A refinement of a gap theorem for gradient shrinking Ricci solitons

Takumi Yokota

Kyoto University / Universität Münster

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Introduction

Definition

A triple (M, g, f) with $f \in C^\infty(M)$ is called a gradient shrinking Ricci soliton if

$$\mathrm{Ric}(g) + \mathrm{Hess}\, f = \frac{1}{2\lambda} g \text{ for } \exists\, \lambda > 0.$$

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A family $(M, g(t)), t \in I \subset \mathbb{R}$ is called a Ricci flow when it satisfies

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)).$$

It is called an ancient solution if it exists for $\forall t \in (-\infty, 0]$.

It is convenient to use the reverse time $\tau := -t$.

For $\forall (M, g, f)$, $g_0(\tau) := (\tau/\lambda)\varphi_t^* g, \tau \in (0, \infty)$ is a Ricci Flow.

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Gaussian density

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We define the Gaussian density of a gradient shrinking Ricci soliton (M^n, g, f), i.e., $\mathrm{Ric} + \mathrm{Hess}\, f = \frac{1}{2\lambda} g$, as

$$\Theta(M) := \int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f} d\mu_g.$$

Note: We always normalize f so that

$$\lambda(R + |\nabla f|^2) \equiv f.$$

Here R denotes the scalar curv. of g. We know $R \geq 0$ (Zhang 2009).

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Theorem (Cao–Zhou 2010, cf. Haslhofer–Müller 2011)

For $\forall$*gradient shrinking Ricci soliton* (M^n, g, f) *and* $\forall p \in M$, $\exists c_1, c_2$ *and* $C > 0$ *such that*

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2$$

for $\forall x \in M$ *with* $r(x) := d(x,p)/\sqrt{\lambda} \gg 1$, *and*

$$\mathrm{Vol}(B_p(r)) \leq Cr^n \text{ for } \forall r > 0.$$

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Main Theorem = Gap Theorem for Ricci solitons

Theorem (Y. 2011)

For $\forall n \geq 2$, $\exists\, \epsilon_n > 0$ *such that:*
Any gradient shrinking Ricci soliton (M^n, g, f) *with* $\Theta(M) > 1 - \epsilon_n$ *is the Gaussian soliton* $(\mathbb{R}^n, g_E, |\cdot|^2/4)$.

Corollary (Conjectured by Carrillo–Ni)

Any gradient shrinking Ricci soliton (M^n, g, f) *with* $\Theta(M) \geq 1$ *is the Gaussian soliton* $(\mathbb{R}^n, g_E, |\cdot|^2/4)$.

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Remarks

- Main Thm was proved in [Y. 2009] for (M^n, g, f) with $\mathrm{Ric}(g) \geq -\exists K$, which was dropped in [Y. 2011].
- Carrillo–Ni (2009) proved a Log–Sobolev ineq. for gradient shrinking Ricci solitons with $\Theta(M)$ as the best const.
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Remarks

- For the proof of Main Theorem, we need
  - Perelman's reduced volume $\tilde{V}(\tau) := \int_M (4\pi\tau)^{-n/2} \mathrm{e}^{-\ell} d\mu$,
  - a gap theorem for ancient solutions, and
  - the estimate "$f \approx \ell$" for $\forall (M^n, g, f)$.

Definition of reduced volume (Perelman 2002)

Let $(M^n, g(\tau)), \tau := -t \in [0, T)$ be a backward RF.
Fix $p, q \in M$ and $[\tau_1, \tau_2] \subset [0, T)$.

- $\mathcal{L}$-length of a curve $\gamma : [\tau_1, \tau_2] \to M$:

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(|\dot{\gamma}(\tau)|^2_{g(\tau)} + R(\gamma(\tau), \tau) \right) d\tau$$

- $\mathcal{L}$-distance between (p, τ_1) and (q, τ_2):

$$L_{(p,\tau_1)}(q, \tau_2) := \inf_{\gamma} \mathcal{L}(\gamma),$$

  where inf is taken for γ with $\gamma(\tau_1) = p$ & $\gamma(\tau_2) = q$.

- Reduced volume based at $(p, 0)$:

$$\tilde{V}_{(p,0)}(\tau) := \int_M (4\pi\tau)^{-n/2} \exp\left(\frac{-1}{2\sqrt{\tau}} L_{(p,0)}(\cdot, \tau) \right) d\mu_{g(\tau)}.$$

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Existence of minimal $\mathcal{L}$-geodesics

Lemma

For $\forall (M, g(\tau)), \tau \in [0, T)$ *and* $\forall (p, 0), (q, \tau) \in M \times [0, T)$, *if* $\mathrm{Ric} \geq -\exists\, K$ *on* $M \times [0, T)$, *then*

$$\exists\, \gamma : [0, \tau] \to M \text{ from } p \text{ to } q \text{ s.t. } \mathcal{L}(\gamma) = L_{(p,0)}(q, \tau).$$

Proof: Since $\frac{1}{2}\frac{\partial}{\partial \tau} g(\cdot) = \mathrm{Ric}(g(\cdot)) \geq -K g(\cdot)$ on $M \times [0, T)$,

$$g(\tau) \geq \mathrm{e}^{-2K\tau} g(0).$$

Then, $$L_{(p,0)}(q, \tau) := \inf_{\gamma} \mathcal{L}(\gamma) \geq c \cdot d^2_{g(0)}(p, q) - C.$$

Hence, $\{\gamma_i\}$ with $\mathcal{L}(\gamma_i) \to L_{(p,0)}(q, \tau)$ remains in a bounded region, and subconverges to a minimal $\mathcal{L}$-geodesic $\exists\, \gamma$. □

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Monotonicity of reduced volume

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- $\tilde{V}_{(p,0)}(\tau) = 1 \iff (M^n, g(\cdot))$ *is isometric to* $(\mathbb{R}^n, g_E)$ *on* $[0, \tau]$.

cf.

Theorem (Bishop–Gromov volume comparison)

For $\forall (M^n, g)$ *with* $\mathrm{Ric} \geq 0$ *and* $\forall p \in M$,

- $\tilde{V}_p(r) := \mathrm{Vol}(B_p(r))/\omega_n r^n \nearrow 1$ *as* $r \searrow 0$.
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Corollary (M. Anderson 1990)

Any Ricci flat manifold (M^n, g), *i.e.,* $\mathrm{Ric} \equiv 0$ *with*

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Proof of Gap Theorem for ancient solutions

Suppose $\exists\, (M_i^n, g_i(\tau), p_i), \tau \in [0, \infty)$, $i = 1, 2, \cdots$ s.t.

$$\lim_{\tau \to \infty} \tilde{V}^{g_i}_{(p_i,0)}(\tau) \to 1 \text{ as } i \to \infty.$$

1. Use Perelman's point-picking argument to find "nice" points $(q_i, \tau_i) \in M_i \times [0, \infty)$ and put $h_i(\tau) := Q_i^{-1} g_i(Q_i \tau + \tau_i)$, $\tau \in [0, \infty)$, where $Q_i := |\mathrm{Rm}^{g_i}|(q_i, \tau_i)$.
2. Then, by Hamilton's compactness thm & Perelman's no-local collapsing thm,

$$(M_i^n, h_i, q_i) \underset{\text{in } C^\infty}{\longrightarrow} \exists\, (M_\infty^n, h_\infty(\tau), q_\infty) \text{ as } i \to \infty$$

   with $|\mathrm{Rm}^{h_\infty}|(q_\infty, 0) = 1$.
3. On the other hand, $\tilde{V}^{h_\infty}_{(q_\infty,0)}(\tau) = 1$ and hence $(M_\infty^n, h_\infty(\tau))$ is isometric to $(\mathbb{R}^n, g_E)$. Contradiction! □

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Proof of Main Thm

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Main Thm would follow if we could apply

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For $\forall p \in M$, $\forall \tau \in (0,\infty)$, *and* $\ell_{(p,0)}(\cdot,\tau) := \frac{1}{2\sqrt{\tau}} L_{(p,0)}(\cdot,\tau)$,

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Put $g(\tau) := g_0(\tau + \lambda), \tau \in [0, \infty)$.

Problem 2: $\mathrm{Ric}(g(\tau))$ may not be bounded below on $M \times [0, \infty)$.

In spite of this, we have

Proposition (Y. 2011)

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Proof of Proposition: We only have to prove ∃ of minimal $\mathcal{L}$-geodesics. This follows from

$$\ell^g_{(p,0)}(q,\tau) \approx f_{\tau+\lambda}(q) \approx \frac{1}{4\lambda} d^2_{g(0)}(\varphi_{\tau+\lambda}(q), p)$$

for $\forall (p,0), (q,\tau) \in M \times [0,\infty)$. Second ≈ is due to Cao–Zhou. □

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