Nonnegatively Curved Homogeneous spaces

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For M^n, $n \geq 4$, we have no classification.

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- We can mod out by a closed subgroup of G on left:

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- We can further mod out by another closed subgroup acting on the right:

$$\begin{array}{c} G \\ \downarrow \\ K\backslash G/H \end{array}$$

Spaces of positive sectional curvature

Homogeneous spaces which admit a homogeneous metric of positive sectional curvature are classified:

1. rank one symmetric spaces
2. even-dimensional examples, found by Wallach (1972): $W^6 = SU(3)/T^2$, $W^{12} = Sp(3)/(Sp(1))^3$, and $W^{24} = F_4/Spin(8)$.
3. odd-dimensional examples, found by Bérard-Bergery (1976): the Berger spaces $B^7 = SO(5)/SO(3)$ (here $SO(3)$ is maximal subgroup) and $B^{13} = SU(5)/Sp(2) \cdot S^1$.

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Dimensions: $n = 6$, 7, 12, 13, and 24, only (as well as compact rank-one symmetric spaces).

Spaces of nonnegative sectional curvature

There are many more examples of manifolds with nonnegative sectional curvature.

All known examples obtained by one of these constructions:

- Take an isometric quotient of a compact Lie group with a biinvariant metric, or
- Apply a gluing procedure referred to as a *Cheeger deformation*, generalized by Grove and Ziller.

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  A Cheeger deformation is still a quotient, where we mod out by an isometric group action:
  G acts by isometries on M. We have a fibration

$$M \times G \to (M \times G)/\Delta G \cong M.$$

  The action of G (on the product $M \times G$) is $g \star (p, h) = (gp, gh)$.
  On $M \times G$, deform by scaling in the direction of the orbits of G. Get a submersion metric on the base space M.

Spaces of nonnegative sectional curvature

A piece of the big question:

- On a given manifold, how large is the set of nonnegatively curved metrics?

- Schwachhöfer and Tapp investigated a deformation of a normal homogeneous metric g_0 on a compact homogeneous space G/H.

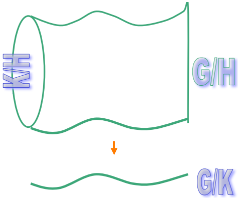
Space of invariant metrics

- Schwachhöfer and Tapp prove that the family of invariant metrics is star-shaped with respect to any normal homogeneous metric.
- Invariant metrics are identified with their corresponding symmetric matrices, which are parametrized by their inverses.
- Thus the problem of determining all invariant metrics with nonnegative curvature reduces to determining how long nonnegative curvature is maintained when deforming along a linear path (starting at a normal homogeneous metric).

Riemannian submersions of homogeneous spaces

Joint work with Andreas Kollross

- Start with a homogeneous space G/H with $H < K < G$, where G is a compact, simply connected Lie group (or $G = SO(N)$) endowed with a biinvariant metric g_0.



- We have a fibration $K/H \to G/H \to G/K$.

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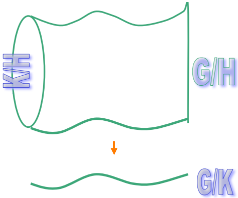

- We have a fibration $K/H \to G/H \to G/K$.
- For parameter t we define a family of metrics on G/H:

$$g_t = \left(\frac{1}{1-t}\right) g_0(X^{\mathfrak{m}}, Y^{\mathfrak{m}}) + g_0(X^{\mathfrak{s}}, Y^{\mathfrak{s}})$$

Here $t < 1$ means that we are enlarging the fiber.

Fibration metrics

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ ($\mathfrak{s}$ is the horizontal component)
$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ ($\mathfrak{m}$ is the vertical component)

Theorem

(Schwachhöfer-Tapp) (1) The metric g_t has nonnegative curvature for small $t > 0$ if and only if there exists some $C > 0$ such that for all X and Y in $\mathfrak{p}$,

$$|[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}}| \leq C|[X, Y]|. \qquad (*)$$

(2) In particular, if (K, H) is a symmetric pair, then g_t has nonnegative curvature for small $t > 0$, and in fact for all $t \in (-\infty, 1/4]$.

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What does their theorem tell us?

- Part (1) has an 'if and only if': very strong!
- But we don't know when $(*)$ holds. In fact, for a given triple (H, K, G) we don't know how to find the constant C or even whether any such constant exists.
- Part (2) is the observation that (K, H) a symmetric pair means $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \Rightarrow [\mathfrak{m}, \mathfrak{m}]^{\mathfrak{m}} = 0$, so that the inequality $(*)$ holds trivially.
- Question: When does $(*)$ hold, aside from the case that (K/H) is a symmetric pair?

A clue

Consider two chains:

$$\mathrm{SU}(2) \subset \mathrm{SO}(4) \subset \mathrm{G}_2 \qquad \widetilde{\mathrm{SU}(2)} \subset \mathrm{SO}(4) \subset \mathrm{G}_2$$

Here $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset \mathrm{G}_2$, and $\mathrm{SU}(2)$, $\widetilde{\mathrm{SU}(2)}$ are not conjugate in G_2. For both, the base is $\mathrm{G}_2/\mathrm{SO}(4)$; the fibers are isometric to S^3.

$$\begin{array}{ccc} S^3 & \longrightarrow & \mathrm{G}_2/\mathrm{SU}(2) \\ & & \downarrow \\ & & \mathrm{G}_2/\mathrm{SO}(4) \end{array} \qquad\qquad \begin{array}{ccc} S^3 & \longrightarrow & \mathrm{G}_2/\widetilde{\mathrm{SU}(2)} \\ & & \downarrow \\ & & \mathrm{G}_2/\mathrm{SO}(4). \end{array}$$

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Condition $(*)$ holds for the first chain and cannot hold for the second chain.

Some results: Rank(G)=Rank(G/K)

In this class, the Satake diagram of G/K is the same as the Dynkin diagram of G, but with uniform multiplicity one. That is, $\mathfrak{s}$ contains a maximal abelian subalgebra of $\mathfrak{g}$.

Theorem (1)

Assume (G, K) is a symmetric pair such that $\mathrm{rk}(G/K) = \mathrm{rk}(G)$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the corresponding Cartan decomposition. Let $\mathfrak{t} \subset \mathfrak{s}$ be a maximal abelian subalgebra of $\mathfrak{g}$. [Choose a root space decomposition as above and assume there is a subset $S_+ \subset R_+$ such that the Lie algebra $\mathfrak{h}$ is spanned by X_α, $\alpha \in S_+$.] Then the triple (H, K, G) satisfies condition $(*)$ if and only if (K, H) is a symmetric pair.

More about the rank(G/K) = rank(G) case

- In fact, the case above exactly corresponds to the existence of a closed symmetric subalgebra $\mathfrak{l} \subset \mathfrak{g}$, such that $\mathfrak{h} = \mathfrak{l} \cap \mathfrak{k}$ and $\mathrm{rk}(\mathfrak{l}) = \mathrm{rk}(\mathfrak{g})$.
- While condition $(*)$ fails for the triples $H \subsetneq K \subsetneq G$ where (K, H) is not a symmetric pair, it must hold for the triples $H \subsetneq L \subsetneq G$, since (L, H) is a symmetric pair.
- Thus the total space G/H has a direction in which nonnegative curvature can be extended, but only by deforming in the direction of fibers L/H over base G/L, not by deforming in the direction of fibers K/H over base G/K.

A corollary of examples

The following chains (H, K, G) of compact Lie groups do not fulfill condition $(*)$:

1. $\mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \times \mathrm{SO}(n_3) \subset \mathrm{SO}(n) \subset \mathrm{SU}(n)$, $n_i \geq 1$, $n_1 + n_2 + n_3 = n$.
2. $[\mathrm{SO}(n_1+1) \times \mathrm{SO}(n_2) \times \mathrm{SO}(n_3)] \times [\mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \times \mathrm{SO}(n_3)] \subset \mathrm{SO}(n+1) \times \mathrm{SO}(n) \subset \mathrm{SO}(2n+1)$, $n_i \geq 1$, $n_1 + n_2 + n_3 = n$.
3. $\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3) \subset \mathrm{U}(n) \subset \mathrm{Sp}(n)$, $n_i \geq 1$, $n_1 + n_2 + n_3 = n$
4. $[\mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \times \mathrm{SO}(n_3)] \times [\mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \times \mathrm{SO}(n_3)] \subset \mathrm{SO}(n) \times \mathrm{SO}(n) \subset \mathrm{SO}(2n)$, where $n_i \geq 1$, $n_1 + n_2 + n_3 = n$.
5. $\mathrm{SO}(3) \cdot \mathrm{SO}(3) \cdot \mathrm{SO}(3) \subset \mathrm{Sp}(4) \subset \mathrm{E}_6$.
6. $\mathrm{SO}(3) \cdot \mathrm{SO}(6) \subset \mathrm{SU}(8)/\{\pm 1\} \subset \mathrm{E}_7$.
7. $\mathrm{SO}(3) \cdot \mathrm{Sp}(4) \subset \mathrm{SO}'(16) \subset \mathrm{E}_8$.
8. $\mathrm{SO}(3) \cdot \mathrm{SO}(3) \subset \mathrm{Sp}(3) \cdot \mathrm{Sp}(1) \subset \mathrm{F}_4$.

Some results: Rank(H)=Rank(K)=Rank(G)

Theorem (2)

Let G be a simple compact Lie group and let $H \subsetneq K \subsetneq G$ be closed subgroups. If $\mathrm{rk}(H) = \mathrm{rk}(K) = \mathrm{rk}(G)$ then either (K, H) is a symmetric pair or there exist elements $X, Y \in \mathfrak{p}$ such that $[X, Y] = 0$ and $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$.

A simple observation: Extending beyond equal ranks

Lemma

Let $H \subsetneq K \subsetneq G$ be a chain of compact groups for which there exists a pair of vectors $X, Y \in \mathfrak{p}$ such that $[X, Y] = 0$ but $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$. Let $G \subseteq G'$ and $H' \subsetneq H$ each be closed subgroups. Then condition $(*)$ fails for the chain $H' < K < G'$.

The same pair of commuting vectors $X, Y \in \mathfrak{p}$ is also a pair of commuting vectors in $\mathfrak{p}'$, with $[X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}]^{\mathfrak{m}'} \neq 0$.

Regular subgroups

Theorem (3)

Let G be a compact Lie group. Let $H \subsetneq K \subsetneq G$ be connected compact Lie groups such that H, K are regular subgroups of G. If the triple (H, K, G) satisfies condition $(*)$ then for each simple ideal $\mathfrak{g}_i$ of $\mathfrak{g}$ one of the following is true.

1. $\mathfrak{g}_i \cap \mathfrak{k} = \mathfrak{g}_i$, *i.e. the simple ideal* $\mathfrak{g}_i$ *is contained in* $\mathfrak{k}$.
2. $\mathfrak{g}_i \cap \mathfrak{k} \neq \mathfrak{g}_i$ *and* $(\mathfrak{g}_i \cap \mathfrak{k}, \mathfrak{g}_i \cap \mathfrak{h})$ *is a symmetric pair, possibly such that* $\mathfrak{g}_i \cap \mathfrak{k}$ *is contained in* $\mathfrak{h}$.
3. $\mathfrak{g}_i \cong \mathfrak{so}(2n+1)$, $\mathfrak{g}_i \cap \mathfrak{k} \cong \mathfrak{so}(2n)$ *and* $\mathfrak{g}_i \cap \mathfrak{h} \cong \mathfrak{su}(n)$.
4. $\mathfrak{g}_i \cong \mathfrak{sp}(n)$ *where all but one simple ideal of* $\mathfrak{g}_i \cap \mathfrak{k}$ *is contained in* $\mathfrak{h}$ *and the one simple ideal not contained in* $\mathfrak{h}$ *is isomorphic to* $\mathfrak{sp}(1)$.
5. $\mathfrak{g}_i \cong \mathrm{Lie}(\mathrm{G}_2)$, $\mathfrak{g}_i \cap \mathfrak{k} \cong \mathfrak{so}(4)$ *and* $\mathfrak{g}_i \cap \mathfrak{h} \cong \mathfrak{su}(2)$ *such that* $\mathfrak{g}_i \cap \mathfrak{h}$ *is contained in a subalgebra* $\mathfrak{su}(3) \subset \mathfrak{g}_i$.

Remark

For items (1), (2) (3) and (5) above, we know that condition $(*)$ holds for the chains $(\mathfrak{h} \cap \mathfrak{g}_i, \mathfrak{k} \cap \mathfrak{g}_i, \mathfrak{g}_i)$.

If condition $(*)$ holds also for each chain of regular subgroups $(H, K, G) = (\mathrm{Sp}(n), \mathrm{Sp}(1)^n, \mathrm{Sp}(1)^{n-1})$ with $n \geq 2$, then the previous theorem can be improved to "if and only if".

G simple, low-dimensional

Theorem (4)

Let G be a simple compact Lie group of dimension at most 15. Then the homogeneous space G/H with fibration metric g_t corresponding to a chain (H, K, G) of nested compact Lie groups admits nonnegative sectional curvature for small $t > 0$ if and only if one of the following holds:

(i) (K, H) *is a symmetric pair, or more generally,* $[\mathfrak{m}, \mathfrak{m}]^{\mathfrak{m}} = 0$*;*

(ii) *the chain* (H, K, G) *is one of* $(\mathrm{SU}(2), \mathrm{SO}(4), \mathrm{SO}(5))$ *or* $(\mathrm{SU}(2), \mathrm{SO}(4), \mathrm{G}_2)$ *where in the second case the subgroup* $\mathrm{SU}(2)$ *is such that* $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset \mathrm{G}_2$.

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- $\mathrm{SU}(2) \subset \mathrm{SO}(4) \subset \mathrm{G}_2$, where $\mathrm{SU}(2)$ is contained in $\mathrm{SU}(3) \subset \mathrm{G}_2$,
- $\mathrm{G}_2 \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(p+8)$, where $p \in \{0, 1\}$, and
- $\mathrm{SU}(3) \subset \mathrm{SO}(6) \subset \mathrm{SO}(7)$.

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- $\mathrm{SU}(3) \subset \mathrm{SO}(6) \subset \mathrm{SO}(7)$.

The third example is one of a family:

$$\begin{array}{ccc} \mathrm{SO}(2n)/\mathrm{SU}(n) & \longrightarrow & \mathrm{SO}(2n+1)/\mathrm{SU}(n) \\ & & \downarrow \\ & & \mathrm{SO}(2n+1)/\mathrm{SO}(2n) \end{array}$$

(We prove for all $n \geq 2$.)

Open Questions

Example: $H = (\mathrm{Sp}(1))^3 \subset K = (\mathrm{Sp}(1))^4 \subset G = \mathrm{Sp}(4)$.
On the Lie algebra level,

$$\mathfrak{h} = (\mathfrak{sp}(1))^3 \oplus \mathrm{Id} \ \subset \ \mathfrak{k} = (\mathfrak{sp}(1))^4 \ \subset \ \mathfrak{g} = \mathfrak{sp}(4).$$

Let $\mathfrak{s}$ denote the complement to $\mathfrak{k}$ in $\mathfrak{g}$; let $\mathfrak{m}$ denote the complement to $\mathfrak{h}$ in $\mathfrak{k}$; i.e., $\mathfrak{k} \oplus \mathfrak{s} = \mathfrak{sp}(4)$ and $\mathfrak{h} \oplus \mathfrak{m} = (\mathfrak{sp}(1))^4$. Write $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{s}$. Note that $\mathfrak{m} \cong \mathfrak{sp}(1)$ is itself a subalgebra, so that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$.

Does there exist a pair of vectors X and Y in $\mathfrak{p}$ such that $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$ yet $[X, Y] = 0$?

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -\bar{x}_{12} & 0 & x_{23} & x_{24} \\ -\bar{x}_{13} & -\bar{x}_{23} & 0 & x_{34} \\ -\bar{x}_{14} & -\bar{x}_{24} & -\bar{x}_{34} & x_{44} \end{pmatrix} \quad Y = \begin{pmatrix} 0 & y_{12} & y_{13} & y_{14} \\ -\bar{y}_{12} & 0 & y_{23} & y_{24} \\ -\bar{y}_{13} & -\bar{y}_{23} & 0 & y_{34} \\ -\bar{y}_{14} & -\bar{y}_{24} & -\bar{y}_{34} & y_{44} \end{pmatrix}$$

are elements of $\mathfrak{p}$ where $x_{12}, \ldots, x_{34}$ and $y_{12}, \ldots, y_{34}$ parametrize the $\mathfrak{s}$-component, while x_{44}, y_{44} parametrize the $\mathfrak{m}$-component.

Non-regular subgroups

Are there any examples of chains (H, K, G) satisfying condition $(*)$ which contain non-regular subgroups?