Rigidity and Stability of Einstein Metrics

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The Einstein-Hilbert Action and its Variation

We consider a compact connected n-dimensional $\mathbb{C}^\infty$-manifold M^n, $n \geq 3$ and the set of all Riemannian metrics on it, denoted by $\mathcal{M}$. The Einstein-Hilbert action (or total scalar curvature) is defined by

$$\begin{aligned} &\mathcal{M} \to \mathbb{R} \\ &g \mapsto S(g) := \int_M \mathrm{scal}_g \, \mathrm{dvol}_g. \end{aligned}$$

The first variation of the action is given by

$$S'(h) = \int_M \langle (1/2)\mathrm{scal}_g \cdot g - \mathrm{ric}_g, h \rangle_g \, \mathrm{dvol}_g.$$

Thus, $g \in \mathcal{M}$ is a critical metric for S if and only if $(1/2)\mathrm{scal}_g \cdot g - \mathrm{ric}_g = 0$, i.e. g is Ricci-flat. If we restrict the action to $\mathcal{M}_c$, the set of all metrics of volume c, we obtain more critical points in this subset:

Proposition 1. *A metric $g \in \mathcal{M}_c$ is critical for $S|_{\mathcal{M}_c}$ if and only if it is Einstein i.e. $\mathrm{ric}_g = \mu \cdot g$ for some $\mu \in \mathbb{R}$.*

Now we wish to determine the sign of the second variation of S at Einstein metrics. It is well known that the second variation is positive in the direction of conformal pertubations. In fact, if $h = f \cdot g$ for some $f \in C^\infty$, the variational formula reads

$$S''_g(h) = \frac{n-2}{2} \int_M \langle f, (n-1)\Delta_g f - n\mu f \rangle \, \mathrm{dvol}_g.$$

The positivity of $S''_g(h)$ follows from the Lichnerowicz eigenvalue estimate on Δ_g.
If $h = L_X g$ for some vector field X, the second variation vanishes, since S is a Riemannian functional, i.e. $S(\varphi^* g) = S(g)$ for any diffeomorphism φ on M.
It remains to consider the second variation on tensors, that are trace-free and divergence-free at each point $p \in M$. These tensors are often called transverse-traceless or TT. On TT-tensors, the second variation is given by

$$S''_g(h) = -\frac{1}{2} \int_M \langle h, \nabla^* \nabla h - 2\mathring{R} h \rangle \, \mathrm{dvol}_g.$$

We denote $\mathcal{G} := \nabla^* \nabla - 2\mathring{R}$.

Stability of Einstein Metrics

Definition 2. Let (M, g) an Einstein manifold and consider the operator $\mathcal{G}$ on TT-tensors. If $\mathcal{G}$ is positive semidefinite, (M, g) is said to be stable. If $\mathcal{G}$ is positive definite, we call (M, g) strictly stable. Elements in the kernel of $\mathcal{G}$ are called infinitesimal Einstein deformations.

If g_t is a curve of einstein metrics through g_0, then its derivative g'_0 is in the kernel of $\mathcal{G}$. Therefore, an Einstein metric is isolated in the set of Einstein metrics, if $\ker \mathcal{G}$ is trivial.
For example, the flat torus (T^n, g_{eukl}) is stable but not strictly stable. The kernel of $\mathcal{G}$ consists of all parallel symmetric $(0,2)-$ tensors and is of dimension $\frac{n(n-1)}{2} - 1$. The sphere with its standard metric is strictly stable, hence isolated as an Einstein metric.
A product of two Einstein metrics with positive scalar curvature is unstable. In the case of positive scalar curvature, many unstable Einstein metrics are known. In constrast, no unstable Einstein metrics of nonpositive scalar curvature are known. It is conjectured that all such Einstein manifolds are stable.

Stability and Sectional Curvature Bounds

In the seventies, stablity of Einstein metrics was studied by N. Koiso ([3],[4]). A consequence of his work was the following

Theorem 3 (Koiso)**.** *Let (M^n, g) be an Einstein manifold. If the sectional curvature is negative or if it is positive and $\frac{n-2}{3n}$-pinched, (M, g) is strictly stable.*

The pinching assumption is very strong, since for $n > 8$, all such pinched manifolds are quotients of the sphere. In my work, I try to weaken the curvature assumptions. So far, I have extended the stability result to weakly pinched manifolds:

Theorem 4. *Let (M^n, g) be an Einstein manifold, which is weakly $\frac{n-2}{3n}$-pinched. If $n \geq 8$, (M, g) is strictly stable.*

For manifolds of nonpositive sectional curvature, the following was obtained:

Theorem 5. *Let (M, g) a non-flat Einstein manifold with Einstein constant μ and nonpositive sectional curvature. If $K_{min} > \frac{2}{n}\mu$, (M, g) is strictly stable. If $K_{min} \geq \frac{2}{n}\mu$ and $\ker \mathcal{G}$ is nontrivial, M is even-dimensional and there exists an orthogonal splitting $TM = \mathcal{E} \oplus \mathcal{F}$. The subbundles $\mathcal{E}$ and $\mathcal{F}$ are of the same dimension and both integrable. The corresponding integral submanifolds are flat.*

Stability is not given if we just assume $K \geq 0$. In this case, we obtain a (nonpositive) lower bound on the smallest Eigenvalue of $\mathcal{G}$.

Proposition 6. *Let (M, g) an Einstein manifold with constant μ and sectional curvature $K \geq 0$. Then, the lowest eigenvalue of $\mathcal{G}$ satisfies*

$$\lambda \geq -2\mu.$$

Equality holds if and only if M is locally isometric to a product.

Stability and the Weyl Tensor

By the above, constant curvature metrics strictly stable in the non-flat case. We can also estimate the lowest Eigenvalue of $\mathcal{G}$:

Proposition 7. *Let (M, g) be a Riemannian manifold of constant curvature K. Then (M, g) is stable. It is strictly stable, if $K \neq 0$. The lowest eigenvalue of $\mathcal{G}$ satisfies the estimate*

$$\lambda \geq \max\{2K(n+1), -K(n-2)\}.$$

For a general Einstein-manifold, the difference from being of constant curvature is measured by the Weyl curvature tensor. Therefore, it seems convenient to search stability criterions involving the Weyl curvature.
Let $b : M \to \mathbb{R}$ be the largest eigenvalue of the operator $\mathring{W}$ acting on traceless symmetric $(0,2)$-tensors,

$$\mathring{W} h(X, Y) = \sum_{i=1}^n h(W(e_i, X)Y, e_i).$$

Theorem 8. *Let (M, g) be Einstein with constant μ. If*

$$\|b\|_\infty \leq \max\left\{\mu \frac{n+1}{2(n-1)}, -\mu \frac{n-2}{n-1}\right\}$$

(M, g) is stable. If the strict inequality holds, (M, g) is strictly stable.

If the Einstein constant μ is positive, we are able to prove a criterion involving an integral of the Weyl curvature:

Theorem 9. *Let (M, g) be a non-conformally flat Einstein manifold with positive Einstein constant μ . If*

$$\|b\|_{L^{n/2}} \leq \mu \cdot Vol(M, g)^{2/n} \frac{n+1}{2(n-1)} \left(\frac{4(n-1)}{n(n-2)} + 1\right)^{-1},$$

(M, g) is stable.

Since the $L^{n/2}$-norm of the Weyl tensor is conformally invariant and any non-conformally flat Einstein metric is a Yamabe metric in its conformal class, we can deduce the following:

Corollary 10. *Let (M, g) be a Riemannian metric and let $Y([g])$ be the Yamabe invariant of the conformal class of g. If*

$$\|W\|_{L^{n/2}} \leq Y([g]) \frac{n+1}{2n(n-1)} \left(\frac{4(n-1)}{n(n-2)} + 1\right)^{-1},$$

any Einstein metric in $[g]$ is stable.

In dimension 4, we would obtain that a positive Einstein-manifold is stable if $\|W\|_{L^2} \leq \frac{\mu}{3}\mathrm{Vol}(M, g)^{1/2}$. Unfortunately, this is of no use. Gursky and LeBrun ([2]) proved that if the Weyl-tensor of any Einstein 4-manifold satisfies the upper bound

$$\|W\|_{L^2} < \frac{4\mu}{\sqrt{6}} \mathrm{Vol}(M, g)^{1/2}$$

then $W \equiv 0$. In higher dimensions, our criterion is not ruled out by such isolation results.

Stability in Dimension Six

In dimension six, we can use the Gauss-Bonnet formula to prove the following:

Theorem 11. *Let (M, g) be a positive Einstein six-manifold with $\mathrm{vol}(M) = 1$ and non-vanishing Weyl tensor. If*

$$\left(144 - \frac{12 \cdot 7^2 \cdot 3^2}{5 \cdot 11^2}\right) \frac{\mu^3}{25} \leq 384\pi^3 \chi(M) + 48(W, W \circ W)_{L^2}$$

(M, g) is stable. Here, W is considered as an endomorphism on 2-forms and the L^2-scalar product is taken with respect to the norm on $End(\Lambda^2 M)$.

Here the assumptions are made such that, by using Gauss-Bonnet, we get a small L^3-norm of the Weyl-tensor.

Stability of Kähler-Einstein metrics

It is well known that all Kähler-Einstein metrics of nonpositive scalar curvature are stable. This assertion is wrong in the case of positive scalar curvature. Here we can make use of the Bochner curvature tensor. For Kähler-Einstein metrics, the Bochner tensor measures the difference from being of constant holomorphic sectional curvature. One can see that a positive Kähler-Einstein metric is stable if its Bochner tensor is small. As we did for the Weyl tensor, we proved two criterions involving the L^∞- and the $L^{n/2}$-norm of the Bochner tensor, respectively.

Open Questions

- Can we weaken the pinching assumption of Theorem 4?
- Are all Einstein manifolds of nonpositive scalar curvature stable?

References

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