Fourier inversion and Paley–Wiener theorems for rank one Riemannian symmetric superspaces

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joint work with Wolfgang Palzer

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Plan of the talk

- Basic super stuff
- Spherical superfunctions
- Leading asymptotics
- Asymptotic expansion
- Plancherel theorem for OSp
- Paley–Wiener theorem

Supermanifolds

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X complex supermanifold $:\Leftrightarrow$ $X \cong_{loc} \mathbb{A}^{p|q}_{hol} := (\mathbb{C}^p, \mathcal{H}_{\mathbb{C}^p} \otimes_{\mathbb{C}} \bigwedge(\mathbb{C}^q)^*)$

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For instance, $V = TX_0$, $V = S$ spinor bundle (X_0 spin^c), ...

Theorem (Batchelor). All *cs* manifolds are obtained in this way (i.e. are split).

But: Complex smf $\mathrm{Gr}(1|1, 2|2)$ is not split (Penkov, Wells et al.).
Moreover: Maps are not the same.

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$$\{\text{complex/}cs\text{ Lie supergroups}\} \rightleftharpoons \left\{ \begin{array}{c} \text{(Harish-Chandra) pairs } (\mathfrak{g}, G_0), \\ \mathfrak{g} \text{ Lie superalgebra, } G_0 \text{ Lie group} \end{array} \right\}$$

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$$\mathrm{GL}(p|q, \mathbb{C})(T) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{O}(T)^{p \times q} \;\middle|\; \begin{array}{c} A, D \text{ even, } B, C \text{ odd} \\ A, D \text{ invertible} \end{array} \right\}$$

$$\mathrm{OSp}(p|2q, J, \mathbb{C})(T) = \left\{ g \in \mathrm{GL}(p|2q, \mathbb{C}) \;\middle|\; g^{st^3} J g = J \right\}$$

Here, we let:

J	matrix of supersymmetric form
$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st^3} := \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}$	order 4 automorphism

Cs supergroups from complex supergroups

$G_{\mathbb{C}}$	complex Lie supergroup with pair $(\mathfrak{g}, G_{\mathbb{C},0})$
G_0	real form of $G_{\mathbb{C},0}$

$\rightsquigarrow$ cs Lie supergroup G with pair $(\mathfrak{g}, G_0)$

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$$\mathrm{U}_{cs}(m,n|r,s)(T) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{O}(T)^{(m+n)\times(r+s)} \;\middle|\; \begin{matrix} A \in \mathrm{U}(m,n)(T) \\ D \in \mathrm{U}(r,s)(T) \end{matrix} \right\}$$

$$\mathrm{SOSp}^{+}_{cs}(m,n|2q)(T) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{OSp}(m+n|2q,J,\mathbb{C})(T) \;\middle|\; \begin{matrix} A \in \mathrm{SO}^{+}(m,n)(T) \\ D \in \mathrm{USp}(2q)(T) \end{matrix} \right\}$$

Here, we let:

$$J = \begin{pmatrix} -\mathbb{1}_m & 0 & 0 \\ 0 & \mathbb{1}_n & 0 \\ 0 & 0 & J_q \end{pmatrix}, \quad J_q := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \cdots \otimes \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

$$\mathrm{USp}(2q) := \mathrm{U}(2q) \cap \mathrm{Sp}(2q,\mathbb{C})$$

Riemannian symmetric superspaces

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Examples: Today, we will consider the following “rank one” cases:

G	K
$\mathrm{U}_{cs}(1, 1+p\|q)$	$\mathrm{U}(1) \times \mathrm{U}_{cs}(1+p\|q)$
$\mathrm{SOSp}^+_{cs}(1, 1+p\|2q)$	$\mathrm{SOSp}_{cs}(1+p\|2q)$
$\mathrm{GL}_{cs}(1\|1)$	$\mathrm{U}_{cs}(1\|1)$

Even these are quite surprising.

Symmetric superfunctions

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$\mathfrak{a}$	Cartan subspace
$M = Z_K(\mathfrak{a})$	centraliser of Cartan
K/M	geodesic supersphere at infinity
$\lvert D\dot{k}\rvert$	K-invariant Berezinian density on K/M
$H : G \to \mathbb{A}(\mathfrak{a}_{\mathbb{R}})$	Iwasawa A projection
$\varrho = \frac{1}{2}\,\mathrm{str}_{\mathfrak{n}}\,\mathrm{ad}_{\mathfrak{g}}\lvert_{\mathfrak{a}}$	half sum of positive roots

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Definition.

$$\phi_\lambda(g) := \int_{K/M} |D\dot{k}|\, e^{(\lambda-\varrho)(H(gk))}, \quad \lambda \in \mathfrak{a}^*.$$

These are eigenfunctions of the Laplacian.

Leading asympotics: c-function

Fix basis $h_0 \in \mathfrak{a}$, $\alpha(h_0) = 1$, α indecomposable positive root, identify $\lambda \equiv \lambda(h_0)$.

$$c(\lambda) := \lim_{t \to \infty} e^{-(\lambda-\varrho)t} \phi_\lambda(e^{th_0}).$$

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Theorem (A–Palzer). In each of the cases listed above, $c(\lambda)$ exists for $\Re\lambda > 0$, and

$$c(\lambda) = c_0 \frac{2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}(\lambda + \frac{m_\alpha}{2} + 1)\right)\Gamma\left(\frac{1}{2}(\lambda + \frac{m_\alpha}{2} + m_{2\alpha})\right)} \qquad (c_0 \equiv c_0(\varrho))$$

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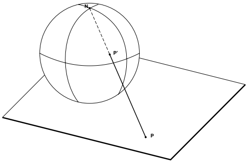
Theorem (A–Schmittner). For G/K reductive of even type, $c(\lambda)$ exists for $\Re\lambda > 0$, and

$$c(\lambda) = c_0 \prod_{\langle\alpha,\alpha\rangle\neq 0} 2^{-\lambda_\alpha}\frac{\Gamma(\lambda_\alpha)}{\Gamma(\frac{1}{2}(\lambda_\alpha+\frac{m_\alpha}{2}+1))\Gamma(\frac{1}{2}(\lambda_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))} \prod_{\langle\alpha,\alpha\rangle=0}\langle\lambda,\alpha\rangle^{-\frac{m_\alpha}{2}}$$

the product over indecomposable roots, where $\lambda_\alpha := \langle\lambda,\alpha\rangle\langle\alpha,\alpha\rangle^{-1}$ for $\langle\alpha,\alpha\rangle \neq 0$.

Comments on the proof

One uses stereographic coordinates $k : \bar{N} \to K/M$



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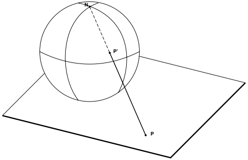

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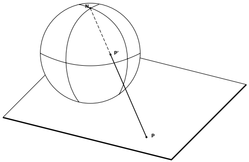

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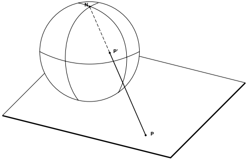

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4. A similar statement holds for SOSp^+.
5. For $\Re\lambda > -\varrho$, derivatives and integral can be exchanged.
6. This gives the assertion.

Asymptotic expansion

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In the other cases, we have an asymptotic expansion

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Corollary (A–Palzer). For $m_\alpha \leq 0$ even, the series terminates and for OSp,

$$\phi_\lambda(e^t) \simeq e^{(\lambda-\varrho)t} P_{-\varrho}^{(-\lambda, 2\varrho-1)}(1-2e^{-2t}),$$

where $P_n^{(\alpha,\beta)}$ are Jacobi polynomials.

Wave packet transform

In what follows, assume that $G = \mathrm{SOSp}^+_{cs}(1, 1 + p|2q)$, $p > 0$.

Definition. The Paley–Wiener space is

$$\mathrm{PW}_R := \left\{ \varphi \in \mathrm{Hol}(\mathfrak{a}^*) \;\middle|\; \begin{array}{c} \varphi(\lambda) = \varphi(-\lambda) \\ \forall k \geqslant 0 : \|\varphi\|_{k,R} := \sup_{\lambda\in\mathfrak{a}^*} (1 + |\lambda|)^k |\varphi(\lambda)| e^{-R|\Re\lambda|} < \infty \end{array} \right\}$$

For $\varphi \in \mathrm{PW}_R$, let the *wave packet transform* be

$$\mathcal{J}\varphi(g) := \int_{i\mathfrak{a}^*_{\mathbb{R}}} \frac{d\lambda}{|c(\lambda)|^2} \phi_\lambda(g)\varphi(\lambda).$$

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Proposition (A–Palzer). Let$\|\varphi\|_{n,R} < \infty$ for some $n > \varrho$, $R \geq 0$. Then

$$\mathcal{J}\varphi(e^t) = 4\pi \sum_{k<-\varrho} \mathrm{res}_{\lambda=\varrho+k} \frac{\Phi_\lambda(e^t)}{c(\lambda)c(-\lambda)} \varphi(\varrho+k), \quad t > R$$

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Corollary (A–Palzer). We have

$$\tilde{\mathcal{J}}(\mathrm{PW}_R) \subseteq \mathcal{O}_R(G/K) := \{ f \in \mathcal{O}(G/K) \mid \mathrm{supp}\, f \subseteq B_R(o) \}$$

$$\tilde{\mathcal{J}}\varphi(g) := \mathcal{J}\varphi(g) - 4\pi \sum_{k < -\varrho} \mathrm{res}_{\lambda = \varrho + k} \frac{\Phi_\lambda(g)}{c(\lambda)c(-\lambda)} \varphi(\varrho + k)$$

Fourier inversion and Paley–Wiener theorem

Proposition (A–Palzer). If $\varrho < 0$, then $\mathcal{J}1$ exists and

$$\mathcal{J}1(e^t) = -\frac{2\pi}{c_0\Gamma(1-\varrho)\Gamma(-2\varrho)}\partial_{y=0}^{-2\varrho-1}\frac{(1-2y^2\cosh t+y^4)^{-\varrho}}{(1-y)^2}.$$

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$$\mathcal{J}1(e^t) = -\frac{2\pi}{c_0\Gamma(1-\varrho)\Gamma(-2\varrho)}\partial_{y=0}^{-2\varrho-1}\frac{(1-2y^2\cosh t+y^4)^{-\varrho}}{(1-y)^2}.$$

Define

$$\mathcal{F}f(\lambda,k) := \int_{G/K}|D\dot{g}|f(g)e^{(\lambda-\varrho)(H(g^{-1}k))}$$

$$\mathcal{J}\varphi(g) := \int_{i\mathfrak{a}_{\mathbb{R}}^*}\frac{d\lambda}{|c(\lambda)|^2}\int_{K/M}|D\dot{k}|\varphi(\lambda,k)e^{(-\lambda-\varrho)(H(g^{-1}k))}$$

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Theorem (A–Palzer). Let $\mathcal{J}1 := 0$ for $\varrho \geqslant 0$. Then for any $f \in \mathcal{O}_c(G/K)$

$$\mathcal{J}\mathcal{F}f = C_0 f + (f * \mathcal{J}1)$$

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Corollary (A–Palzer). Under the above assumptions

$$\mathcal{F}(\mathcal{O}_R(G/K)^K) = \mathrm{PW}_R.$$

Condensed matter physics application

For a long, thin wire with impurities, at low temperatures, the mean conductance $\langle c \rangle$ is

$$\langle c \rangle = \int_{G/K} |D\dot{g}| \, |f_t(g)|^2|_{t=s/2}, \quad \partial_t f_t = \Delta f_t.$$

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In cases of higher rank, one obtains for $s \gg 1$ (Zirnbauer, PRL 1992)

$$\langle c \rangle \simeq \begin{cases} 2^{-4}\pi^{7/2} s^{-3/2} e^{-s}, & \text{orthogonal} \\ 2^{1/2}\pi^{3/2} s^{-3/2} e^{-s/2} & \text{unitary} \\ 1/2 + 2^5 3^{-2} \pi^{3/2} s^{-3/2} e^{-s/4} & \text{symplectic} \end{cases}$$

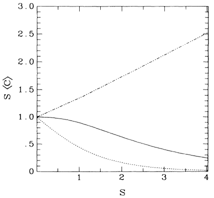

FIG. 1. The product $s/\langle c \rangle$ as a function of $s = L/\xi + 1/\gamma$ for the case of orthogonal symmetry (dotted line), unitary symmetry (solid line), and symplectic symmetry (dash-dotted line).

Illustration from Zirnbauer, PRL 69, no. 10 (1992)

Thank you for your attention.