Geometries modelled on some rank two symmetric spaces

Ilka Agricola, Julia Becker-Bender, Thomas Friedrich, Jos Höll

Philipps Universität Marburg

Cartan's work on isoparametric hypersurfaces

Dfn. M^{m-1} immersed into $\mathbb{R}^m, S^m$, or H^m is called an *isoparametric hypersurface* if its principal curvatures are constant [$\Rightarrow$ constant mean curvature]. Set $p :=$ # of different principal curvatures

Thm. In $S^{n-1} \subset \mathbb{R}^n$ [Cartan 1938-40]:
- If $p = 1$: M^{n-2} is a hypersphere in S^{n-1}
- If $p = 2$: $M^{n-2} = S^p(r) \times S^p(s)$ for $p + q = n - 2,\ r^2 + s^2 = 1$
- If $p = 3$: M^{n-2} is a tube of constant radius over a generalized Veronese embedding of $\mathbb{KP}^2$ into S^{n-1} for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

Thus, for $p = 3$, n must be $5, 8, 14$, or 26 !

Construction: use harmonic homogeneous polynomial F of degree p on $\mathbb{R}^n$ satisfying
$$\|\mathrm{grad}\, F\|^2 = p^2\|x\|^{2p-2}$$
The level sets of $F\big|_{S^{n-1}}$ define an isoparametric hypersurface family. For $p = 3$, Cartan described explicitly the polynomial F.

Link to geometry: F can be understood as a symmetric rank p tensor Υ, and each level set M will be invariant under the stabilizer of Υ!

If $M^{n-2} \subset S^{n-1} = \mathrm{SO}(n)/SO(n-1)$ is an orbit of $G \subset \mathrm{SO}(n)$, then it is isoparametric (because it is homogeneous):

classif. of all $G \subset \mathrm{SO}(n)$ s.t. $\mathrm{codim}\big\|_{S^{n-1}}$ (princ. G-orbit)=1 or, equiv., $\mathrm{codim}\big\|_{\mathbb{R}^n}=2$	$\Rightarrow$	classif. of homogeneous isopar. hypersurfaces in S^{n-1}

Needed: a classification of all irreducible reps. of $G \subset \mathrm{SO}(n)$ on $\mathbb{R}^n$ with codimension 2 principal orbits.

Thm.[Hsiang² / Lawson, 1970/71] These are exactly the isotropy representations of rank 2 symmetric spaces.

The proof produces a list, and it turns out to coincide with the list of isotropy representations.

Thm.[Takagi-Takahashi, 1972] Let $M^n = G/H$ compact symmetric space, $\mathrm{rk} = 2$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$.
- An H-orbit M of a unit vector in $S^{n-1} \subset \mathfrak{p}$ is an isoparametric hypersurface.
- The principal curvatures and their multiplicities are computed from the root data, for example: The order of the Weyl group is $2p \Rightarrow$ only $p = 1, 2, 3, 4, 6$ are possible

$\Rightarrow$ In the case $p = 3$, there are **4 symmetric spaces yielding isoparametric hypersurfaces:**

$$\mathrm{SU}(3)/SO(3),\ \mathrm{SU}(3),\ \mathrm{SU}(6)/\mathrm{Sp}(3),\ E_6/F_4$$

Description of their isotropy representations

Let $\mathbb{R}^n$ $(n = 5, 8, 14, 26)$ be $\mathrm{Her}_0(\mathbb{K}^3)$, the Hermitian trace-free endomorphisms on $\mathbb{K}^3$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ with the conjugation action of $H_n = \mathrm{SO}(3), \mathrm{SU}(3), \mathrm{Sp}(3)$, or F_4, resp.
Define for $X, Y, Z \in \mathbb{R}^n$ a symmetric 3-tensor by polarisation from tr:
$$\Upsilon(X,Y,Z) := 2\sqrt{3}[\mathrm{tr}\, X^3 + \mathrm{tr}\, Y^3 + \mathrm{tr}\, Z^3] - tr(X+Y)^3 - \mathrm{tr}\,(X+Z)^3 - \mathrm{tr}\,(Y+Z)^3 + \mathrm{tr}\,(X+Y+Z)^3.$$
For $\mathbb{K} = \mathbb{H}, \mathbb{O}$, a second tensor is obtained as $\tilde{\Upsilon}(X,Y,Z) := \Upsilon(\bar{X}, \bar{Y}, \bar{Z})$ – it is not conjugate to Υ under $\mathrm{SO}(n)$.

Thm. For $n = 5, 8, 14, 26$: $H_n = \{A \in \mathrm{SO}(n) : A^*\Upsilon = \Upsilon\}$ and for any basis $V_1, \ldots V_n$ of $\mathbb{R}^n \cong \mathrm{Her}_0(\mathbb{K}^3)$
- Υ is totally symmetric,
- Υ is trace-free, i.e. $\sum_i \Upsilon(X, V_i, V_i) = 0$,
- Υ satisfies the identity (g: metric) $\displaystyle\mathop{\mathfrak{S}}_{X,Y,Z} \sum_i \Upsilon(X,Y,V_i)\Upsilon(Z,U,V_i) = \mathop{\mathfrak{S}}_{X,Y,Z} g(X,Y)g(Z,U)$

In particular: Υ determines g!

N.B. For $n = 8, 14$, $\exists$ an alternative tensor reducing $\mathrm{SO}(n)$ to H_n: $n = 8$: a 3-form, $n = 14$: a 5-form

H_n-structures on Riemannian manifolds

Dfn. For $n = 5, 8, 14, 26$: A n-mnfd with a H_n-structure is a Riemannian mnfd (M^n, g) with a reduction of the frame bundle $\mathcal{R}(M^n)$ to H_n and thus has automatically a 3-tensor Υ with the properties above!

Dfn. A H_n-mnfd is called *integrable* if $\nabla^g\Upsilon = 0$ (∇^g: Levi-Civita conn.) ($\Rightarrow$ $\mathrm{Hol}_0(\nabla^g) \subset H_n$).

Thm. [Nurowski, 2007] An integrable H_n-structure is isometric to one of the symmetric spaces G_n/H_n, i.e.
$$\mathrm{SU}(3)/SO(3),\ \mathrm{SU}(3),\ \mathrm{SU}(6)/\mathrm{Sp}(3),\ E_6/F_4,$$
or one of their non-compact dual symmetric spaces.

I. Agricola, J. Becker-Bender, M. Bobinski, S. Chiossi, A. Fino, T. Friedrich and P. Nurowski looked at the case $n = 5$. The case $n = 8$ was studied by N. Hitchin, C. Puhle and I. Witt. I. Agricola, T. Friedrich and J. Hoell looked at the 14 dimensional case.

Topological existence conditions: the case $H_5 = \mathrm{SO}(3)$

$\exists$ two non-equivalent embeddings $\mathrm{SO}(3) \to \mathrm{SO}(5)$: as upper diagonal block matrices: '$\mathrm{SO}(3)_{\mathrm{st}}$' and by the irreducible 5-dim. representation of SO(3): '$\mathrm{SO}(3)_{\mathrm{ir}}$'.
Dfn. Kervaire semi-characteristics:
$$k(M^5) := \sum_{i=0}^{2} \dim_{\mathbb{R}}\big(H^{2i}(M^5;\mathbb{R})\big) \mod 2\,, \qquad \hat{\chi}_2(M^5) := \sum_{i=0}^{2} \dim_{\mathbb{Z}_2}\big(H_i(M^5;\mathbb{Z}_2)\big) \mod 2\,.$$
Thm. [Lusztig-Milnor-Peterson 1969] $k(M^5) - \hat{\chi}_2(M^5) = w_2(M^5) \cup w_3(M^5)$.
In particular, if M^5 is spin, then $k(M^5) = \hat{\chi}_2(M^5)$.

Thm. [Thomas 1967; Atiyah 1969] A cpt. oriented 5-mnfd admits a $\mathrm{SO}(3)_{st}$-structure iff $w_4(M^5) = 0,\ k(M^5) = 0$.

Topological existence conditions for $\mathrm{SO}(3)_{ir}$-structures were investigated in [ABF11]:

Exa: $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ has an $\mathrm{SO}(3)_{ir}$-structure and $k(M^5) = 1$ and $\hat{\chi}_2(M^5) = 0$. In particular, $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ does not admit any $\mathrm{SO}(3)_{st}$-structure!

Prop. M^5 admits an $\mathrm{SO}(3)_{ir}$-structure iff there exists a 3-dim. real bundle E^3 such that $T(M^5) = S^2_0(E^3)$.

Thm. Suppose that $T(M^5) = S^2_0(E^3)$. Then $p_1(M^5) = 5 \cdot p_1(E^3)$; in particular, $p_1(M^5)/5 \in H^4(M^5;\mathbb{Z})$ is integral. $w_1(M^5) = w_4(M^5) = w_5(M^5) = 0$, $w_2(M^5) = w_2(E^3)$ and $w_3(M^5) = w_3(E^3)$.

Conjecture: M^5 admits an $\mathrm{SO}(3)_{ir}$-structure iff $w_4(M^5) = 0$, $\hat{\chi}_2(M^5) = 0$, $\frac{p_1(M^5)}{5} \in H^4(M^5;\mathbb{Z})$.

Can only prove: **Thm.** A compact, s.c. spin mnfd admitting a $\mathrm{SO}(3)_{ir}$- or $\mathrm{SO}(3)_{st}$-str. is parallelizable.

Cor. S^5 has none of both SO(3)-structures.
Exa: The connected sums $(2l+1)\#(S^2 \times S^3)$ are s.c., spin and admit a $\mathrm{SO}(3)_{st}$-structure.

A rather sophisticated construction yields:

Thm. There exist mnfds $p\mathbb{CP}^2 \# q\overline{\mathbb{CP}^2}$ such that *every* S^1-bundle over them admits a SO_{ir}-structure. (for example: $(p,q) = (21,1), (43,3), (197,17)\ldots$)

Topological existence conditions: the case $H_{14} = \mathrm{Sp}(3)$ [AFH12]

From $H^*(B\mathrm{Sp}(3), \mathbb{Z}) = \mathbb{Z}[q_4, q_8, q_{12}]$ (with $q_i \in H^i$), one deduces:

Thm. Every compact 14-dimensional mnfd with a Sp(3)-structure satisfies $\chi(M) = 0$ and $w_i(M) = 0$ except for $i = 4, 8, 12$.

In particular, it is orientable and spin; for example, S^{14} has no Sp(3)-structure.

Open problem: sufficient and necessary conditions !

Some non-compact examples: use isom. $\mathrm{Spin}(5) \cong \mathrm{Sp}(2) \subset \mathrm{Sp}(3)$ and the decomposition
$\mathbb{R}^{14} \overset{\mathrm{Spin}(5)}{=} \mathbb{R} \oplus \mathbb{R}^5 \oplus \Delta_5$ (the 5-dim. spin rep.)
Every S^1-bundle M^{14} over one of the following
- spin bundle of a 5-dim. spin mnfd X^5 (= 8-dim VB)
- associated bundle $\mathcal{R}(Y^8) \times_{\mathrm{Spin}(5)} \mathbb{R}^5$ over an 8-dim. mnfd Y^8 with an Sp(2)-structure (hyper-Kähler, quaternionic-Kähler etc.) carries a Sp(3)-structure.

Characteristic connections and types of H_n-structures

General philosophy: Given a mnfd M^n with G-structure ($G \subset \mathrm{SO}(n)$), replace ∇^g by a *metric connection ∇ with torsion that preserves the geometric structure!*
$$\text{torsion: } T(X,Y,Z) := g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$
Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)
$$\Rightarrow \quad g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \tfrac{1}{2}T(X,Y,Z)$$
If existent, this connection is called the *'characteristic connection'*.

Thm:[AFH12] The characteristic connection is unique if the action of G on $\mathbb{R}^n$ is not the adjoint representation (proof makes heavy use of skew holonomy theorem).

Let $T \in \Lambda^3(M)$ be the torsion of the char. connection ∇. Decompose $\Lambda^3(\mathbb{R}^n)$ under H_n-action, for example:
$$\Lambda^3(\mathbb{R}^5) \cong \Lambda^2(\mathbb{R}^5) \cong \mathfrak{so}(5) = \mathfrak{so}(3)_{\mathrm{ir}} \oplus V^7, \quad \Lambda^3(\mathbb{R}^{14}) \cong \mathfrak{sp}(3) \oplus V^{70} \oplus V^{84} \oplus V^{189}$$
We say that a H_n-structure is of type $X, Y \oplus Z \ldots$ if $T \in X, Y \oplus Z \ldots \subset \Lambda^3(M)$ and of general type if T has contributions in all parts of $\Lambda^3(M)$.

Homogeneous examples: the case $H_5 = \mathrm{SO}(3)$

Exa 1: 'twisted' Stiefel mnfd $V^{\mathrm{ir}}_{2,4} = \mathrm{SO}(3) \times \mathrm{SO}(3)/\mathrm{SO}(2)_{\mathrm{ir}}$
Recall: classical Stiefel manifold $V^{\mathrm{st}}_{2,4} = \mathrm{SO}(4)/\mathrm{SO}(2)$: Carries an $\mathrm{SO}(3)_{\mathrm{st}}$ structure, an Einstein-Sasaki metric, 2 Riemannian Killing spinors [Jensen 75, Friedrich 1981]

Consider now $H := \mathrm{SO}(2) \subset \mathrm{SO}(3)_{\mathrm{ir}}, H \ni A \longmapsto (A, A^2) \in \mathrm{SO}(3) \times \mathrm{SO}(3) =: G$, $V^{\mathrm{ir}}_{2,4} := \mathrm{SO}(3) \times \mathrm{SO}(3)/\mathrm{SO}(2)_{\mathrm{ir}}$.
With $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, its isotropy rep. decomposes $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, thus the metric has three parameters $\alpha, \beta, \gamma > 0$.

Thm.[ABF11]
- If $\alpha\beta + 4\gamma\alpha - 25\beta\gamma = 0$, there exists a characteristic connection for the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure.
- Its holonomy is $\mathrm{SO}(2)_{\mathrm{ir}}$ and its torsion is parallel.
- The metric of a $\mathrm{SO}(3)_{\mathrm{ir}}$ structure with char. conn. is naturally reductive if and only if $\alpha = 5\beta = 5\gamma$.
- $\exists_1$ Einstein metric, not nat. reductive.
- $\exists$ two invariant almost contact metric structures. Both admit a unique characteristic connection.
- The contact structure is Sasakian (but never Einstein) if and only if $\alpha = 25\beta^2 = 100\gamma^2$; it is in addition an $\mathrm{SO}(3)_{\mathrm{ir}}$ structure for $(\alpha, \beta, \gamma) = (\frac{25}{36}, \frac{1}{6}, \frac{1}{12})$.

Exa 2: $W^{\mathrm{ir}} = \mathbb{R} \times (\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2)/\mathrm{SO}(2)_{\mathrm{ir}}$
Decompose again $\mathfrak{m} = \mathfrak{n}^{\mu} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with the same Ansatz for the metric.

Thm.[ABF11]
- $\forall\ \alpha, \beta, \gamma > 0$ s.t. $\alpha \geq 12\gamma$, the $\mathrm{SO}(3)_{\mathrm{ir}}$ structure admits a characteristic connection.
- Its holonomy is $\mathrm{SO}(3)_{\mathrm{ir}} \subset \mathrm{SO}(5)$. Its torsion is *not* parallel, but it is divergence-free, $\delta T^{\alpha\beta\gamma} = 0$.
- The metric of the $\mathrm{SO}(3)_{\mathrm{ir}}$ str. with char. conn. is never naturally reductive and never Einstein.
- $\nexists$ a compatible contact structure.

Consequence: $\mathrm{SO}(3)_{\mathrm{ir}}$ structures are different from contact str. and define a new type of geometry on 5-mnfds.

Homogeneous examples: the case $H_{14} = \mathrm{Sp}(3)$

Exa 1: Higher Aloff-Wallach mnfd $M^{14} = \mathrm{SU}(4)/S^1$
Embed S^1 as $\mathrm{diag}(e^{-it}, e^{-it}, e^{it}, e^{-it}) \subset \mathrm{SU}(4)$.
The splitting $\mathfrak{su}(4) = \mathbb{R} \oplus \mathfrak{m}^{14}$ leads to the decomposition $\mathfrak{m}^{14} = \bigoplus_{i=1}^{4} V_i \oplus \bigoplus_{j=1}^{6} W_j$, $\dim V_i = 2$, $\dim W_j = 1$ under the action of S^1. There are invariant metrics g depending on $\alpha_1, \ldots, \alpha_{10}$.

Thm.[AFH12]
- $\exists$ a 3-dim. space of metrics, depending on $\alpha, \beta, \gamma > 0$, such that the Sp(3)-structure admits a char. connection.
- If there exists a characteristic connection, its torsion is parallel.
- The Sp(3)- structure is always of general type, i.e. its torsion has contributions in all summands of $\Lambda^3(M)$.
- The Riemannian Ricci curvature has Eigenvalues $6\alpha - \beta$, $6\alpha - \gamma$, $6\alpha - \beta - \gamma$, each with multiplicity 4 and 4β, 4γ with mult. 1. Thus its scalar curvature is
$$scal^{g_{\alpha,\beta,\gamma}} = \frac{2(18\alpha - \beta - \gamma)}{\alpha^2}$$
and the metric is never Einstein.

Exa 2: The homogeneous space $M^{14} = \mathrm{SU}(5)/\mathrm{Sp}(2)$ as a mnfd M^{14} is the same as $\mathrm{SU}(6)/\mathrm{Sp}(3)$, but not symmetric.
- $\mathfrak{su}(5) = \mathfrak{sp}(2) \oplus \mathfrak{m}^{14}$, $\mathfrak{m}^{14} = \mathbb{R} \oplus \mathbb{R}^5 \oplus \Delta_5$ (recall $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$)
- 3 deformation parameters α, β, γ in the metric.

Thm.[AFH12]
- All metrics admit a characteristic connection for the Sp(3)-structures.
- The characteristic connection has full holonomy Sp(3) if $\alpha \neq \beta$.
- The Sp(3)-structure is of type $\mathfrak{sp}(3)$ if $\alpha = \frac{1}{4}(\sqrt{15\beta\gamma} - \beta)$, of type V^{189} if $\alpha = \frac{1}{12}(9\beta - \sqrt{15\beta\gamma})$, integrable if $\beta = 2\alpha$ and $\gamma = \frac{6}{5}\alpha$, and of general type otherwise.
- The torsion is parallel if either $\beta = \alpha$ or ($\beta = 2\alpha$ and $\gamma = \frac{6}{5}\alpha$).
- The Riemannian curvature tensor has then 3 EV's of mult. 1, 5 and 8 given by 5γ, $\frac{8\alpha^2+\beta^2}{\beta}$ and $10\alpha - \frac{5}{4}(\beta + \gamma)$.

In particular, the metric is Einstein if $\beta = \sqrt{2}\alpha = \frac{1}{\sqrt{8}-1}\gamma$. In this case the Ricci tensor is
$$Ric^{g_{\alpha\beta\gamma}} = \frac{5}{2\alpha^2} g_{\alpha\beta\gamma}.$$

[ABF11] I. Agricola, J. Becker-Bender, T. Friedrich *On the topology and the geometry of* SO(3)-*manifolds*, Ann. Global Anal. Geom. 40 (2011), pp. 67-84.
[AFH12] I. Agricola, T. Friedrich, J. Höll, Sp(3) *structures on 14-manifolds* to appear.