The Ricci flow and its solitons for homogeneous manifolds and the Alekseevskii conjecture

Jorge Lauret,
UN Córdoba, Argentina

Marburg, July 6th, 2012

Contents

Contents

Contents

The space of homogeneous manifolds

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k},\mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k},\mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle\cdot,\cdot\rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k},\mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k},\mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu/K_\mu$ manifold).

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective).

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold$)$.

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective$)$.

(iv) $\langle \cdot, \cdot \rangle$ is $\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective).

(iv) $\langle \cdot, \cdot \rangle$ is $\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle\cdot,\cdot\rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

- (i) μ Jacobi ✓, $\mu(\mathfrak{k},\mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k},\mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu,\ K_\mu)$.
- (ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu/K_\mu$ manifold).
- (iii) $\{Z \in \mathfrak{k} : \mu(Z,\mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu/K_\mu$ almost-effective).
- (iv) $\langle\cdot,\cdot\rangle$ is $\mathrm{ad}_\mu\,\mathfrak{k}|_\mathfrak{p}$-invariant $(\rightsquigarrow g_\mu\ G_\mu$-invariant), $g_\mu(o) = \langle\cdot,\cdot\rangle)$.

$q = \dim\mathfrak{k}$, $n = \dim\mathfrak{p}$,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective).

(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant $(\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle)$.

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{ \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)} \checkmark \right\} \subset \mathcal{L}_{q+n}.$$

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

- (i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.
- (ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold$)$.
- (iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective$)$.
- (iv) $\langle \cdot, \cdot \rangle$ is $\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant $(\rightsquigarrow g_\mu$ G_μ-invariant$)$, $g_\mu(o) = \langle \cdot, \cdot \rangle)$.

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{ \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)} \checkmark \right\} \subset \mathcal{L}_{q+n}.$$

$\mathcal{H}_{q,n} \leftrightarrow$ all simply connected Riemannian homogeneous space of dimension n with a q-dimensional isotropy (up to isometry).

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold$)$.

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective$)$.

(iv) $\langle \cdot, \cdot \rangle$ is $\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant $(\rightsquigarrow g_\mu$ G_μ-invariant$)$, $g_\mu(o) = \langle \cdot, \cdot \rangle)$.

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{ \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)✓} \right\} \subset \mathcal{L}_{q+n}.$$

$\mathcal{H}_{q,n} \leftrightarrow$ all simply connected Riemannian homogeneous space of dimension n with a q-dimensional isotropy (up to isometry).

$\mathcal{H}_{0,n} = \mathcal{L}_n$ variety of Lie algebras

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.
(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold).
(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective).
(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{ \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)} \checkmark \right\} \subset \mathcal{L}_{q+n}.$$

$\mathcal{H}_{q,n} \leftrightarrow$ all simply connected Riemannian homogeneous space of dimension n with a q-dimensional isotropy (up to isometry).

$\mathcal{H}_{0,n} = \mathcal{L}_n$ variety of Lie algebras $\leftrightarrow$ left-invariant metrics on all n-dimensional s.c. Lie groups.

Geometric structures

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$,

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}, J] = 0$,

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) $Nijenhuis(J, \mu) = 0$,

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) *Nijenhuis*$(J, \mu) = 0$, polynomial on μ,

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) $Nijenhuis(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) $Nijenhuis(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.
- Fix $(J, \langle\cdot,\cdot\rangle)$,

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) $Nijenhuis(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.
- Fix $(J, \langle \cdot, \cdot \rangle)$, $\langle J\cdot, J\cdot \rangle = \langle \cdot, \cdot \rangle$,

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) $Nijenhuis(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.
- Fix $(J, \langle\cdot,\cdot\rangle)$, $\langle J\cdot, J\cdot\rangle = \langle\cdot,\cdot\rangle$, $\quad + \quad$ (iv)', (iv)

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) $Nijenhuis(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.
- Fix $(J, \langle\cdot,\cdot\rangle)$, $\langle J\cdot, J\cdot\rangle = \langle\cdot,\cdot\rangle$, $\quad + \quad$ (iv)', (iv)
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ (almost-) hermitian homogeneous space.

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\operatorname{ad}_\mu \mathfrak{k}|_\mathfrak{p}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) $Nijenhuis(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.
- Fix $(J, \langle \cdot, \cdot \rangle)$, $\langle J \cdot, J \cdot \rangle = \langle \cdot, \cdot \rangle$, $\quad + \quad$ (iv)', (iv)
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ (almost-) hermitian homogeneous space.
- Analogously: Symplectic, Hyper-complex, etc.

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) *Nijenhuis*$(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.
- Fix $(J, \langle\cdot,\cdot\rangle)$, $\langle J\cdot, J\cdot\rangle = \langle\cdot,\cdot\rangle$, $\quad + \quad$ (iv)', (iv)
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ (almost-) hermitian homogeneous space.
- Analogously: Symplectic, Hyper-complex, etc.

Deformation theory, convergence, invariants, critical points of functionals, evolution equations ???

Geometric structures

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

- Fix $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -I$, $\quad + \quad$ (iv)' $[\mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}, J] = 0$,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ almost-complex homogeneous space.
- $+$ (v) *Nijenhuis*$(J, \mu) = 0$, polynomial on μ,
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ complex homogeneous space.
- Fix $(J, \langle \cdot, \cdot \rangle)$, $\langle J\cdot, J\cdot \rangle = \langle \cdot, \cdot \rangle$, $\quad + \quad$ (iv)', (iv)
  $\mu \in \mathcal{H}_{q,n} \rightsquigarrow (G_\mu/K_\mu, J)$ (almost-) hermitian homogeneous space.
- Analogously: Symplectic, Hyper-complex, etc.

Deformation theory, convergence, invariants, critical points of functionals, evolution equations ???

(only $q = 0$ and μ nilpotent has been explored).

Subsets of $\mathcal{H}_{q,n}$

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

Subsets of $\mathcal{H}_{q,n}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$$

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g}),$

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathrm{GL}_{q+n} \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ by

$$(h \cdot \mu)(X, Y) = h\mu(h^{-1}X, h^{-1}Y), \qquad \forall X, Y \in \mathfrak{g}.$$

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathrm{GL}_{q+n} \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ by

$$(h \cdot \mu)(X, Y) = h\mu(h^{-1}X, h^{-1}Y), \qquad \forall X, Y \in \mathfrak{g}.$$

$$\mu \in \mathcal{H}_{q,n},\ h := \begin{bmatrix} h_q & 0 \\ 0 & h_n \end{bmatrix} \in \mathrm{GL}_{q+n}, \qquad [h_n^t h_n, \mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}] = 0.$$

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathrm{GL}_{q+n} \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ by

$$(h \cdot \mu)(X, Y) = h\mu(h^{-1}X, h^{-1}Y), \qquad \forall X, Y \in \mathfrak{g}.$$

$\mu \in \mathcal{H}_{q,n}$, $h := \begin{bmatrix} h_q & 0 \\ 0 & h_n \end{bmatrix} \in \mathrm{GL}_{q+n}, \qquad [h_n^t h_n, \mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}] = 0.$

$\Rightarrow h \cdot \mu \in \mathcal{H}_{q,n}$.

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathrm{GL}_{q+n} \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ by

$$(h \cdot \mu)(X, Y) = h\mu(h^{-1}X, h^{-1}Y), \qquad \forall X, Y \in \mathfrak{g}.$$

$\mu \in \mathcal{H}_{q,n}$, $h := \begin{bmatrix} h_q & 0 \\ 0 & h_n \end{bmatrix} \in \mathrm{GL}_{q+n}, \qquad [h_n^t h_n, \mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}] = 0.$

$\Rightarrow h \cdot \mu \in \mathcal{H}_{q,n}$.

- $G_{h\cdot\mu}/K_{h\cdot\mu}$ and G_μ/K_μ are equivariantly diffeomorphic.

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$,

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathrm{GL}_{q+n} \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ by

$$(h \cdot \mu)(X, Y) = h\mu(h^{-1}X, h^{-1}Y), \qquad \forall X, Y \in \mathfrak{g}.$$

$$\mu \in \mathcal{H}_{q,n},\ h := \begin{bmatrix} h_q & 0 \\ 0 & h_n \end{bmatrix} \in \mathrm{GL}_{q+n}, \qquad [h_n^t h_n, \mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}] = 0.$$

$\Rightarrow h \cdot \mu \in \mathcal{H}_{q,n}$.

- $G_{h\cdot\mu}/K_{h\cdot\mu}$ and G_μ/K_μ are equivariantly diffeomorphic.
- $(G_{h\cdot\mu}/K_{h\cdot\mu}, g_{h\cdot\mu})$ is equivariantly isometric to $\left(G_\mu/K_\mu, g_{\langle h_n\cdot, h_n\cdot\rangle}\right)$

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g},$

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathrm{GL}_{q+n} \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ by

$$(h \cdot \mu)(X, Y) = h\mu(h^{-1}X, h^{-1}Y), \qquad \forall X, Y \in \mathfrak{g}.$$

$\mu \in \mathcal{H}_{q,n}$, $h := \begin{bmatrix} h_q & 0 \\ 0 & h_n \end{bmatrix} \in \mathrm{GL}_{q+n}, \qquad [h_n^t h_n, \mathrm{ad}_\mu \mathfrak{k}|_\mathfrak{p}] = 0.$

$\Rightarrow h \cdot \mu \in \mathcal{H}_{q,n}$.

- $G_{h\cdot\mu}/K_{h\cdot\mu}$ and G_μ/K_μ are equivariantly diffeomorphic.
- $(G_{h\cdot\mu}/K_{h\cdot\mu}, g_{h\cdot\mu})$ is equivariantly isometric to $\left(G_\mu/K_\mu, g_{\langle h_n\cdot, h_n\cdot\rangle}\right)$ ($\rightsquigarrow$ all G_μ-invariant metrics on G_μ/K_μ),

Subsets of $\mathcal{H}_{q,n}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\quad \mathcal{H}_{q,n} \subset \mathcal{L}_{q+n} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$,

Notation: $\mathrm{GL}_{q+n} := \mathrm{GL}_{q+n}(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$, same for $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathrm{GL}_{q+n} \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ by

$$(h \cdot \mu)(X, Y) = h\mu(h^{-1}X, h^{-1}Y), \qquad \forall X, Y \in \mathfrak{g}.$$

$$\mu \in \mathcal{H}_{q,n},\ h := \begin{bmatrix} h_q & 0 \\ 0 & h_n \end{bmatrix} \in \mathrm{GL}_{q+n}, \qquad [h_n^t h_n, \mathrm{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}] = 0.$$

$\Rightarrow h \cdot \mu \in \mathcal{H}_{q,n}$.

- $G_{h\cdot\mu}/K_{h\cdot\mu}$ and G_μ/K_μ are equivariantly diffeomorphic.
- $(G_{h\cdot\mu}/K_{h\cdot\mu}, g_{h\cdot\mu})$ is equivariantly isometric to $\left(G_\mu/K_\mu, g_{\langle h_n\cdot, h_n\cdot\rangle}\right)$
  ($\rightsquigarrow$ all G_μ-invariant metrics on G_μ/K_μ),
  ($\Rightarrow \mathrm{O}(n) \cdot \mu \subset \mathcal{H}_{q,n}$ all equivariantly isometric to g_μ).

Convergence

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k,\ \lambda \in \mathcal{H}_{q,n}$.

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism}\right\}.$$

Theorem (JL 2010)

$\mu_k,\ \lambda \in \mathcal{H}_{q,n}$.

- $\mu_k \to \lambda \Rightarrow (G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *infinitesimally*

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$.

- $\mu_k \to \lambda \Rightarrow (G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *infinitesimally (only the germs of the metrics at the origins are involved).*

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k,\ \lambda \in \mathcal{H}_{q,n}$.

- $\mu_k \to \lambda \Rightarrow (G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *infinitesimally (only the germs of the metrics at the origins are involved).* $(\sim$ *converse* ✓ $)$

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism}\right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$.

- $\mu_k \to \lambda \Rightarrow (G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *infinitesimally (only the germs of the metrics at the origins are involved).* ($\sim$ *converse* ✓) *Assume* $\mu_k \to \lambda$ *and* $\inf_k r_{\mu_k} > 0$, *then,*

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup \left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$.

- $\mu_k \to \lambda \Rightarrow (G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *infinitesimally* *(only the germs of the metrics at the origins are involved).* $(\sim$ *converse* $\checkmark)$
  Assume $\mu_k \to \lambda$ *and* $\inf_k r_{\mu_k} > 0$, *then,*
- $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *locally*

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$.

- $\mu_k \to \lambda \Rightarrow (G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *infinitesimally (only the germs of the metrics at the origins are involved).* $(\sim$ *converse* $\checkmark)$
  Assume $\mu_k \to \lambda$ *and* $\inf_k r_{\mu_k} > 0$, *then,*
- $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *locally (smooth on fixed open neighborhoods of the origins).*

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$.

- $\mu_k \to \lambda \Rightarrow (G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *infinitesimally (only the germs of the metrics at the origins are involved).* $(\sim$ *converse* $\checkmark)$

  Assume $\mu_k \to \lambda$ *and* $\inf_k r_{\mu_k} > 0$, *then,*
- $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *locally (smooth on fixed open neighborhoods of the origins).*
- $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ *pointed (or Cheeger-Gromov), after passing to a subsequence.*

Convergence

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$,

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$, $\mu \in \mathcal{H}_{0,n} = \mathcal{L}_n \rightsquigarrow (G_\mu, \langle\cdot,\cdot\rangle)$ left-invariant metric.

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$, $\mu \in \mathcal{H}_{0,n} = \mathcal{L}_n \rightsquigarrow (G_\mu, \langle \cdot, \cdot \rangle)$ left-invariant metric.

Theorem (JL 2010)

$\mu_k \in \mathcal{H}_{0,n} = \mathcal{L}_n$, *the following are equivalent.*

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$, $\mu \in \mathcal{H}_{0,n} = \mathcal{L}_n \rightsquigarrow (G_\mu, \langle\cdot,\cdot\rangle)$ left-invariant metric.

Theorem (JL 2010)

$\mu_k \in \mathcal{H}_{0,n} = \mathcal{L}_n$, *the following are equivalent.*

- $\mu_k \to \lambda$.

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$, $\mu \in \mathcal{H}_{0,n} = \mathcal{L}_n \rightsquigarrow (G_\mu, \langle\cdot,\cdot\rangle)$ left-invariant metric.

Theorem (JL 2010)

$\mu_k \in \mathcal{H}_{0,n} = \mathcal{L}_n$, the following are equivalent.

- $\mu_k \to \lambda$.
- $(G_{\mu_k}, \langle\cdot,\cdot\rangle) \to (G_\lambda, \langle\cdot,\cdot\rangle)$ *infinitesimally.*

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$, $\mu \in \mathcal{H}_{0,n} = \mathcal{L}_n \rightsquigarrow (G_\mu, \langle \cdot, \cdot \rangle)$ left-invariant metric.

Theorem (JL 2010)

$\mu_k \in \mathcal{H}_{0,n} = \mathcal{L}_n$, *the following are equivalent.*

- $\mu_k \to \lambda$.
- $(G_{\mu_k}, \langle \cdot, \cdot \rangle) \to (G_\lambda, \langle \cdot, \cdot \rangle)$ *infinitesimally.*
- $(G_{\mu_k}, \langle \cdot, \cdot \rangle) \to (G_\lambda, \langle \cdot, \cdot \rangle)$ *locally.*

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$, $\mu \in \mathcal{H}_{0,n} = \mathcal{L}_n \rightsquigarrow (G_\mu, \langle\cdot,\cdot\rangle)$ left-invariant metric.

Theorem (JL 2010)

$\mu_k \in \mathcal{H}_{0,n} = \mathcal{L}_n$, the following are equivalent.

- $\mu_k \to \lambda$.
- $(G_{\mu_k}, \langle\cdot,\cdot\rangle) \to (G_\lambda, \langle\cdot,\cdot\rangle)$ *infinitesimally*.
- $(G_{\mu_k}, \langle\cdot,\cdot\rangle) \to (G_\lambda, \langle\cdot,\cdot\rangle)$ *locally*.
- $(G_{\mu_k}, \langle\cdot,\cdot\rangle) \to (G_\lambda, \langle\cdot,\cdot\rangle)$ *pointed, provided G_λ is compact.*

Convergence

Assume $\mathfrak{k} = 0$, i.e. $\mathfrak{g} = \mathfrak{p}$, $\mu \in \mathcal{H}_{0,n} = \mathcal{L}_n \rightsquigarrow (G_\mu, \langle \cdot, \cdot \rangle)$ left-invariant metric.

Theorem (JL 2010)

$\mu_k \in \mathcal{H}_{0,n} = \mathcal{L}_n$, the following are equivalent.

- $\mu_k \to \lambda$.
- $(G_{\mu_k}, \langle \cdot, \cdot \rangle) \to (G_\lambda, \langle \cdot, \cdot \rangle)$ *infinitesimally*.
- $(G_{\mu_k}, \langle \cdot, \cdot \rangle) \to (G_\lambda, \langle \cdot, \cdot \rangle)$ *locally*.
- $(G_{\mu_k}, \langle \cdot, \cdot \rangle) \to (G_\lambda, \langle \cdot, \cdot \rangle)$ *pointed, provided G_λ is compact.*
- $g_{\mu_k} \to g_\lambda$ *smoothly on $\mathbb{R}^n \equiv \mathfrak{g}$, provided all μ_k are completely solvable (e.g. nilpotent).*

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which infinitesimally converges to another Aloff-Wallach space λ, but such that it does not admit any pointed or local convergent subsequence.

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which infinitesimally converges to another Aloff-Wallach space λ, but such that it does not admit any pointed or local convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which infinitesimally converges to another Aloff-Wallach space λ, but such that it does not admit any pointed or local convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$.

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which infinitesimally converges to another Aloff-Wallach space λ, but such that it does not admit any pointed or local convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$. μ_k is actually isometric to a convergent sequence in $\mathcal{H}_{1,3}$.

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which infinitesimally converges to another Aloff-Wallach space λ, but such that it does not admit any pointed or local convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$. μ_k is actually isometric to a convergent sequence in $\mathcal{H}_{1,3}$.
- A sequence $\mu_k \in \mathcal{H}_{1,5}$ of homogeneous metrics on $S^3 \times S^2$ converging to $\lambda \notin \mathcal{H}_{1,5}$.

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which infinitesimally converges to another Aloff-Wallach space λ, but such that it does not admit any pointed or local convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$. μ_k is actually isometric to a convergent sequence in $\mathcal{H}_{1,3}$.
- A sequence $\mu_k \in \mathcal{H}_{1,5}$ of homogeneous metrics on $S^3 \times S^2$ converging to $\lambda \notin \mathcal{H}_{1,5}$. However, λ can be viewed as an element of $\mathcal{H}_{2,4}$, giving rise to a collapsing of the μ_k with bounded curvature to a metric on $S^2 \times S^2$.

Ricci operator

Ricci operator

$\mu \in \mathcal{H}_{q,n}$,

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \operatorname{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\operatorname{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\operatorname{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y\rangle = \mathrm{tr}\, \mathrm{ad}_\mu X\, \mathrm{ad}_\mu Y$

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \frac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X\, \mathrm{ad}_\mu Y$ (Killing form),

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X \, \mathrm{ad}_\mu Y$ (Killing form),

$H_\mu \in \mathfrak{p}$, $\quad \langle H_\mu, X \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X$

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X \, \mathrm{ad}_\mu Y$ (Killing form),

$H_\mu \in \mathfrak{p}$, $\quad \langle H_\mu, X \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X$ (unimodularity)

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X\, \mathrm{ad}_\mu Y$ (Killing form),

$H_\mu \in \mathfrak{p}$, $\quad \langle H_\mu, X \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X$ (unimodularity)

$\mathrm{tr}\, M_\mu E = \frac{1}{4} \langle \pi(E) \mu_{\mathfrak{p}}, \mu_{\mathfrak{p}} \rangle, \quad \forall E \in \mathrm{End}(\mathfrak{p})$

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X\, \mathrm{ad}_\mu Y$ (Killing form),

$H_\mu \in \mathfrak{p}$, $\quad \langle H_\mu, X \rangle = \mathrm{tr}\, \mathrm{ad}_\mu X$ (unimodularity)

$\mathrm{tr}\, M_\mu E = \frac{1}{4} \langle \pi(E)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}} \rangle, \quad \forall E \in \mathrm{End}(\mathfrak{p})$ (moment map

for the linear action $\mathrm{GL}(\mathfrak{p}) \circlearrowleft \Lambda^2 \mathfrak{p}^* \otimes \mathfrak{p}$)

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y\rangle = \mathrm{tr}\, \mathrm{ad}_\mu X \, \mathrm{ad}_\mu Y$ (Killing form),

$H_\mu \in \mathfrak{p}$, $\quad \langle H_\mu, X\rangle = \mathrm{tr}\, \mathrm{ad}_\mu X$ (unimodularity)

$\mathrm{tr}\, M_\mu E = \frac{1}{4}\langle \pi(E)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}\rangle, \quad \forall E \in \mathrm{End}(\mathfrak{p})$ (moment map

for the linear action $\mathrm{GL}(\mathfrak{p}) \circlearrowleft \Lambda^2\mathfrak{p}^* \otimes \mathfrak{p}$)

M_μ and H_μ depends only on $\mu_{\mathfrak{p}}$, where $\quad \forall X, Y \in \mathfrak{p}$,

Ricci operator

$\mu \in \mathcal{H}_{q,n}$, $\quad \mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ given by

$$\boxed{\mathrm{Ric}_\mu = M_\mu - \tfrac{1}{2} B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})},$$

$\langle B_\mu X, Y\rangle = \mathrm{tr}\,\mathrm{ad}_\mu X\,\mathrm{ad}_\mu Y$ (Killing form),

$H_\mu \in \mathfrak{p}$, $\quad \langle H_\mu, X\rangle = \mathrm{tr}\,\mathrm{ad}_\mu X$ (unimodularity)

$\mathrm{tr}\, M_\mu E = \frac{1}{4}\langle \pi(E)\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}\rangle, \quad \forall E \in \mathrm{End}(\mathfrak{p})$ (moment map for the linear action $\mathrm{GL}(\mathfrak{p}) \circlearrowleft \Lambda^2\mathfrak{p}^* \otimes \mathfrak{p}$)

M_μ and H_μ depends only on $\mu_{\mathfrak{p}}$, where $\quad \forall X, Y \in \mathfrak{p}$,

$$\mu(X,Y) = \mu_{\mathfrak{k}}(X,Y) + \mu_{\mathfrak{p}}(X,Y), \qquad \mu_{\mathfrak{k}}(X,Y) \in \mathfrak{k}, \quad \mu_{\mathfrak{p}}(X,Y) \in \mathfrak{p}.$$

Ricci flow

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

$\Rightarrow g(t) \longleftrightarrow \langle\cdot,\cdot\rangle_t$: $\mathrm{Ad}(K_{\mu_0})$-invariant inner product on $\mathfrak{p}$ solving the ODE:

$$\frac{d}{dt}\langle\cdot,\cdot\rangle_t = -2\,\mathrm{Rc}(\langle\cdot,\cdot\rangle_t), \qquad \langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle,$$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

$\Rightarrow g(t) \longleftrightarrow \langle\cdot,\cdot\rangle_t$: $\mathrm{Ad}(K_{\mu_0})$-invariant inner product on $\mathfrak{p}$ solving the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\cdot,\cdot\rangle_t = -2\,\mathrm{Rc}(\langle\cdot,\cdot\rangle_t), \qquad \langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle,$$

$t \in (T_-, T_+), \quad -\infty \leq T_- < 0 < T_+ \leq T_+.$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

$\Rightarrow g(t) \longleftrightarrow \langle\cdot,\cdot\rangle_t$: $\mathrm{Ad}(K_{\mu_0})$-invariant inner product on $\mathfrak{p}$ solving the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\cdot,\cdot\rangle_t = -2\,\mathrm{Rc}(\langle\cdot,\cdot\rangle_t), \qquad \langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle,$$

$t \in (T_-, T_+), \quad -\infty \leq T_- < 0 < T_+ \leq T_+.$

Ricci flow on $\mathcal{H}_{q,n}$???

Bracket flow

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{bmatrix}\right)\mu},$$

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}),$$

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}),$$

$$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n}.$$

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}),$$

$$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n}.$$

$\mu_0 \in \mathcal{H}_{q,n} \Rightarrow \mu(t) \in \mathcal{H}_{q,n}$ for all t,

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}),$$

$$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n}.$$

$\mu_0 \in \mathcal{H}_{q,n} \Rightarrow \mu(t) \in \mathcal{H}_{q,n}$ for all t,

$\mu(t) \rightsquigarrow (G_{\mu(t)}/K_{\mu(t)}, \langle\cdot,\cdot\rangle)$ curve of homogeneous spaces.

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad \text{(recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\text{)},$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ such that

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ equivariant diffeomorphism determined by the Lie group isomorphism between G_{μ_0} and $G_{\mu(t)}$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$
$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *equivariant diffeomorphism determined by the Lie group isomorphism between* G_{μ_0} *and* $G_{\mu(t)}$ *with derivative* $\tilde{h} := \left[\begin{smallmatrix} I & 0 \\ 0 & h \end{smallmatrix}\right] : \mathfrak{g} \longrightarrow \mathfrak{g}$*, where* $h(t) := d\varphi(t)|_o : \mathfrak{p} \longrightarrow \mathfrak{p}$*,*

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *equivariant diffeomorphism determined by the Lie group isomorphism between* G_{μ_0} *and* $G_{\mu(t)}$ *with derivative* $\tilde{h} := \left[\begin{smallmatrix} I & 0 \\ 0 & h \end{smallmatrix}\right] : \mathfrak{g} \longrightarrow \mathfrak{g}$*, where* $h(t) := d\varphi(t)|_o : \mathfrak{p} \longrightarrow \mathfrak{p}$*,*

(i) $\frac{\mathrm{d}}{\mathrm{d}t} h = -h\,\mathrm{Ric}(\langle\cdot,\cdot\rangle_t), \quad h(0) = I.$

(ii) $\frac{\mathrm{d}}{\mathrm{d}t} h = -\,\mathrm{Ric}_{\mu(t)}\, h, \quad h(0) = I.$

$$(M,g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n}, \qquad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ such that

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ equivariant diffeomorphism determined by the Lie group isomorphism between G_{μ_0} and $G_{\mu(t)}$ with derivative $\tilde{h} := \left[\begin{smallmatrix} I & 0 \\ 0 & h \end{smallmatrix}\right] : \mathfrak{g} \longrightarrow \mathfrak{g}$, where $h(t) := d\varphi(t)|_o : \mathfrak{p} \longrightarrow \mathfrak{p}$,

(i) $\frac{\mathrm{d}}{\mathrm{d}t} h = -h \operatorname{Ric}(\langle\cdot,\cdot\rangle_t), \quad h(0) = I.$

(ii) $\frac{\mathrm{d}}{\mathrm{d}t} h = -\operatorname{Ric}_{\mu(t)} h, \quad h(0) = I.$

The following conditions also hold:

(iii) $\langle\cdot,\cdot\rangle_t = \langle h\cdot, h\cdot\rangle.$

(iv) $\mu(t) = \tilde{h}\mu_0(\tilde{h}^{-1}\cdot, \tilde{h}^{-1}\cdot).$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.
- $\mu(t_k) \to \lambda \in \mathcal{H}_{q,n}$ (or a suitable normalization)

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.
- $\mu(t_k) \to \lambda \in \mathcal{H}_{q,n}$ (or a suitable normalization) $\Rightarrow$ convergence or subconvergence $g_{\mu_k} \to g_\lambda$ (infinitesimal, local or pointed).

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.
- $\mu(t_k) \to \lambda \in \mathcal{H}_{q,n}$ (or a suitable normalization) $\Rightarrow$ convergence or subconvergence $g_{\mu_k} \to g_\lambda$ (infinitesimal, local or pointed).
- $\mu(t)|_{\mathfrak{k}\times\mathfrak{g}} \equiv \mu_0|_{\mathfrak{k}\times\mathfrak{g}}$.

Application to nilmanifolds

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu$$

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\,\mathrm{grad}(\mathrm{tr}\,\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \mathrm{tr}\,\mathrm{Ric}_\mu^2$ on the sphere.

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0, \infty)$)*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$, $\operatorname{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\operatorname{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\operatorname{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\operatorname{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0, \infty)$ and $\|\operatorname{Rm}(g(t))\| \leq \frac{C}{t}$).*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0,\infty)$ and $\|\mathrm{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0,\infty)$ and $\|\mathrm{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*
- *After rescaling ($\mathrm{R} \equiv -1$),*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\operatorname{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\operatorname{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\operatorname{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0, \infty)$ and $\|\operatorname{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*
- *After rescaling ($\mathrm{R} \equiv -1$), $g(t)$ converges to a Ricci soliton metric g_∞,*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0, \infty)$ and $\|\mathrm{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*
- *After rescaling ($\mathrm{R} \equiv -1$), $g(t)$ converges to a Ricci soliton metric g_∞, which is also invariant under a transitive nilpotent Lie group, though possibly non-isomorphic to G.*

Other applications

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

[Payne 2010] Qualitative behavior of bracket flow solutions for nilmanifolds.

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

[Payne 2010] Qualitative behavior of bracket flow solutions for nilmanifolds.

[Glickenstein-Payne 2010] Ricci flow of 3-dim unimodular Lie groups.

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

[Payne 2010] Qualitative behavior of bracket flow solutions for nilmanifolds.

[Glickenstein-Payne 2010] Ricci flow of 3-dim unimodular Lie groups.

[Arroyo 2012] Application to Ricci flow of 4-dim homogeneous manifolds and to Ricci flow of solvmanifolds.

Example in dim $= 3$

Example in dim = 3

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

Example in dim = 3

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

$$\begin{cases} \mu(X_3, Z_1) = X_2, \\ \mu(Z_1, X_2) = X_3, \\ \mu(X_2, X_3) = aX_1 + bZ_1. \end{cases}$$

Example in dim $= 3$

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

$$\begin{cases} \mu(X_3, Z_1) = X_2, \\ \mu(Z_1, X_2) = X_3, \\ \mu(X_2, X_3) = aX_1 + bZ_1. \end{cases}$$

Bracket flow: $$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = (-\frac{3}{2}a^2 + 2b)a, \\ \frac{\mathrm{d}}{\mathrm{d}t} b = (-a^2 + 2b)b. \end{cases}$$

Example in dim = 3

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

$$\begin{cases} \mu(X_3, Z_1) = X_2, \\ \mu(Z_1, X_2) = X_3, \\ \mu(X_2, X_3) = aX_1 + bZ_1. \end{cases}$$

Bracket flow:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = (-\frac{3}{2}a^2 + 2b)a, \\ \frac{\mathrm{d}}{\mathrm{d}t} b = (-a^2 + 2b)b. \end{cases}$$

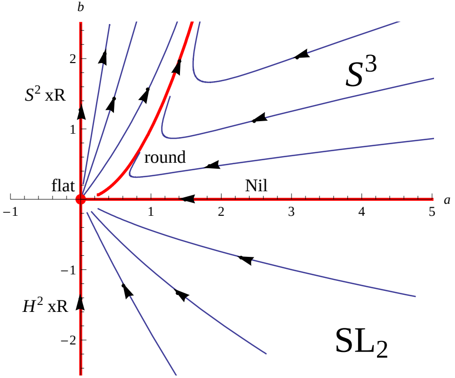


Figure: Phase plane for the ODE system

Example in dim = 3

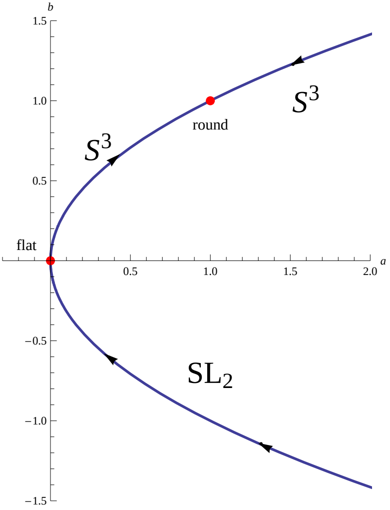


Figure: Volume-normalized bracket flow

Example in dim = 3

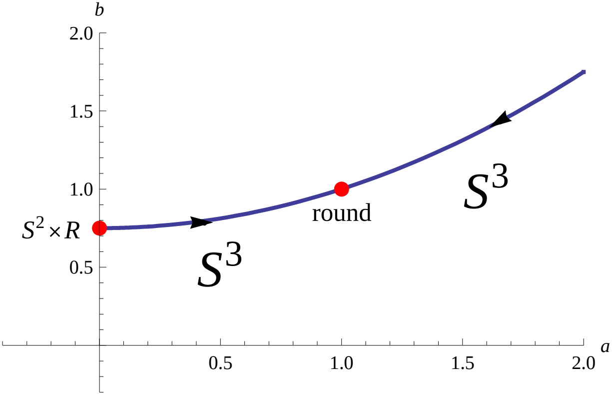


Figure: R-normalized bracket flows: $R \equiv \frac{3}{2}$.

Example in dim $= 3$

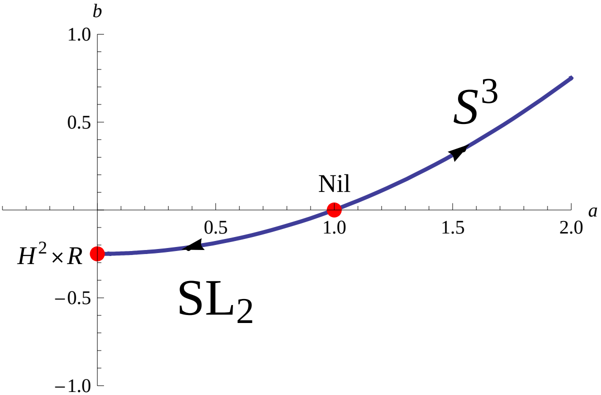


Figure: R-normalized bracket flows: $R \equiv -\frac{1}{2}$.

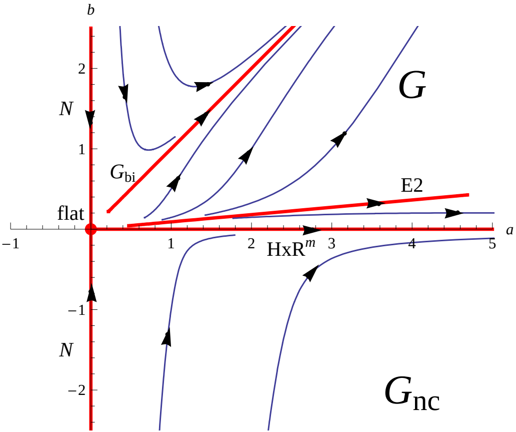
Example in simple Lie groups

Example in simple Lie groups

Bracket flow: $$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = \frac{1}{4}(\alpha a^2 + (1-\alpha) b^2) a, \\ \\ \frac{\mathrm{d}}{\mathrm{d}t} b = -\frac{1}{4}(\alpha a^2 + (3-\alpha) b^2 - 4ab) b. \end{cases}$$

Example in simple Lie groups

$$\text{Bracket flow:} \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = \frac{1}{4}(\alpha a^2 + (1-\alpha) b^2) a, \\ \frac{\mathrm{d}}{\mathrm{d}t} b = -\frac{1}{4}(\alpha a^2 + (3-\alpha) b^2 - 4ab) b. \end{cases}$$



Example in simple Lie groups

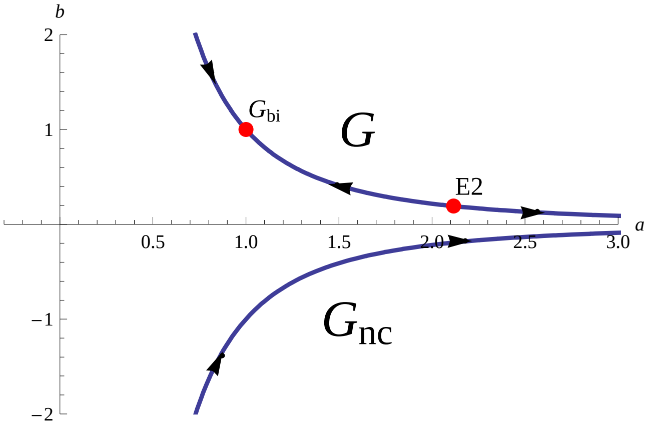


Figure: Volume-normalized bracket flow

Example in simple Lie groups

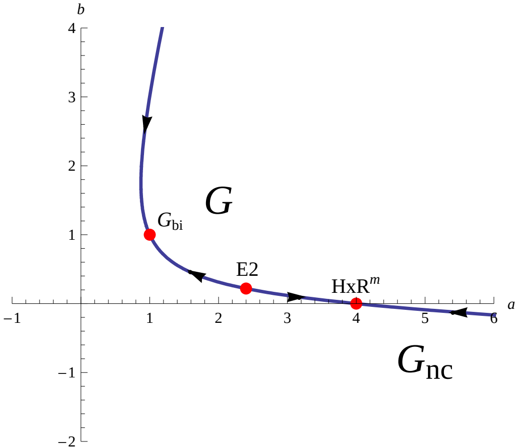


Figure: R-normalized bracket flow: $R \equiv 2$

Example in simple Lie groups

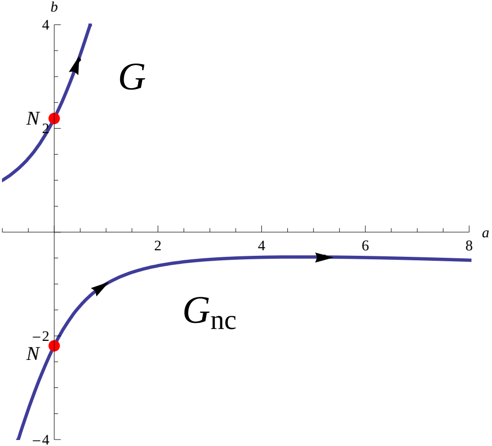


Figure: R-normalized bracket flow: $R \equiv -3$

Ricci solitons

Ricci solitons

(M, g) complete Riemannian manifold,

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie]

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space)

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space) homogeneous Ricci soliton must be noncompact,

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space) homogeneous Ricci soliton must be noncompact, expanding ($c < 0$),

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space) homogeneous Ricci soliton must be noncompact, expanding ($c < 0$), non-gradient.

Algebraic solitons

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Leftrightarrow\ \exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Leftrightarrow\ \exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + Symm(D_{\mathfrak{p}}).$$

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Leftrightarrow \exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + Symm(D_{\mathfrak{p}}).$$

algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Leftrightarrow \exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + Symm(D_{\mathfrak{p}}).$$

algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}}$$

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Leftrightarrow \exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + Symm(D_{\mathfrak{p}}).$$

algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}} \Leftrightarrow \mu(t) = (-2ct + 1)^{-1/2} \cdot [\cdot, \cdot].$$

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Leftrightarrow \exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + Symm(D_{\mathfrak{p}}).$$

algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}} \Leftrightarrow \mu(t) = (-2ct + 1)^{-1/2} \cdot [\cdot,\cdot].$$

solvsoliton: algebraic soliton with $K = e$ and G solvable.

Algebraic solitons

$(G/K, g_{\langle\cdot,\cdot\rangle})$: connected homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Leftrightarrow \exists\, c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + Symm(D_{\mathfrak{p}}).$$

algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}} \Leftrightarrow \mu(t) = (-2ct + 1)^{-1/2} \cdot [\cdot,\cdot].$$

solvsoliton: algebraic soliton with $K = e$ and G solvable.

nilsoliton: algebraic soliton with $K = e$ and G nilpotent.

Classification and structure of nilsolitons ($\leftrightarrow$ GIT): [Fernandez Culma, Jablonski, Nikolayevsky, Payne, Will, JL, ...]

Classification and structure of nilsolitons ($\leftrightarrow$ GIT): [Fernandez Culma, Jablonski, Nikolayevsky, Payne, Will, JL, ...]

Classification and structure of solvsolitons: [Lafuente, Will, JL, ...]

Classification and structure of nilsolitons ($\leftrightarrow$ GIT): [Fernandez Culma, Jablonski, Nikolayevsky, Payne, Will, JL, ...]

Classification and structure of solvsolitons: [Lafuente, Will, JL, ...]

Theorem (Jablonski 2012)

- *Any (nonflat) Ricci soliton solvmanifold is isometric to a simply connected solvsoliton.*

Classification and structure of nilsolitons ($\leftrightarrow$ GIT): [Fernandez Culma, Jablonski, Nikolayevsky, Payne, Will, JL, ...]

Classification and structure of solvsolitons: [Lafuente, Will, JL, ...]

Theorem (Jablonski 2012)

- *Any (nonflat) Ricci soliton solvmanifold is isometric to a simply connected solvsoliton.*
- *Any homogeneous Ricci soliton (M, g) is a semi-algebraic soliton with respect to its full isometry group $G = \mathrm{Isom}(M, g)$.*

Open questions:

Open questions:

- Is any homogeneous Ricci soliton isometric to an algebraic soliton ??

Open questions:

- Is any homogeneous Ricci soliton isometric to an algebraic soliton ??

- Is any algebraic soliton isometric to a solvsoliton ??

Open questions:

- Is any homogeneous Ricci soliton isometric to an algebraic soliton ??
- Is any algebraic soliton isometric to a solvsoliton ??

Alekseevskii's conjecture [Besse, 80's]. Any Einstein connected homogeneous Riemannian manifold of negative scalar curvature is diffeomorphic to a Euclidean space.

Open questions:

- Is any homogeneous Ricci soliton isometric to an algebraic soliton ??
- Is any algebraic soliton isometric to a solvsoliton ??

Alekseevskii's conjecture [Besse, 80's]. Any Einstein connected homogeneous Riemannian manifold of negative scalar curvature is diffeomorphic to a Euclidean space.

Theorem (Lafuente-JL 2012)

Any example of an algebraic soliton which is not a solvsoliton gives rise to a counterexample to the Alekseevskii conjecture.

Structure of semi-algebraic solitons

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is* $\perp$*-Killing,*

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is* $\perp$*-Killing, and consider the orthogonal decomposition* $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$*, where* $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$*.*

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\perp$-Killing, and consider the orthogonal decomposition $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. Then $(G/K, g_{\langle \cdot,\cdot \rangle})$ is a semi-algebraic soliton if and only if the following conditions hold:

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\perp$-Killing, and consider the orthogonal decomposition $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. Then $(G/K, g_{\langle\cdot,\cdot\rangle})$ is a semi-algebraic soliton if and only if the following conditions hold:

- $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{k} \oplus \mathfrak{h}$. *In particular,* $\mathfrak{u} := \mathfrak{k} \oplus \mathfrak{h}$ *is a reductive Lie subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{n}$.

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\perp$-Killing, and consider the orthogonal decomposition $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. Then $(G/K, g_{\langle\cdot,\cdot\rangle})$ is a semi-algebraic soliton if and only if the following conditions hold:

- $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{k} \oplus \mathfrak{h}$. *In particular,* $\mathfrak{u} := \mathfrak{k} \oplus \mathfrak{h}$ *is a reductive Lie subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{n}$.
- $\mathrm{Ric}_{\mathfrak{u}} = cI + C_{\mathfrak{h}}$, *where* $\mathrm{Ric}_{\mathfrak{u}}$ *is the Ricci operator of* U/K *with* $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{h}$

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is* $\perp$*-Killing, and consider the orthogonal decomposition* $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$*, where* $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$*. Then* $(G/K, g_{\langle\cdot,\cdot\rangle})$ *is a semi-algebraic soliton if and only if the following conditions hold:*

- $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{k} \oplus \mathfrak{h}$. *In particular,* $\mathfrak{u} := \mathfrak{k} \oplus \mathfrak{h}$ *is a reductive Lie subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{n}$.
- $\operatorname{Ric}_{\mathfrak{u}} = cI + C_{\mathfrak{h}}$*, where* $\operatorname{Ric}_{\mathfrak{u}}$ *is the Ricci operator of* U/K *with* $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{h}$ *and* $\langle C_{\mathfrak{h}} Y, Y\rangle = \operatorname{tr} Symm(\operatorname{ad} Y|_{\mathfrak{n}})^2$, $\forall Y \in \mathfrak{h}$.

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\perp$-Killing, and consider the orthogonal decomposition $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. Then $(G/K, g_{\langle\cdot,\cdot\rangle})$ is a semi-algebraic soliton if and only if the following conditions hold:

- $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{k} \oplus \mathfrak{h}$. *In particular,* $\mathfrak{u} := \mathfrak{k} \oplus \mathfrak{h}$ *is a reductive Lie subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{n}$.
- $\mathrm{Ric}_{\mathfrak{u}} = cI + C_{\mathfrak{h}}$, *where* $\mathrm{Ric}_{\mathfrak{u}}$ *is the Ricci operator of* U/K *with* $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{h}$ *and* $\langle C_{\mathfrak{h}} Y, Y\rangle = \mathrm{tr}\, Symm(\mathrm{ad}\, Y|_{\mathfrak{n}})^2$, $\forall Y \in \mathfrak{h}$.
- $\mathrm{Ric}_{\mathfrak{n}} = cI + D_1$, *for some* $D_1 \in \mathrm{Der}(\mathfrak{n})$, *where* $\mathrm{Ric}_{\mathfrak{n}}$ *denotes the Ricci operator of* $(\mathfrak{n}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}})$ *(nilsoliton).*

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is* $\perp$*-Killing, and consider the orthogonal decomposition* $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$*, where* $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$*. Then* $(G/K, g_{\langle\cdot,\cdot\rangle})$ *is a* *semi-algebraic soliton* *if and only if the following conditions hold:*

- $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{k} \oplus \mathfrak{h}$*. In particular,* $\mathfrak{u} := \mathfrak{k} \oplus \mathfrak{h}$ *is a reductive Lie subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{n}$*.*
- $\operatorname{Ric}_{\mathfrak{u}} = cI + C_{\mathfrak{h}}$*, where* $\operatorname{Ric}_{\mathfrak{u}}$ *is the Ricci operator of* U/K *with* $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{h}$ *and* $\langle C_{\mathfrak{h}} Y, Y \rangle = \operatorname{tr} Symm(\operatorname{ad} Y|_{\mathfrak{n}})^2$, $\forall Y \in \mathfrak{h}$*.*
- $\operatorname{Ric}_{\mathfrak{n}} = cI + D_1$*, for some* $D_1 \in \operatorname{Der}(\mathfrak{n})$*, where* $\operatorname{Ric}_{\mathfrak{n}}$ *denotes the Ricci operator of* $(\mathfrak{n}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}})$ *(nilsoliton).*
- $\sum[\operatorname{ad} Y_i|_{\mathfrak{n}}, (\operatorname{ad} Y_i|_{\mathfrak{n}})^t] = 0$*, where* $\{Y_i\}$ *is any orthonormal basis of* $\mathfrak{h}$

Structure of semi-algebraic solitons

Theorem (Lafuente-JL 2011)

Assume $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\perp$-Killing, and consider the orthogonal decomposition $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. Then $(G/K, g_{\langle\cdot,\cdot\rangle})$ is a semi-algebraic soliton if and only if the following conditions hold:

- $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{k} \oplus \mathfrak{h}$. *In particular,* $\mathfrak{u} := \mathfrak{k} \oplus \mathfrak{h}$ *is a reductive Lie subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{n}$.
- $\operatorname{Ric}_{\mathfrak{u}} = cI + C_{\mathfrak{h}}$, *where* $\operatorname{Ric}_{\mathfrak{u}}$ *is the Ricci operator of* U/K *with* $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{h}$ *and* $\langle C_{\mathfrak{h}} Y, Y\rangle = \operatorname{tr} Symm(\operatorname{ad} Y|_{\mathfrak{n}})^2$, $\forall Y \in \mathfrak{h}$.
- $\operatorname{Ric}_{\mathfrak{n}} = cI + D_1$, *for some* $D_1 \in \operatorname{Der}(\mathfrak{n})$, *where* $\operatorname{Ric}_{\mathfrak{n}}$ *denotes the Ricci operator of* $(\mathfrak{n}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}})$ *(nilsoliton).*
- $\sum[\operatorname{ad} Y_i|_{\mathfrak{n}}, (\operatorname{ad} Y_i|_{\mathfrak{n}})^t] = 0$, *where* $\{Y_i\}$ *is any orthonormal basis of* $\mathfrak{h}$ *($\Rightarrow$ $(\operatorname{ad} Y|_{\mathfrak{n}})^t \in \operatorname{Der}(\mathfrak{n})$ for all $Y \in \mathfrak{h}$).*