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Quaternionic Contact Hypersurfaces of Hyper-Kähler manifolds

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Quaternionic Manifolds

Definition. A pair $(K, \mathcal{Q})$ of a smooth $4n$-manifold K and a three dimensional subbundle $\mathcal{Q} \subset \mathrm{End}(TK)$ is called Quaternionic Manifold if

i) $\mathcal{Q} = \mathrm{span}\{J_1, J_2, J_3\}$

$$(J_1)^2 = (J_2)^2 = (J_3)^2 = -Id, \qquad J_1 J_2 = -J_2 J_1 = J_3;$$

ii) There exists a torsion-free connection $\hat{\nabla}$ on TK with $\hat{\nabla}_X \mathcal{Q} \subset \mathcal{Q}$, $X \in TK$.

- The above definition resembles the definition of a complex manifold.

- Unlike the complex manifolds, quaternionic manifolds can be distinguished locally by a curvature tensor.

Hyper-Surfaces of Quaternionic Manifolds

Let $M \subset K$ be any hyper-surface of the quaternionic manifold $(K, \mathcal{Q})$.

Dfn. We define $H \subset TM$ to be the maximal $\mathcal{Q}$-invariant distribution on M.

- If f is any defining function for M, i.e. $M = f^{-1}(0)$ and $df|_M \neq 0$, then

$$H = \{ X \in TM : df(J_1 X) = df(J_2 X) = df(J_3 X) = 0 \}.$$

- Thus H is always a smooth codimension 3 distribution on M.

Quaternionic Contact Hyper-Surfaces of Quaternionic Manifolds

Definotion. We say that a hyper-surface M of a quaternionic manifold $(K, \mathcal{Q} = \{J_1, J_2, J_3\})$ is a QC-hyper-surface if

i) $\hat{\nabla} df(X, X) \neq 0, \quad X \in H, \quad \text{unless} \quad X = 0,$

ii) $\hat{\nabla} df(J_s X, J_s Y) = \hat{\nabla} df(X, Y), \qquad X, Y \in H, \qquad s = 1, 2, 3,$

where $H \subset TM$ is the maximal $\mathcal{Q}$-invariant distribution on M, $\hat{\nabla}$ is any torsion-free quaternionic connection of $(K, \mathcal{Q})$, and f is any defining function for M.

Examples of QC-Hyper-Surfaces

- Consider the field of the quaternions

$$\mathbb{H} \;=\; \mathrm{span}_{\mathbb{R}}\{1, i, j, k\},$$

where $i^2 = j^2 = -k^2 = -1$ and $i \cdot j = -j \cdot i = k$.

- Consider the flat quaternionic manifold $K := \mathbb{H}^{n+1}$ with its standard quaternionic structure $\mathcal{Q} = \mathrm{span}\{J_1, J_2, J_3\}$.

$$J_1(x) := -x \cdot i, \qquad J_2(x) := -x \cdot j, \qquad J_3(x) := -x \cdot k.$$

- As a torsion free quaternionic connection $\hat{\nabla}$ we take the flat connection here. It clearly holds $\hat{\nabla}_X \mathcal{Q} \subset \mathcal{Q}$.

Let $x = \begin{pmatrix} q_1 \\ \cdots \\ q_n \\ p \end{pmatrix} \in \mathbb{H}^n \times \mathbb{H}$.

We have the following three basic examples of QC hyper-surfaces of $\mathbb{H}^n \times \mathbb{H}$

- M_1 : $\sum_{a=1}^n |q_a|^2 + \mathbb{R}e(p) = 0$
- M_2 : $\sum_{a=1}^n |q_a|^2 - |p|^2 = -1$
- M_3 : $\sum_{a=1}^n |q_a|^2 + |p|^2 = 1$ (the Sphere).

- Let $Sp(1) := \{z \in \mathbb{H} \ : \ |z| = 1\}$.

- The quaternionic affine group $GL(n+1, \mathbb{H}) \times Sp(1) \rtimes \mathbb{H}^{n+1}$ acts on the vector space $\mathbb{H}^{n+1}$ by

$$\phi(x) \ = \ A \cdot x \cdot \bar{z} \ + \ y,$$

where $\phi = (A, z, y) \in GL(n+1, \mathbb{H}) \times Sp(1) \rtimes \mathbb{H}^{n+1}$.

- If M is a QC-hyper-surface, then $\phi(M)$ is a QC-hyper-surface as well.

- Thus the three examples M_1, M_2 and M_3 determine three orbits of QC-hyper-surfaces of $\mathbb{H}^{n+1}$.

Theorem. If M is a connected QC-hyper-surface of $\mathbb{H}^{n+1}$ then there exists a transformation $\phi \in GL(n+1,\mathbb{H}) \times Sp(1) \rtimes \mathbb{H}^{n+1}$ such that $\phi(M)$ is an open set of one of the hyper-surfaces

- $M_1 : \quad \sum_{a=1}^{n} |q_a|^2 + \mathbb{R}e(p) = 0$
- $M_2 : \quad \sum_{a=1}^{n} |q_a|^2 - |p|^2 = -1$
- $M_3 : \quad \sum_{a=1}^{n} |q_a|^2 + |p|^2 = 1.$

Let $(K, \mathcal{Q} = \{J_1, J_2, J_3\})$ be a quaternionic manifold and $M \subset K$ be a QC-hyper-surface, i.e. we have

- $\hat{\nabla} df(J_s X, J_s X) + \hat{\nabla} df(X, Y) = 0, \quad X, Y \in H$
- $\hat{\nabla} df|_H$ is positive or negative definite on H.

If we define:

- Metric $g := \hat{\nabla} df|_H$ on H.
- Three 1-forms η_1, η_2, η_3 on M given by

$$\eta_s(u) := -df(J_s u), \qquad u \in TM.$$

Then it holds: $g(J_s X, J_s Y) = g(X, Y)$ and $d\eta_s(X, Y) = g(J_s X, Y)$ for any $X, Y \in H$.

Abstract Quaternionic Contact Manifolds

Definition. A pair (M, H) of a $(4n+3)$-manifold M and a $4n$-distribution H on M is called Quaternionic Contact Manifold if locally there exists a smooth field $(\eta_1, \eta_2, \eta_3, I_1, I_2, I_3, g)$, where

- η_1, η_2, η_3 are 1-forms on M with common kernel H
- $I_1, I_2, I_3 \in \text{End}(H)$ satisfy

$$(I_1)^2 = (I_2)^2 = (I_3)^2 = -Id, \qquad I_1 I_2 = -I_2 I_1 = I_3$$

- $g \in H^* \otimes H^*$ is symmetric and positive definite,

and all these satisfy the equations

$$d\eta_s(X, Y) = g(J_s X, Y) \qquad X, Y \in H.$$

Conformal Infinity

Let (M, H) be a QC manifold.

If $(\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3, \hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{g})$ and $(\eta_1, \eta_2, \eta_3, I_1, I_2, I_3, g)$ are two admissible sets in an open neighborhood $U \subset M$ then

$$(\hat{I}_1, \hat{I}_2, \hat{I}_3) = (I_1, I_2, I_3)\Psi, \quad (\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3) = \mathcal{F}(\eta_1, \eta_2, \eta_3)\Psi, \quad \hat{g} = \mathcal{F}g,$$

where $\mathcal{F} : U \to \mathbb{R}^+$ and $\Psi : U \to SO(3)$.

Dfn. We say that a Riemannian metric G defined on $M \times (0, \epsilon)$ with coordinates (x, ρ) has as conformal infinity the QC-manifold (M, H) if there exists an admissible set $(\eta_1, \eta_2, \eta_3, I_1, I_2, I_3, g)$ such that

$$G \;\sim\; \frac{1}{\rho^2}((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2 + d\rho^2) \;+\; \frac{1}{\rho}g,$$

when ρ tends to zero.

Theorem. [O.Biquard, 2000] Each real analytic QC-manifold (M, H) is the conformal infinity of a unique quaternionic Kähler metric G defined on a neighborhood of M.

Example. The quaternionic hyperbolic metric on the unit-ball in $\mathbb{H}^{n+1}$ has as conformal infinity the unit-sphere S^{4n+3} with the QC distribution H being induced by the imbedding of S^{4n+3} into the quaternionic manifold $\mathbb{H}^{n+1}$.

Theorem. [D.Duchemin, 2006] Each real analytic QC-manifold (M, H) can be imbedded as a QC-hyper-surface in an appropriate quaternionic manifold (K, Q).

Typical examples of QC-manifolds are provided by the 3-Sasakian geometry.

Recall: A Riemannian $(4n+3)$-manifold (M,h) is called 3-Sasaki if there exist 3-Killing vector fields ξ_1,ξ_2,ξ_3 such that

- $h(\xi_i,\xi_j)=\delta_{ij},\ \ i,j=1,2,3$
- $[\xi_i,\xi_j]=-2\xi_k$, for any cyclic permutation (i,j,k) of $(1,2,3)$
- $(D_X\tilde{I}_i)Y=h(\xi_i,Y)X-h(X,Y)\xi_i,\ \ i=1,2,3,\ \ X,Y\in TM$,

where $\tilde{I}_i(X):=D_X\xi_i$ and D denotes the Levi-Civita connection of the Riemannian metric h.

We construct a **QC-structure** on M out of the 3-Sasakian one by setting $H=\{\xi_1,\xi_2,\xi_3\}^{\perp}$.

QC-Hyper-Surfaces of Hyper-Kähler Manifolds

Let $(K, \mathcal{Q})$ be a quaternionic manifold and G be any $\mathcal{Q}$-compatible Riemannian metric on K.

Dfn. $(K, \mathcal{Q}, G)$ is called a hyper-Kähler manifold if there exists a frame $\{J_1, J_2, J_3\}$ of $\mathcal{Q}$ which is parallel with respect to the Levi-Civita connection of G.

- A trivial example of a hyper-Kähler manifold is provided by $\mathbb{H}^{n+1}$ with its flat metric.

From now on we will assume: (M, H) is a QC-hyper-surface of a hyper-Kähler manifold (K, J_1, J_2, J_3, G).

Furthermore:

- Let D be the Levi-Civita connection of G
- Let N be the unit-normal vector field of the imbedding
- $II(X,Y) := -G(D_X N, Y)$, $X, Y \in TM$ is the second fundamental form.

Then it holds:

- $II|_H$ is symmetric and negative definite
- $II(J_s X, J_s Y) = II(X,Y), \quad s = 1,2,3, \quad X, Y \in H.$

Note that we make no assumption about $II(J_s N, X)$, $s = 1,2,3$, $X \in H$.

• The key point in our method is proving that for each QC-hyper-surface (M, H) there exists a function $f : M \to \mathbb{R}$ for which it holds

$$II(J_sN, J_sX) = -f^{-1}df(X), \quad s = 1, 2, 3, \quad X \in H.$$

• The function f is obtained by performing a certain volume normalization on M by comparing $II|_H$ with the hyper-Kähler metric $G|_H$. For this purpose we use the following lemma

Lemma. Let $\mathcal{H}^{4n}$ be a real vector space with a prescribed hyper-complex structure (J_1, J_2, J_3). Assume that we are given two positive definite inner products $\hat{g}$ and g on $\mathcal{H}^{4n}$ compatible with (J_1, J_2, J_3).

If we set

$$\hat{\gamma}_i(X, Y) := \hat{g}(I_jX, Y) + \sqrt{-1}\,\hat{g}(I_kX, Y) \qquad \text{and}$$
$$\gamma_i(X, Y) := g(I_jX, Y) + \sqrt{-1}\,g(I_kX, Y),$$

then there exists a positive constant μ such that $(\hat{\gamma}_s)^n = \mu(\gamma_s)^n$, $s = 1, 2, 3$.

Note that the Levi-Civita connection of the hyper-Kähler metric G induces a connection in the bundle $TK|_M \to M$.

Theorem. If (M, H) is a QC-hyper-surface of a hyper-Kähler manifold $(K, \mathcal{Q})$ then it holds:

- The second fundamental form II extends in an unique way to a symmetric J_s-invariant section Δ of the bundle $(T^*K \otimes T^*K)|_M \to M$.

- The section $f\Delta$ is parallel with respect to the Levi-Civita connection of the hyper-Kähler metric G.

Let $(K, \mathcal{Q}, G)$ be a hyper-Kähler manifold with Riemannian curvature tensor R.

Theorem. If $M \subset K$ is a QC-hyper-surface with normal vector N, then at each point of M it holds

$$R(X, Y)N = 0, \qquad X, Y \in TK.$$

Thank You for Your Attention!