Constructions of generalized complex structures in dimension four.

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Connecting two areas of research.

Generalized complex geometry. Introduced by Nigel Hitchin, and developed/studied by his students Marco Gualtieri and Gil Cavalcanti among others.

Inequivalent smooth structures on 4-manifolds. Work of Fintushel - Stern, Gompf, Baldridge - Kirk, Akhmedov - D. Park, and Yazinski on the construction of manifolds homeomorphic, but not diffeomorphic to $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$.

Produce: An increase in our understanding of generalized complex structures

1) a myriad of existence results

2) unveiling generalized complex structures with interesting traits

Some reasons to consider dimension four...

(Dated) Question. Do there exist generalized complex manifolds, which are neither complex nor symplectic?

Dimension two. Surfaces are Kähler $\Rightarrow$ they admit a generalized complex structure.

Dimension four. Known obstructions to the existence of complex and symplectic structures:

Kodaira's classification and Seiberg-Witten theory

Theorem (Taubes). Suppose a symplectic 4-manifold (X, ω) satisfies $b_2^+(X) > 1$, then

$$SW_X(\pm c_1(X, \omega)) \neq 0.$$

Vanishing theorem. Suppose $b_2^+ > 1$. If X admits a Riemannian metric g with $Scal_g > 0$, then $SW_X \equiv 0$ for all $Spin^{\mathbb{C}}$-structures.

"State of the art" of existence results.

1) **A myriad of homeomorphism types of almost complex 4-manifolds admit a generalized complex structure, z.B.**

$$(2m+1)\mathbb{CP}^2\#n\overline{\mathbb{CP}^2},\ (2m+1)(S^2\times S^2) \text{ or } E(n)\#2m(S^2\times S^2)$$

Theorem (Gualtieri-Cavalcanti '08, T'10). Let X be a closed almost complex simply connected 4-manifold. If $\omega_2(X)=0$, suppose the signature of X is non-positive. There exists a manifold Y homeomorphic to X, which admits a generalized complex structure that has multiple type change loci. If $b_2^+\geq 3$, then Y is neither complex nor symplectic.

Many other π_1's: Similar statement for cyclic fundamental groups using Ian Hambleton's results with Kreck and Teichner.

...and regarding the nature of these structures:

2) Existence of several generalized complex structures on a same manifold z. B.

$$L(p,1) \times S^1 \text{ and } T^2 \times S^2$$

with different properties. **New ones have multiple type change loci.**

Type change locus = structural feature of generalized complex structures presented today: the type of structure jumps from symplectic to complex along a codimension two submanifold.

Theorem (T'10). There exist generalized complex structures with arbitrary many type change loci.

Quick recall about generalized complex structures.

Definition. The canonical bundle of $\mathcal{J}$ is the complex line sub-bundle

$$K \subset \wedge^{\bullet} T^*_{\mathbb{C}} M$$

annihilated by the $+i$-eigenspace of $\mathcal{J}$.

Proposition (Gualtieri). A generator $\rho \in K_x$ has the form

$$\rho = e^{B+i\omega} \wedge \Omega,$$

where $\Omega = \theta_1 \wedge \cdots \wedge \theta_n$ for $\{\theta_i\}_I$ a basis for $L \cap T^*_{\mathbb{C}} M$, and B, ω are real and imaginary components of a complex 2-form. The $+i$-eigenbundle is $L \subset T_{\mathbb{C}} M \oplus T^*_{\mathbb{C}} M$

Definition. Let $\rho = e^{B+i\omega} \wedge \Omega$ be a generator of the canonical bundle K of a generalized complex structure $\mathcal{J}$ at a point $x \in M$. The type of $\mathcal{J}$ at x is the degree of Ω.

Type jumping phenomena via Poisson structures.

Fundamental example: Holomorphic Poisson structure on $\mathbb{C}^2$ with complex coordinates (z_1, z_2) given by

$$\sigma = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \in H^0(\bigwedge^2 T_{1,0})$$

The corresponding canonical bundle K is generated by the complex differential form

$$\rho = z_1 + dz_1 \wedge dz_2.$$

The spinor equals $dz_1 \wedge dz_2$ along the locus $\{z_1 = 0\}$, and everywhere else it can be written as

$$\rho = z_1 \exp(\frac{dz_1 \wedge dz_2}{z_1}).$$

The form ρ generates a canonical bundle for a generalized complex structure that has type 2 along the locus z_1 and type 0 elsewhere.

Surgery along tori.

Let $T \hookrightarrow X$ be a torus of self-intersection zero, and $\nu(T)$ its tubular neighborhood.

Definition. Surgery on a torus T along the curve γ

$$X_{T,\gamma}(p,q,r) := (X - \nu(T)) \cup_{\varphi} (T^2 \times D^2),$$

where the diffeomorphism $\varphi : T^2 \times \partial D^2 \to \partial(X - \nu(T))$ used to glue the pieces together satisfies

$$\varphi_*([\partial D^2]) = p[S^1_\alpha] + q[S^1_\beta] + r[\mu_T]$$

in $H_1(\partial(X - \nu(T)); \mathbb{Z})$.

Significance of how the meridian is being glued back

Surgeries and geometric structures.

Theorem A. (Luttinger, Auroux-Donaldson-Katzarkov). If X admits a symplectic structure for which T is a Lagrangian torus, then the result of performing an $r = 1$ torus surgery with respect to the Lagrangian framing

$$X_{T,\gamma}(p, q, 1)$$

admits a symplectic structure.

Theorem B. (Cavalcanti-Gualtieri). If X admits a symplectic structure for which T is a symplectic submanifold, then $X_{T,\gamma}(p, q, 0)$ admits a generalized complex structure. The core torus $T^2 \times \{0\}$ of the surgery is a type change locus, which inherits the structure of an elliptic curve.

Significance of how the meridian is being glued back

Sample.

An application of Fintushel - Stern's *reverse-engineering* yields:

Proposition. Let Σ_g be a genus $g \geq 0$ surface. The manifold

$$T^2 \times \Sigma_g$$

admits generalized complex structures $\mathcal{J}_i$ and $\mathcal{J}_n$ such that

$$\mathrm{Type}(\mathcal{J}_i) = i$$

for $i \in \{0, 1, 2\}$, and $\mathcal{J}_n$ has n type change loci for $n \in \mathbb{N}$.

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