common covering:
  Constructions
  Inaudible geometric properties
- Isospectral manifolds with different local geometry:
  Constructions
  Inaudible geometric properties
- Graphs, plane domains, and broken drums
- Open questions

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Laplace Spectrum versus Geodesic Length Spectrum

Riemann Surfaces (H. Huber 1959)

M_1 and M_2 Riemann surfaces.

$$\mathrm{spec}(M_1) = \mathrm{spec}(M_2)$$

$$\Longleftrightarrow$$

They have the same geodesic length spectrum.

Generic Manifolds (Y. Colin de Verdière 1973, J. Duistermaat–V. Guillemin 1975)

For generic Riemannian manifolds M,

$\mathrm{spec}(M)$ determines the geodesic length spectrum.

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But . . .

Let $\mathrm{spec}_p(M)$ denote the spectrum of the Hodge Laplacian on p-forms.

(Duistermaat–Guillemin) Generically $\mathrm{spec}_p(M)$ also determines the geodesic length spectrum.

Contrast

(R. Miatello–J.P. Rossetti 2003) Counterexamples: Flat 4-manifolds M_1 and M_2 with $\mathrm{spec}_1(M_1) = \mathrm{spec}_1(M_2)$ but different lengths of closed geodesics.

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Heat Invariants

Asymptotic expansion as $t \to 0^+$:

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} \sim (4\pi t)^{-\frac{n}{2}}(a_0 + a_1 t + a_2 t^2 + \dots)$$

So $Spec(M)$ determines:

- $n = dim(M)$
- $a_0 = vol(M)$
- $a_1 = \frac{1}{6}\int_M scal_g$. In dimension 2, this gives $\chi(M)$
- $360a_2 = 5\int_M scal_g^2 - 2\int_M \|Ric_g\|^2 - 10\int_M \|R_g\|^2$

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Manifolds with boundary

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- $a_{\frac{1}{2}} = const(vol(\partial M_{Dir}) - vol(\partial M_{Neum}))$

Thus in either the Dirichlet or Neumann cases, the boundary volume is audible.

Contrast

(M. Levitin, L. Parnovsky, and I. Polterovich 2005) For mixed boundary conditions, total boundary volume is not audible.

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Question

Are the heat invariants the only spectral invariants given by integrals of curvature expressions?

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Methods of construction

- Direct computation using generating functions
  Lens spaces (A. Ikeda 1980)
  Flat manifolds (R. Miatello, R. Podestá, J. P. Rossetti)
- Representation theoretic methods
  Prototype: Sunada's Theorem

General representation technique

Theorem

(DeTurck-G 1989)
(M, g) Riemannian manifold.

G Lie group acting isometrically on M.

H_1 and H_2 discrete subgroups of G acting freely and properly discontinuously on M with $H_i \backslash M$ compact.

Assume H_1 and H_2 are representation equivalent; i.e.,

$$L^2(H_1 \backslash G) \simeq L^2(H_2 \backslash G)$$

as G-modules.
Then

$$spec\,(H_1 \backslash M) = spec\,(H_2 \backslash M).$$

Sunada's technique (G finite)

Proposition

(Gassman) Assume G is finite. Then H_1 and H_2 are representation equivalent subgroups of G

$$\Longleftrightarrow$$

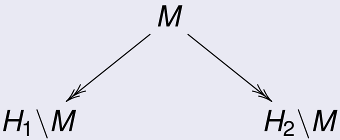
H_1 and H_2 are almost conjugate in G; i.e.,

$$\#([x] \cap H_1) = \#([x] \cap H_2)$$

for every G-conjugacy class $[x]$.

Sunada's Theorem (1985)

Let M be a compact Riemannian manifold and let G be a finite group of isometries of M. Suppose H_1 and H_2 are almost conjugate subgroups of G.



Then

1. spec $(\Gamma_1 \backslash M)$ = spec $(\Gamma_2 \backslash M)$.
2. $H_1 \backslash M$ and $H_2 \backslash M$ have the same geodesic length spectrum.

We will say $H_1 \backslash M$ and $H_2 \backslash M$ are *Sunada isospectral*.

Example

$G = SL(3, \mathbf{Z}_2)$

$$H_1 = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$$

$[G : H_i] = 7$

G can be generated by two elements.

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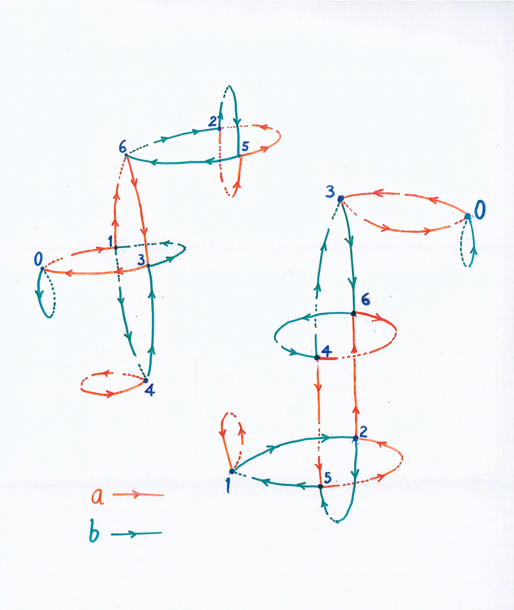
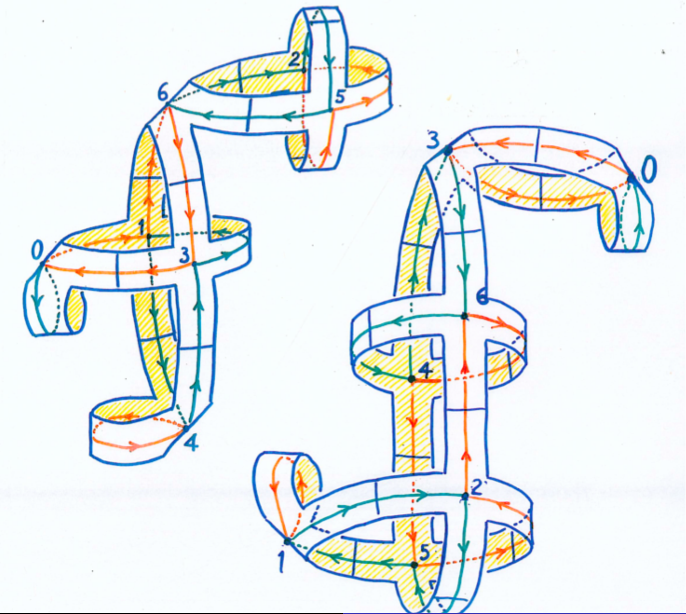
Schreier graphs



Isospectral flat surfaces (P. Buser)



Isospectral Riemann Surfaces and Other Locally Symmetric Spaces

- (Brooks-Gornet-Gustafson 1998) Collections of $g^{c\log(g)}$ mutually isospectral Riemann surfaces of genus g.
- (D. B. McReynolds, preprint) Analogous result for locally symmetric spaces of every noncompact type.

Contrast

(P. Buser) Any collection of mutually isospectral Riemann surfaces of genus g has at most $\exp(720g^2)$ members.

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Geometry of the examples

Examples of isospectral manifolds with a common covering tell us that:

You can't hear:

- $\pi_1(M)$ (Vignéras, 1982)
- diameter (Buser, 1988)
- whether M (with boundary) is orientable (P. Bérard–D. Webb, 1995)
- Betti numbers (R. Miatello–J.P. Rossetti, 2001)
- whether M has a spin structure (R. Miatello and R. Podestà, 2004)

You can't hear:

- whether (M, g) is Kähler. In fact a Kähler manifold can be isospectral to a manifold that is not topologically Kähler. (R. Miatello–J.P. Rossetti, 2001)
- whether (M, g) is hyper-Kähler (R. Miatello–J.P. Rossetti, 2001)
- whether the geodesic flow of (M, g) is completely integrable (D. Schueth 2008)

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Methods of construction

- Representation theoretic method – Sunada modified (C. Sutton)
- Torus action technique (G, D. Schueth ...)

Spectrum of a submersion with totally geodesic fibers

Suppose

$$\begin{array}{c} M \\ \pi \big\downarrow \\ H\backslash M \end{array}$$

is a Riemannian submersion with totally geodesic fibers. Then eigenfunctions on $H\backslash M$ lift to G-invariant eigenfunctions on M, so

$$\mathrm{spec}(H\backslash M) \subset spec(M)$$

Representation theoretic method (Sunada modified)

Theorem

(Sutton 2002)
Let G be a compact Lie group acting by isometries on a compact Riemannian manifold (M, g). Let H_1 and H_2 be closed subgroups of G that act freely on M. Give $H_i \backslash M$ the Riemannian metric induced by g. Assume:

- *H_1 and H_2 are representation equivalent in G.*
- *The fibers of $M \to H_i \backslash M$ are totally geodesic.*

Then

$$spec(H_1 \backslash M) = spec(H_2 \backslash M).$$

Example

(Sutton 2002)
Isospectral, simply-connected normal homogeneous spaces $H_1\backslash SU(n)$ and $H_2\backslash SU(n)$.

Dimension $\sim 10^{10}$!

Example

(Jinpeng An, Jun Yu, Jiu-Kang Yu 2011) New examples:

dimension 26

Different Homotopy Type!

Idea of Torus Action Method

We will show a pair of manifolds M_1 and M_2 are isospectral by showing that they have *enough isospectral quotients* of the form:

$$\begin{array}{ccc} M_1 & & M_2 \\ \downarrow & & \downarrow \\ K\backslash M_1 & \sim & K\backslash M_2 \end{array}$$

<u>**Contrast with Sunada's technique**</u> In Sunada's Theorem, one fixes M and finds isospectral quotients.

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ \Gamma_1\backslash M & & \Gamma_2\backslash M \end{array}$$

Lemma

Let T be a torus and suppose ρ is a representation of T on a (complex) vector space V.
Then

$$V = \sum_{K<T \text{ of codim } 1} V^K.$$

(The sum is over closed subgroups K, i.e. "subtori". Here V^K denotes the K-fixed vectors in V.)

Proof.

May assume ρ is irreducible. Since T is abelian, $\dim(V) = 1$. Thus

$$\rho : T \to U(1)$$

Let $K = \ker(\rho)$. Then K has co-dimension one and $V = V^K$. □

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Torus action method – simplest form

Theorem

Let M_1 and M_2 be compact Riemannian manifolds. Suppose a torus T acts isometrically and (freely) on M_1 and M_2 and that the (orbits are totally geodesic).

- *Assume: For all subtori $K \leq T$ of codimension ≤ 1,*

$$Spec(K \backslash M_1) = Spec(K \backslash M_2).$$

Then $Spec(M_1) = Spec(M_2)$.

Proof

- Key Hypothesis: For all subtori $K \leq T$ of codimension ≤ 1,

$$Spec(K\backslash M_1) = Spec(K\backslash M_2).$$

- To show: $\text{spec}(M_1) = \text{spec}(M_2)$

T acts on $C^\infty(M_i)$ and commutes with Δ_i. By the lemma,

$$C^\infty(M_i) = \sum_{K<T \text{ of codim } 1} C^\infty(M_i)^K, \quad \text{so}$$

$$spec(M_1) = \coprod_K \text{spec}(\Delta_{M_1}|_{C^\infty(M_1)^K}) = \coprod_K \text{spec}(K\backslash M_i).$$

$$||$$

$$spec(M_2) = \coprod_K \text{spec}(\Delta_{M_2}|_{C^\infty(M_2)^K}) = \coprod_K \text{spec}(K\backslash M_2).$$

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Examples

- (G 2003, Schueth 2003) Isospectral deformations of metrics on spheres and balls
- (Schueth 2001, Proctor 2005) Isospectral deformations of left-invariant metrics on the classical compact simple Lie groups
- (Szabo-G 2003) Isospectral deformations of negatively curved metrics on a manifold with boundary

Contrast

(V. Guillemin–D. Kazhdan 1980, C. Croke–V. Sharafutdinov 1998) Can't isospectrally deform a negatively curved metric on a closed manifold.

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What the examples tell us about geometry

The following properties are not spectral invariants:

- (Z. Szabo 1999) homogeneity
- (Szabo-G 2003) constant scalar curvature
- (Szabo-G) (for manifolds with boundary) constant Ricci curvature
- (Schueth–T. Arias-Marco 2010) local symmetry, weak local symmetry, D'Atri, probablistic commutativity, type $\mathcal{A}$, type $\mathcal{C}$.

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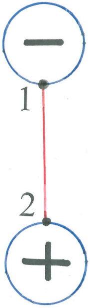
Colored loop-signed Graphs (Peter Herbrich)

(Motivation: Substantial generalization of Sunada's theorem by R. Band–O. Parzanchevsky 2011.)



$$A_{red} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_{blue} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

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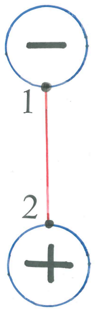

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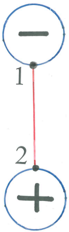
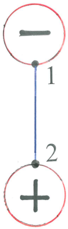
Isospectral Loop-signed edge-colored graphs (Peter Herbrich)





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$$A_{blue} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A'_{blue} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

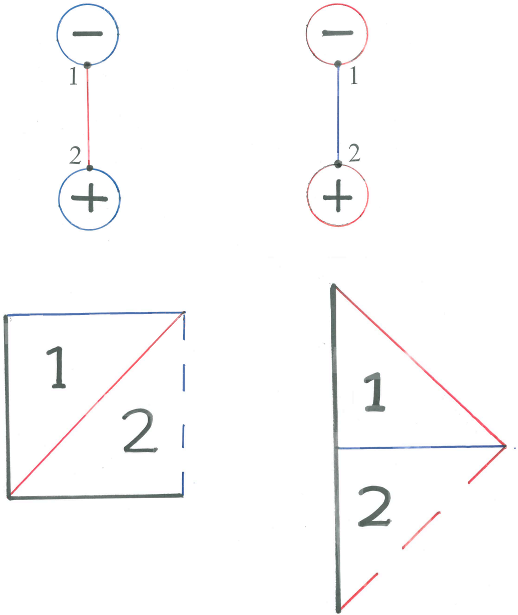
Then

$$BA_{red}B^{-1} = A'_{red}$$

and

$$BA_{blue}B^{-1} = A'_{blue}$$

We say that these loop-signed edge color graphs G and G' are isospectral.


−
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2

Sound-alike Broken Drums (Levitin–Parnovsky–Polterovich))

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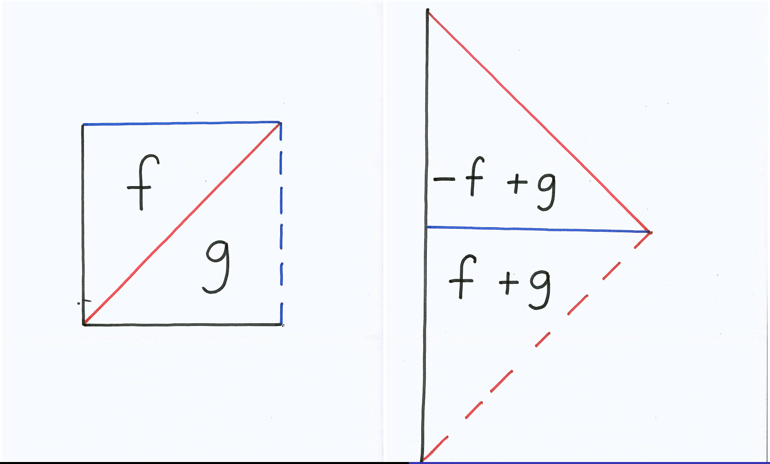

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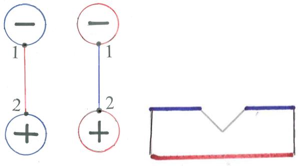
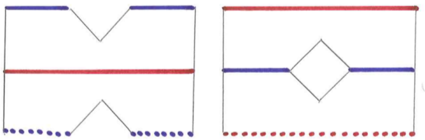


Building block



One has a hole; the other doesn't.

This can't happen with pure boundary conditions.

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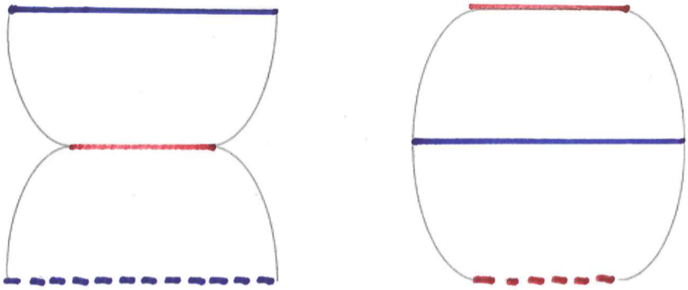
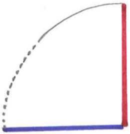


One is convex; the other not.
One is smooth; the other not.

Can this happen with pure boundary conditions?



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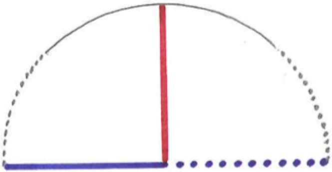
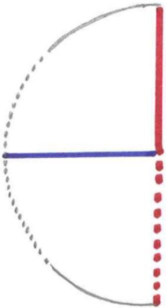
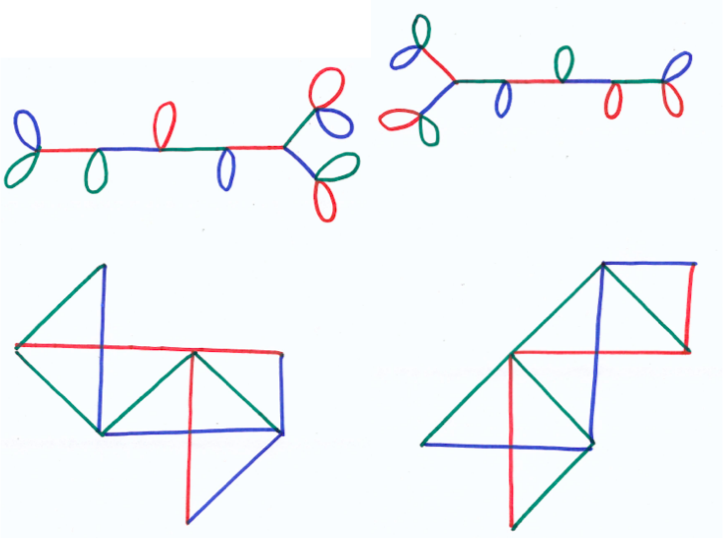




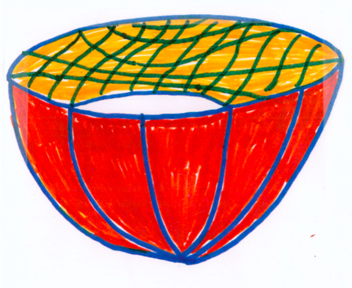
Same membrane but boundary conditions switched.

P. Bérard, P. Buser–K.D. Semmler (simplfication of Webb-Wolpert-G)



A broken drum can sound like an unbroken drum of a different shape!

(Peter Herbrich, preprint)



Outline

- Geometry contained in the spectrum
- Isospectral manifolds with a common covering:
  Constructions
  Inaudible geometric properties
- Isospectral manifolds with different local geometry:
  Constructions
  Inaudible geometric properties
- Graphs, plane domains, and broken drums
- Open questions

Open Questions

• Can you tell from the spectrum whether a metric has constant curvature?

Progress

Constant curvature metrics are spectrally isolated.

- Flat (R. Kuwabara 1980)
- Round (Tanno 1980)
- Hyperbolic (V. Sharafutdinov 2011)

• Can you tell from the spectrum whether a closed manifold is Einstein?

• Can you tell from the spectrum whether a Riemannian manifold is symmetric?

Progress

(D. Schueth–C. Sutton–G 2010) Bi-invariant metrics on compact Lie groups are spectrally isolated among all left-invariant metrics.

"Local" spectral data

• Let M be a possibly noncompact locally homogeneous Riemannian manifold. What information about M is contained in the collection of spectra of arbitrarily small geodesic spheres (or balls)?

Example

(D. Schueth and T. Arias-Marco 2010) The collection of spectra of arbitrarily small geodesic spheres (or balls) determines whether a harmonic manifold is locally symmetric.

Contrast

(Szabo-G 2003) There exist a locally symmetric harmonic manifold M (covered by quaternionic hyperbolic space) and a non-symmetric harmonic manifold M' and arbitrarily small domains $\Omega \subset M$ and $\Omega' \subset M'$ such that $spec(\Omega) = spec(\Omega')$.

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Remarks. (H. Hezari–S. Zelditch, 2010) Analogous result for domains in $\mathbf{R}^n$.

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Thank you!