A variational principle for spinors

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Geometric structures on manifolds and their applications
3rd of July 2012

Talk based on

- *A heat flow for special metrics*
  joint with H. Weiß (München)
- *Energy functionals and soliton equations for* G_2*-forms*
  joint with H. Weiß (München)
- *A spinorial energy functional: critical points and gradient flow*
  joint with B. Ammann (Regensburg) and H. Weiß (München)

Leitmotiv

M^n C^∞-manifold (compact, spin, simply-connected, $n \geq 4$)

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Local question

Is there a natural direction for deforming g towards a special metric?

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Rough classification scheme

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Theorem (Kuiper)

(M^n, g) compact, simply-connected and conformally flat

$\Rightarrow (M^n, g)$ conformally equivalent to (S^n, g_{round})

Generalised Bieberbach theorem

Theorem (Cheeger-Gromoll, Fischer-Wolf)

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Existence of compact simply-connected (irreducible) Ricci-flat manifolds?

Berger's list (Ricci-flat case)

dim M	$\mathrm{Hol}(M, g)$	geometry	examples
n	$\mathrm{SO}(n)$	generic	?
$2m$	$\mathrm{SU}(m)$	Calabi-Yau	Yau
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8	$\mathrm{Spin}(7)$	Spin(7)	Joyce
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- approach special holonomy from variational point of view
- study (negative) gradient flow of the functional

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Proposition (Wang)

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The $\mathcal{E}$-functional

Spin structure $\tilde{P} \to P$ is a 2-fold cover of $\mathrm{GL}_+(n)$-frame bundle $P \to M$ such that fibrewise $0 \to \mathbb{Z}_2 \to \widetilde{\mathrm{GL}}_+(n) \to \mathrm{GL}_+(n) \to 0$

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Universal spinor bundle

$$\Sigma M = \widetilde{P} \times_{\mathrm{Spin}(n)} \Sigma_n \to \odot^2_+ T^*M \to M$$

$$\mathcal{F} = \Gamma(\Sigma M) = \{(g, \phi) \,|\, \phi \in \Gamma(\Sigma_g M)\} \to \mathcal{M} := \{\text{metrics on M}\}$$

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The energy functional

$$\mathcal{E} : \mathcal{N} \to \mathbb{R}, \quad (g,\phi) \mapsto \frac{1}{2}\int_M |\nabla^g \phi|^2_g dv^g$$

Critical points

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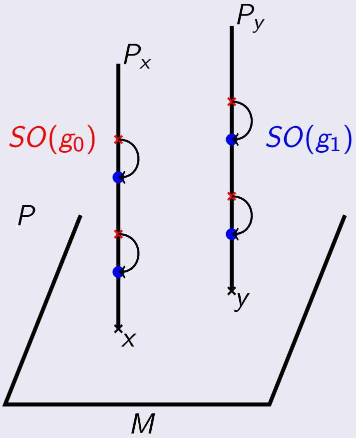
- saddle points exist for $\gamma_M \geq 1$

Sketch of the proof of Theorem A



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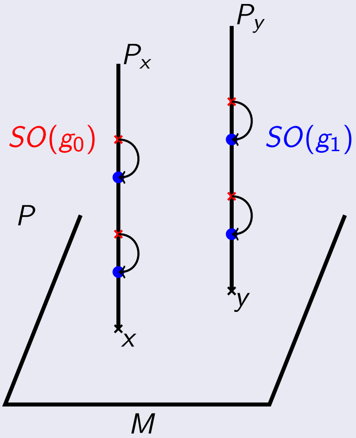

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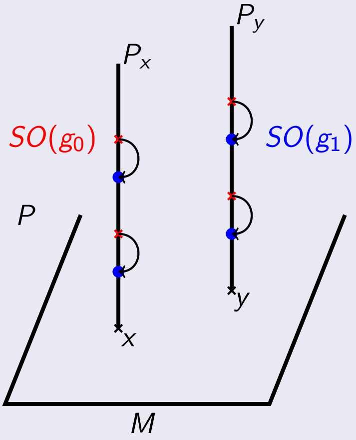

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This lifts to $\widetilde{P}$.

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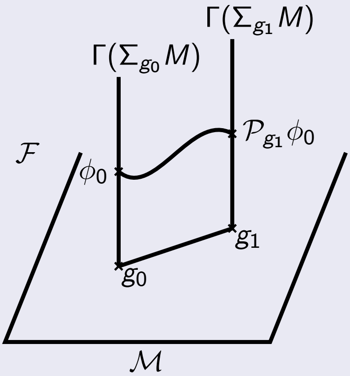
Bourguignon-Gauduchon distribution



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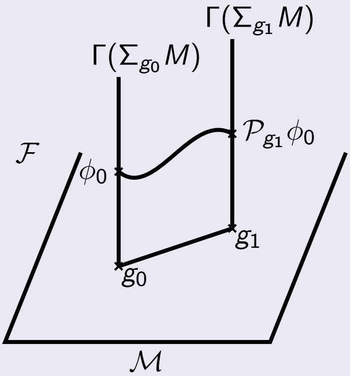

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horizontal distribution

$T_{(g,\phi)}\mathcal{F} \cong \Gamma(\odot^2 T^* M) \oplus \Gamma(\Sigma_g M)$

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- J.-P. Bourguignon and P. Gauduchon, *Spineurs, opérateurs de Dirac et variations de métriques*

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Lemma

negative L^2-gradient $Q = (Q_1, Q_2) = -\mathrm{grad}\,\mathcal{E} : \mathcal{N} \to T\mathcal{N}$ given by

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M. Wang, *Preserving parallel spinors under metric deformations*

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The flow equation

$$\partial_t(g_t, \phi_t) = Q(g_t, \phi_t), \quad (g_0, \phi_0) = (g, \phi) \in \mathcal{N} \qquad \text{(SF)}$$

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Theorem B (Short-time existence and uniqueness)

For all $(g,\phi) \in \mathcal{N}$ there exists a uniquely determined smooth family $(g_t,\phi_t) \in \mathcal{N}$ for $t \in [0,\epsilon]$ such that (SF) holds.

Sketch of the proof of Thm B

DeTurck's trick

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Back to (DF)

solve $\frac{d}{dt}f = -X_0(\tilde{g}_t,\tilde{\phi}_t)\circ f \Rightarrow (g_t,\phi_t) = f_t^*(\tilde{g}_t,\tilde{\phi}_t)$ solves (SF)

Further functionals

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- $\mathcal{E}^{1/8}(g,\phi) = \frac{1}{2}\int_M |D^g\phi|^2 dv^g$

Premoduli spaces

Theorem C (Smoothness of the critical set)

M simply-connected, $(\bar{g}, \bar{\phi})$ critical and irreducible
$\Rightarrow \mathrm{Crit}(\mathcal{E})$ is smooth at $(\bar{g}, \bar{\phi})$ and $\widetilde{Q}_{\bar{X}}^{-1}(0)$ is a smooth slice for $\widetilde{Diff}_0(M)$-action on $\mathrm{Crit}(\mathcal{E})$.

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- index theory (Wang's stability theorem)
- R. Goto, *Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler,* G_2 *and* $\mathrm{Spin}(7)$ *structures*

  (cf. also J. Nordström)

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Theorem C (Smoothness of the critical set)

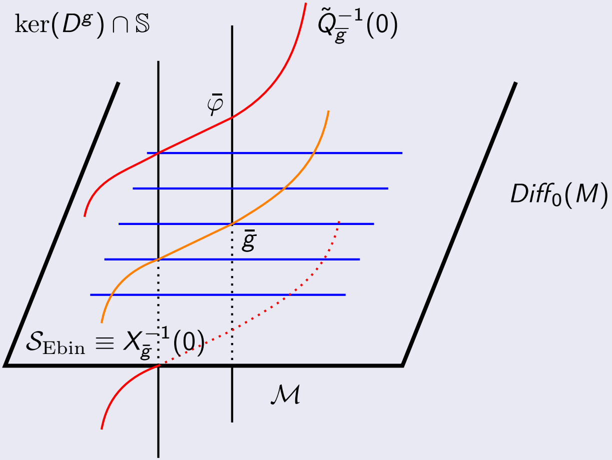
M simply-connected, $(\bar{g}, \bar{\phi})$ critical and irreducible
$\Rightarrow \mathrm{Crit}(\mathcal{E})$ is smooth at $(\bar{g}, \bar{\phi})$ and $\widetilde{Q}_{\bar{X}}^{-1}(0)$ is a smooth slice for $\widetilde{Diff}_0(M)$-action on $\mathrm{Crit}(\mathcal{E})$.

If $\dim M = 4, 6, 7$ (or 8)
$\Rightarrow \mathcal{E}$ Morse-Bott

- index theory (Wang's stability theorem)
- R. Goto, *Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler,* G_2 *and* $\mathrm{Spin}(7)$ *structures*
  (cf. also J. Nordström)
- $\widetilde{Q}_{\bar{X}}^{-1}(0) = Q^{-1}(0) \cap \bar{X}^{-1}(0)$

Sketch of the proof of Theorem C

Generalised Ebin slice



Examples of the flow

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(g_0, ϕ_0) Killing spinor

$\Rightarrow$ Flow dies in finite time, e.g. $S^7 \to pt$.

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Stability of Dirichlet flow on positive forms

initial condition sufficiently close to a critical point

$\Rightarrow$ (SF) exists for all times and converges modulo diffeomorphisms to a critical point.

Metrics of special holonomy

special holonomy $\leftrightarrow$ closed forms of special algebraic type

G_2-manifolds

- (M^7, Ω) G_2-manifold, $\Omega_p \in \mathcal{O} = \mathrm{GL}_+(7)/\mathrm{G}_2 \subset \Lambda^3 T_p^* M$
- Ω reduces $P = P_{\mathrm{GL}_+(7)} \to M$ to $P_{\mathrm{G}_2} \to M$ which extends to $P_{\mathrm{SO}(7)} \to M$, hence induces metric g_Ω
- ∇^{g_Ω} reduces to $P_{\mathrm{G}_2} \Leftrightarrow \nabla^{g_\Omega}\Omega = 0 \Leftrightarrow d\Omega = 0,\ d \star_{g_\Omega} \Omega = 0$

- choice of a metric $g_p \in \mathrm{GL}_+(7)/\mathrm{SO}(7)$
- choice of a unit spinor $\phi_p \in S^7 \cong \mathrm{Spin}(7)/\mathrm{G}_2 \subset \Sigma_7^{\mathbb{R}}$
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Flowing positive forms

Dirichlet functional

$$\mathcal{D} : \mathcal{P}(M) \to \mathbb{R}, \quad \Omega \mapsto \frac{1}{2}\int_M (|d\Omega|^2_{g_\Omega} + |d\star_{g_\Omega}\Omega|^2_{g_\Omega})dv^{g_\Omega}$$

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$$\partial_t\Omega = Q(\Omega), \quad \Omega(0) = \Omega_0 \in \mathcal{P}(M) \qquad \text{(DF)}$$

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Theorem B'

For any $\Omega_0 \in \mathcal{P}(M)$ there exists a uniquely determined smooth family $\Omega_t \in \mathcal{P}$ for $t \in [0, \epsilon]$ such that (DF) holds.

Longtime existence and convergence?

Theorem D' (Stability)

$\bar{\Omega}$ be critical and $k > 11/2$

$\Rightarrow$ for all $\epsilon > 0$ there is $\delta > 0$ such that for any Ω_0 with $\|\Omega_0 - \bar{\Omega}\|_{W^{k,2}} < \delta$, the (DDF) $\tilde{\Omega}(t)$ with $\widetilde{\Omega}(0) = \Omega_0$

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Corollary

For initial conditions sufficiently C^∞-close to $\bar{\Omega}$ the Dirichlet flow exists for all times and converges modulo diffeomorphisms to a critical positive form.

Sketch of the proof of Theorem D'

Key facts

Let $\Omega \in \mathcal{P}(M)$ and $L_\Omega := D_\Omega \widetilde{Q}_{\bar{X}}$ (symmetric for $\Omega = \bar{\Omega}$).

- **(Linear stability)** $L_{\bar{\Omega}} \leq 0$

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- **(Integrability)** $\mathcal{M} = \tilde{Q}_{\bar{\Omega}}^{-1}(0)$ smooth near $\bar{\Omega}$.
- **(Coercivity)** $\langle -L_{\bar{\Omega}}\dot{\Omega}, \dot{\Omega}\rangle_{L^2_{\bar{\Omega}}} \geq C\|\dot{\Omega}\|_{W^{1,2}_{\bar{\Omega}}} - \|\dot{\Omega}\|_{L^2_{\bar{\Omega}}}$

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1st step: Implicit function theorem (uses coercivity)

Ω_0 sufficiently close to $\bar{\Omega}$

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- **(Orthogonal projection)** Ω sufficiently $W^{k,2}$-close to $\bar{\Omega}$
  $\Rightarrow \omega' := \Omega - \Omega' \in (T_{\Omega'}\mathcal{M})^{\perp_{L^2}}$ for unique $\Omega' \in \mathcal{M}$

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- **(Remainder term estimate)** $R_{\Omega'}(\omega') = \tilde{Q}_{\bar{\Omega}}(\Omega) - L_{\Omega'}\omega'$
  For $\kappa > 0$ there exists $\epsilon > 0$ such that

$$\|\Omega - \bar{\Omega}\|_{W^{k,2}} < \epsilon \Rightarrow \|R_{\Omega'}(\omega')\|_{L^2} \leq \kappa \|L_{\Omega'}\omega'\|_{L^2}.$$

Sketch of the proof of Theorem D'

For a solution $\tilde{\Omega}(t)$ of (DDF) with $\|\tilde{\Omega}(t) - \bar{\Omega}\|_{W^{k,2}} < \epsilon$ consider

$$\tilde{Q}(t) = \tilde{Q}_{\bar{\Omega}}(\tilde{\Omega}(t))$$

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- Let λ_1 first eigenvalue > 0 of $-L_{\bar{\Omega}}$. **2nd step** $\Rightarrow$

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- **1st step** $\Rightarrow \tilde{\Omega}(t) = \tilde{\Omega}_0 + \int_0^t \tilde{Q}(t)dt \to \tilde{\Omega}_\infty \in \mathcal{M}$