Polar Manifolds and Actions

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Definition of polar action

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Let M be a complete Riemannian manifold with a proper isometric action by a Lie group G. The action is called polar if

(a) There exists an isometric immersion $\sigma\colon \Sigma \to M$ of a connected complete manifold Σ whose image meets all orbits, i.e. $G \cdot \sigma(\Sigma) = M$.

(b) All orbits intersect Σ orthogonally, i.e. $\sigma_*(T_p\Sigma) \perp T_{\sigma(p)}(G \cdot \sigma(p))$ for all $p \in \Sigma$.

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or equivalently

Theorem (H.Boualem)

The proper isometric action by G on M is polar, if and only if the horizontal distribution of $M \to M/G$ is integrable on the regular part.

Polar Group

Definition

Let G act polar on M with section $\sigma\colon \Sigma \to M$. Then we define the polar group $\Pi = N_G(\Sigma)/Z_G(\Sigma)$ where
$N_G(\Sigma) = \{g \in G \mid g(\sigma(\Sigma)) \subset \sigma(\Sigma)\}$ and
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(a) *Π is a discrete subgroup of $N(H)/H$ where H is a principle isotropy group and $\sigma(\Sigma) \subset M^H$.*

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(c) *For all $p \in \Sigma$ we have $\sigma(\Pi \cdot p) = G \cdot \sigma(p) \cap \sigma(\Sigma)$, i.e. the orbits meet the section Σ in the orbits of the polar group. In particular, $M/G \simeq \Sigma/\Pi$ are isometric and hence an orbifold.*

Examples of Polar Actions

Example

(a) $G = SO(2)$ acting on $\mathbb{S}^2(1)$ or $\mathbb{R}^2$ fixing north and south pole, respectively the origin. Section $\Sigma = \mathbb{S}^1$ resp. $\mathbb{R}$ and polar group $\Pi = \mathbb{Z}_2$.

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(d) $M = G/K$ a symmetric space and K acts on T_pM via $d(L_k)_p$ as the isotropy representation. The section $\Sigma = \mathfrak{a} \subset \mathfrak{p}$, with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition, is a maximal abelian subalgebra. The polar group Π is the Weyl group of the symmetric space. These are the so called **s-representations**.

Polar Representations

Theorem (Dadok)

Let G be a compact connected group acting orthogonally on $\mathbb{R}^n$. The representation is polar if and only if it is linearly equivalent to an s-representation, or a subgroup of an s-representation which acts with the same orbits (orbit equivalent).

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There is a short list of such orbit equivalent representations, e.g. $\mathrm{SU}(n) \subset \mathrm{U}(n) \subset \mathrm{SO}(2n)$ acting on $\mathbb{R}^{2n}$ with orbits the spheres centered at the origin, or similarly $G_2 \subset \mathrm{SO}(7)$ acting on R^7.

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Let G act polar on M, then the slice representation of the isotropy group $G_{\sigma(p)}$ on $(T_{\sigma(p)}(G \cdot \sigma(p)))^\perp$ is polar with section $\sigma_*(T_p\Sigma)$.

Polar manifolds

Example

(a) The left action of K on a symmetric space $M = G/K$ is polar. The section is flat if $\operatorname{rank}(M) > 1$ and $\Sigma = \mathbb{S}^k$ or $\mathbb{RP}^k$ if $\operatorname{rank}(M) = 1$.

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(c) If M_i are two G-polar manifolds with section Σ_i with fixed points $p_i \in M_i$ and equivalent isotropy representations at p_i, then $M_1 \# M_2$ is polar with section $\Sigma_1 \# \Sigma_2$.

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(d) Similarly, connected sums along orbits, or along orbit types, are polar under corresponding assumptions.

(e) Cohomogeneity one manifolds, i.e. $\dim(M/G) = 1$ are polar with section $\Sigma \simeq \mathbb{S}^1$ or $\mathbb{R}$, a normal geodesic, and polar group a finite or infinite dihedral or cyclic group (the order may depend on the metric).

Basic Theorems

The geometry of polar actions on symmetric spaces were initially studied extensively by Szenthe, Palais, Terng, Heintze, Thorbergsson, Lu, Olmos , Podesta ect. More recently they have been classified (Kollross, Lytchak, Podesta, Thorbergsson). In particular, if $\mathrm{rank}(M) > 1$, then M is hyperpolar, i.e. the section is flat.

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Theorem (Palais-Terng, Chevalley)

If G acts polar on M with section $\sigma : \Sigma \to M$, then the restriction map $C^\infty(M)^G \to C^\infty(\Sigma)^\Pi$ is an isomorphism, i.e. smooth Π invariant functions on Σ extend uniquely to smooth G-invariant functions on M.

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Theorem (Mendes)

Let G act polar on (M, g), then for any Π-invariant metric g' on Σ, there exists a G-invariant metric $\tilde{g}$ on M such that the G action is again polar with section Σ and $\sigma^*(\tilde{g}) = g'$.

Ellipticity

Example

SO(n) acts polar on $\mathbb{S}^n$ with 2 fixed points and section $\mathbb{S}^1$. Hence SO(n) $\times$ SO(n) acts polar on $\mathbb{S}^n \times \mathbb{S}^n$ with 4 fixed points and section $\mathbb{S}^1 \times \mathbb{S}^1 = T^2$ and hence also polar on the k-fold connected sum $\mathbb{S}^n \times \mathbb{S}^n \# \ldots \# \mathbb{S}^n \times \mathbb{S}^n$ with section $T^2 \# \ldots \# T^2$ a surface of genus k. Furthermore, we can assume that the section has a hyperbolic metric if $k > 1$, and in this case M is also rationally hyperbolic (in the sense of rational homotopy theory).

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Theorem (Grove-Z)

If G acts polar on M and $\sec_\Sigma \equiv 0$ or 1, then M is elliptic.

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Goal Reconstruct the manifold (or construct a new manifold) by prescribing the isotropy groups along the section Σ.

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This is always possible for cohomogeneity one manifolds. If $H \subset \{K_-, K_+\} \subset G$ are Lie group inclusions with $K_-/H \simeq \mathbb{S}^{\ell_-}$ and $K_+/H \simeq \mathbb{S}^{\ell_+}$, then the action of $K_\pm$ on $\mathbb{S}^{\ell_\pm} = \partial \mathbb{D}^{\ell_\pm+1}$ extends to $\mathbb{D}^{\ell_\pm+1}$ and we define a new manifold by

$$M = G \times_{K_-} \mathbb{D}^{\ell_-+1} \cup G \times_{K_+} \mathbb{D}^{\ell_++1}$$

glued along the common boundary G/H.

G acts on M on the left in the first coordinate, with $M/G = [-1, +1]$, and isotropy groups H, K_-, K_+.

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We can think of this as prescribing the isotropy groups along Σ, but only for the orbit types in a fundamental domain of the action of the polar group Π on the normal geodesic Σ.

Coxeter polar actions

Let $R \subset \Pi$ be the normal subgroup generated by reflections along hypersurfaces. Let c be a component of the complement of all reflecting hypersurfaces and C its closure, also called a chamber of the action by the reflection group. Furthermore, one has the subgroup $\Pi_C = \{\gamma \in \Pi \mid \gamma(C) = C\}$.

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A word of caution: The polar group of a Coxeter polar action does not have to be a Coxeter group in the abstract sense (but is always a quotient of a Coxeter group).

Group Graph

Assume that the action by G is Coxeter polar and $C \subset \Sigma$ a chamber with C isometric to M/G. The quotient M/G, and hence also C, is stratified by the orbit types (i.e. union of orbits with the same isotropy up to conjugacy), or better the components of the orbit types, which are totally geodesic submanifolds (possibly non-complete).

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Group Graph $G(C)$, or marking of C : Associate to every component of an orbit type F the pair (K, V_K) where K is the isotropy along F and V_K the slice representation. This is a **vertex** of the graph. Two vertices F_1 and F_2 are connected, if $F_2 \subset \bar{F}_1$ and there is no F_3 with $F_2 \subset \bar{F}_3$ and $F_3 \subset \bar{F}_1$. The corresponding groups are **connected by inclusions**: $K_1 \subset K_2$.

Compatability of group graph: For each vertex (K, V_K), the slice representation V_K is Coxeter polar with group graph the history of K, i.e. all groups eventually ending up in K. The tangent cone of C at q is a chamber of the slice representation whose marking is induced by the marking on C.

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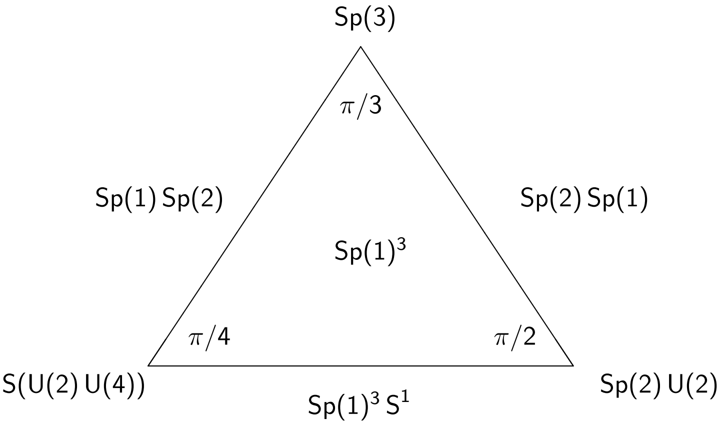
Theorem (Grove-Z)

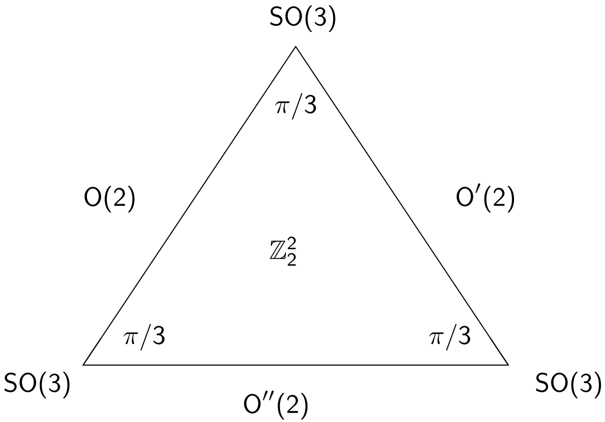
Any set of smooth compatible Coxeter polar data $D = (C, \mathsf{G}(C))$ determines a Coxeter polar G manifold $M(D)$ with orbit space C. Moreover, if M is a Coxeter polar manifold with data D then $M(D)$ is equivariantly diffeomorphic to M.

Cohomogeneity two action of SU(6) on $\mathbb{CP}^{14}$:



Examples

Cohomogeneity two action of SO(3) on SU(3)/ SO(3):



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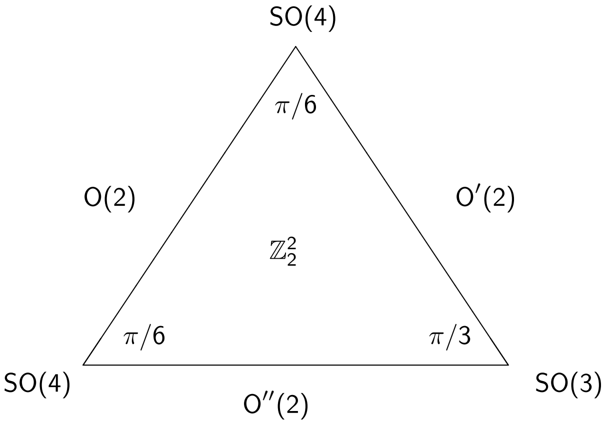
Cohomogeneity two action of $SO(4)$ on M^8:



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Theorem (Fang-Grove-Thorbergsson)

A compact polar manifold with an invariant metric of positive sectional curvature is equivariantly diffeomorphic to a rank one symmetric space.