Homogeneous Einstein metrics on Stiefel manifolds and compact Lie groups.

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(Joint work with Y. Sakane (Osaka) and M. Statha (Patras))

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Introduction

• (M, g) Einstein $\mathrm{Ric}(g) = \lambda g, \ \ \lambda \in \mathbb{R}$.
We study G-invariant Einstein metrics on a homogeneous space G/K.

• General Problem: Find G-invariant Einstein metrics on a homogeneous space G/K and classify them if they are not unique.
Homogeneous Einstein spaces can be divided into three major classes depending on the Einstein constant λ.

Here we consider the case $c > 0$.

Introduction

Examples for the case $\lambda > 0$
(G/K is compact and $\pi_1(G/K)$ is finite).

- Sphere $(S^n = \mathrm{SO}(n+1)/\,\mathrm{SO}(n), g_0)$,
  Complex Projective space $(\mathbb{C}P^n = \mathrm{SU}(n+1)/(S(U(1) \times U(n)))$,
  Symmetric spaces of compact type
  Isotropy irreducible spaces (in these cases G-invariant Einstein metrics are unique)
- Generalized flag manifolds (Kähler C-spaces)
  They admit a finite number of Kähler-Einstein metrics, corresponding to each complex structure.

Introduction

The study of non-Kähler Einstein metrics on generalized flag manifolds was initiated in the 1990's by works of M. Wang, W. Ziller, D. Alekseevsky, M. Kimura, A. Arvanitoyeorgos and continues up to date with works of Y. Sakane, E. Dos-Santos, C. Negreiros, A. Arvanitoyeorgos, I. Chrysikos, W. Wang, G. Zhao. Note that the isotropy representation for these spaces decomposes into a sum of irreducible and non equivalent summands.

When the number of the isotropy summands increases (≥ 4), the construction of the Einstein equation is getting demanding, as well as the procedure towards solving the Einstein equation. The Einstein equation reduces to algebraic systems of equations, which often have coefficients that depend on the rank of classical Lie groups.

To this end, submersion techniques and Gröbner bases methods have been recently (2011, 2012, 2013) used by A. Arvanitoyeorgos, I. Chrysikos and Y. Sakane.

Einstein metrics on Stiefel manifolds - some history

We are interested in homogeneous Einstein metrics on the Stiefel manifolds the $V_k\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-k)$ of orthonormal k-frames in $\mathbb{R}^n$.

- In 1963, S. Kobayashi proved existence of a homogeneous Einstein metric on the unit tangent bundle $T_1S^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2) = V_2\mathbb{R}^n$ (S^1-bundle over a Kähler manifold $\mathrm{SO}(n)/(\mathrm{SO}(n-2)\times \mathrm{SO}(2))$).
- In 1970, A. Sagle proved that the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-k)$ for $k \geq 3$ admit a homogeneous Einstein metric.
- In 1973, G. Jensen proved that the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-k)$ for $k \geq 3$ admit at least two homogeneous Einstein metrics.
  Note that for $n = 3$ $SO(3)$ admits a unique Einstein metric.

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Einstein metrics on Stiefel manifolds - some history

- In 1987 A. Back and W.Y. Hsiang proved that for $n \geq 5$, $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ admits exactly one homogeneous Einstein metric.

  The same result was obtained in 1998 by M. Kerr (the diagonal metrics are the only homogeneous Einstein metrics).

  But notice that

  by a result of D. V. Alekseevsky, I. Dotti, C. Ferraris (1996) $\mathrm{SO}(4)/\mathrm{SO}(2)$ admits exactly two invariant Einstein metrics (one is a diagonal metric and the other is non diagonal metric). Jensen's metric is a diagonal metric. Note that $\mathrm{SO}(4)/\mathrm{SO}(2)$ is diffeomorphic to $S^3 \times S^2$. The non-diagonal Einstein metric comes from the product metric on $S^3 \times S^2$.

Einstein metrics on Stiefel manifolds - some history

- In 2009 A.A., V.V. Dzhepko and Yu. G. Nikonorov proved that for $s > 1$ and $\ell > k \geq 3$, the Stiefel manifolds $\mathrm{SO}(sk+\ell)/\mathrm{SO}(\ell)$ admit at least four $\mathrm{SO}(sk+\ell)$-invariant Einstein metrics that are also $\mathrm{Ad}(\mathrm{SO}(k)^s \times \mathrm{SO}(\ell))$-invariant, two of which are Jensen's metrics.
  In 2007, they have also treated $\mathrm{SO}(k+k+\ell)/\mathrm{SO}(\ell)$ ($\ell > k \geq 3$) as a special case of above and proved that the Stiefel manifold $\mathrm{SO}(k+k+\ell)/\mathrm{SO}(\ell)$ admit at least four $\mathrm{SO}(k+k+\ell)$-invariant Einstein metrics that are also $\mathrm{Ad}(\mathrm{SO}(k) \times \mathrm{SO}(k) \times \mathrm{SO}(\ell))$-invariant.
- Open problem : How many homogeneous Einstein metrics are there on the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$?

Einstein metrics on compact Lie groups - some history

- A compact semisimple Lie group with a bi-invariant metric is Einstein.
- Concerning left-invariant Einstein metrics on compact simple Lie groups, J.E. D'Atri and W. Ziller in 1976 classified all naturally reductive metrics and found a large number of Einstein metrics on compact simple Lie groups which are naturally reductive.
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  Are there non-naturally reductive Einstein metrics on the compact simple Lie groups ?

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For this problem:

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- In 2012 A. A., K. Mori and Y. Sakane have shown the existence of non naturally reductive Einstein metrics on the compact simple Lie groups $\mathrm{SO}(n)$ $(n \geq 11)$, $\mathrm{Sp}(n)$ $(n \geq 3)$, E_6, E_7 and E_8, by using generalized flag manifolds.
- In 2013 (arXiv) Z. Chen and K. Liang proved existence on non naturally reductive Einstein metrics on the compact simple Lie group F_4 .
- Note that in the recent works (2011, 2012) of G.W. Gibbons, H. Lü, A. H. Mujtaba and C. Pope the Einstein metrics found on $SU(n)$, $SO(n)$ and G_2, are naturally reductive.
  The problem is particularly difficult. For example, for the compact Lie groups $SU(3)$ and $SU(2) \times SU(2)$ the number of left-invariant Einstein metrics is unknown.

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Main results

Our main results are the following:

Theorem 1

There exist new homogeneous Einstein metrics on the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ *for* $k \geq 4$ *and* $n \geq 6$.

(Different from the ones found by Jensen and Arvanitoyeorgos, Dzhepko, Nikonorov.)

Theorem 2

There exist non-naturally reductive Einstein metrics on the compact Lie groups $\mathrm{SO}(n)$ $(n \geq 7)$.

Ricci tensor of a compact homogeneous space G/K

- Let G be a compact semisimple Lie group and K be a connected closed subgroup of G.
  Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B ($= -$ Killing form of $\mathfrak{g}$). Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $[\,\mathfrak{k}, \mathfrak{m}\,] \subset \mathfrak{m}$ and a decomposition of $\mathfrak{m}$ into irreducible Ad(K)-modules:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q.$$

- We assume that the $\mathrm{Ad}(K)$-modules $\mathfrak{m}_j$ $(j = 1, \ldots, q)$ are mutually non equivalent.
  Then a G-invariant metric on G/K corresponds to an $\mathrm{Ad}(K)$-invariant inner product

$$\langle\ ,\ \rangle = x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}, \tag{1}$$

  for positive real numbers $x_1, \ldots, x_q$.

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- Note that G-invariant symmetric covariant 2-tensors on G/K are the same form as the metrics.
  In particular, the Ricci tensor r of a G-invariant Riemannian metric on G/K is of the same form as (1).
- Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$).
- We put $A^\gamma_{\alpha\beta} = B\left([e_\alpha, e_\beta], e_\gamma\right)$, so that $[e_\alpha, e_\beta] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set
  $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$.
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Ricci tensor of a compact homogeneous space G/K

- Then, the non-negative number $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the B-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}. \qquad (2)$$

- Let $d_k = \dim \mathfrak{m}_k$. Then we have (Park - Sakane 1997):

Proposition

The components $r_1, \ldots, r_q$ of Ricci tensor r of the metric $< \, , \, >= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}$ on G/K are given by

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1, \ldots, q) \qquad (3)$$

where the sum is taken over $i, j = 1, \ldots, q$.

The isotropy representation contains equivalent summands

When the isotopy representation of a homogeneous space $M = G/K$ contains some equivalent summands, then the description of G-invariant metrics is more difficult.
A typical example is the 5-dimensional Stiefel manifold $V_2\mathbb{R}^4 = \mathrm{SO}(4)/SO(2)$ where the isotropy representation χ is given as

$$\chi = 1 \oplus \lambda_2 \oplus \lambda_2,$$

where $\lambda_2 : \mathrm{SO}(2) \to \mathrm{Aut}(\mathbb{R}^2)$ is the standard representation of $\mathrm{SO}(2)$.
The tangent space decomposes as

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2, \qquad \mathfrak{m}_1 \cong \mathfrak{m}_2.$$

With respect to an adapted basis the invariant metrics are described as

$$\begin{pmatrix} x_0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & \lambda & 0 \\ 0 & 0 & x_1 & 0 & \lambda \\ 0 & \lambda & 0 & x_2 & 0 \\ 0 & 0 & \lambda & 0 & x_2 \end{pmatrix}.$$

The isotropy representation contains equivalent summands $V_k\mathbb{R}^n$

- For the tangent space $\mathfrak{p}$ of the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-4)$

$$\mathfrak{p} = \left(\begin{array}{cccc|c} 0 & a_{12} & a_{13} & a_{14} & A_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & A_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & A_{35} \\ -a_{14} & -a_{24} & a_{34} & 0 & A_{45} \\ \hline -{}^tA_{15} & -{}^tA_{25} & -{}^tA_{35} & -{}^tA_{45} & * \end{array}\right),$$

  we have a decomposition $\mathfrak{p} = \sum_{i<j} \mathfrak{p}_{ij} \oplus \sum_{k<5} \mathfrak{p}_{k5}$ as $\mathrm{Ad}(\mathrm{SO}(n-4))$-submodules, where $\dim \mathfrak{p}_{ij} = 1$ and $\dim \mathfrak{p}_{k5} = n-4$. Note that submodules $\mathfrak{p}_{ij}$ are equivalent each other and submodules $\mathfrak{p}_{k5}$ are equivalent each other.
- For general invariant metrics on the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$, it would be difficult to find homogeneous Einstein metrics.

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The isotropy representation contains equivalent summands

When the isotropy representation of G/K contains some equivalent summands, we will try to search for G-invariant Einstein metrics among more special subsets of invariant metrics.
We will then choose special decompositions of tangent space into irreducible non-equivalent summands, so that we are able to apply the Park-Sakane formula for the Ricci tensor.
Invariant Einstein metrics for homogeneous spaces containing equivalent summands have been studied before by Yu. Nikonorov (2007) and M. Kerr (1998). The seed of the present description lies in previous work by A.A. - V.V. Dzhepko - Yu. Nikonorov (2008, 2009).

The isotropy representation contains equivalent summands

Let G/H be a homogeneous space of a compact Lie groups G and H be a closed subgroup of G.Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$ with respect to some $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, where $\mathfrak{m} \cong T_o(G/H)$. A metric g on G/H is called G-invariant if the diffeomorphism $\tau_g : G/H \to G/H$, $\tau_g(p) = gp$ is a isometry. The following proposition gives a description of G-invariant metrics on a homogeneous spaces.

Proposition

Then there exists a one-to-one correspondence between:

1. G-invariant metrics g on G/H
2. $\mathrm{Ad}^{G/H}$-invariant inner products $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$, that is

$$\langle \mathrm{Ad}^{G/H}(h)X, \mathrm{Ad}^{G/H}(h)Y \rangle = \langle X, Y \rangle, \quad X, Y \in \mathfrak{m}, h \in H$$

3. $\mathrm{Ad}^{G/H}$-equivariant, B-symmetric and positive definite operators $A : \mathfrak{m} \to \mathfrak{m}$ such that

$$\langle X, Y \rangle = B(A(X), Y).$$

We say that the inner product is $\mathrm{Ad}^G(H)$-invariant or simply $\mathrm{Ad}(H)$-invariant

The isotropy representation contains equivalent summands

Thus all $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ can be parametrized by $\mathrm{Ad}(H)$-equivariant, symmetric and positive definite operators $A : \mathfrak{m} \to \mathfrak{m}$. So we have

$$\mathcal{M}^G \leftrightarrow \{A : \mathfrak{m} \to \mathfrak{m} \mid \mathrm{Ad}(H)\text{-equivariant, symmetric and positive definite operators}\}$$

If $\mathfrak{m}$ decomposes into a direct sum of $\mathrm{Ad}(H)$-invariant irreducible non equivalent modules $\mathfrak{m}_i$ $(i = 1, \dots, s)$ that is

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s \tag{1}$$

(i.e. the isotropy representation $\mathrm{Ad}^{G/H} : H \to \mathrm{Aut}(\mathfrak{m})$ of G/H is a direct sum of irreducible **non-equivalent** sub representations $\mathrm{Ad}^{G/H} = \chi_1 \oplus \cdots \oplus \chi_s$), then Schur's lemma implies that all $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ are given by

$$\langle \cdot, \cdot \rangle = x_1(-B)|_{\mathfrak{m}_1} + \cdots + x_s(-B)|_{\mathfrak{m}_s}, \ x_i \in \mathbb{R}^+, \ i = 1, \dots, s.$$

In this case the matrix of the operator A with respect to some $(-B)$-orthonormal basis of $\mathfrak{m}$ is diagonal.

Algebraic description of certain invariant metrics

We consider the normalizer $N_G(H)$ of H in G and the let

$$\Phi = \{\phi = \mathrm{Ad}(n)|_{\mathfrak{m}} : n \in N_G(H)\}.$$

Lemma

It is $\Phi \subset \mathrm{Aut}(\mathfrak{m})$.

Therefore, the action

$$\begin{array}{rcl} \Phi \times \mathcal{M}^G & \longrightarrow & \mathcal{M}^G \\ (\phi\,,\,A) & \longmapsto & \phi \circ A \circ \phi^{-1} \equiv \tilde{A}. \end{array}$$

is well defined (i.e. the operator $\tilde{A}$ is $\mathrm{Ad}(H)$-equivariant, B-symmetric and positive definite).

Algebraic description of certain invariant metrics

The set $(\mathcal{M}^G)^\Phi$ of fixed points of the above action is given by

$$(\mathcal{M}^G)^\Phi = \{A \in \mathcal{M}^G : \phi \circ A \circ \phi^{-1} = A \text{ for all } \phi \in \Phi\},$$

which is the set of all $\mathrm{Ad}(N_G(H))$-invariant inner products on $\mathfrak{m}$.

Observe that, since $H \subset N_G(H)$ then the set of fixed points of the action of

$$\Phi_H = \{\phi = \mathrm{Ad}(h)|_{\mathfrak{m}} : h \in H\} \subset \Phi$$

on $\mathcal{M}^G$ is the set of all $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$.

Notice as well the special case when $H = \{e\}$, so $G/\{e\} \cong G$. Then $N_G(H) = G$, hence the fixed points are the $\mathrm{Ad}(G)$-invariant inner products on $\mathfrak{g}$, i.e. bi-invariant metrics on the Lie group G.

Algebraic description of certain invariant metrics

Now let K be a closed subgroup of G such that $H \subset K \subset N_G(H) \subset G$. Then the set of fixed points of the action of

$$\Phi_K = \{\phi = \mathrm{Ad}(k)|_{\mathfrak{m}} : k \in K\} \subset \Phi$$

on $\mathcal{M}^G$ (this action is non trivial) is the following subset of $\mathcal{M}^G$:

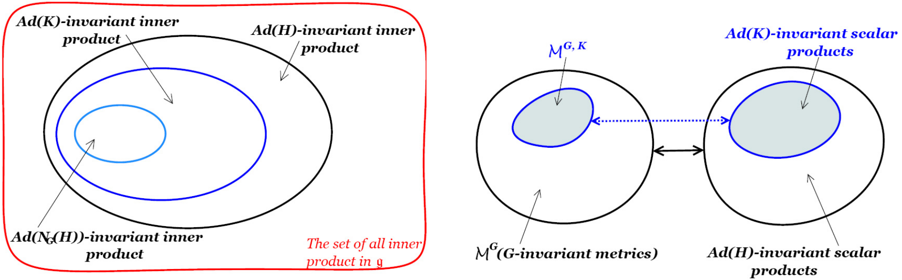
$$(\mathcal{M}^G)^{\Phi_K} = \{A \in \mathcal{M}^G \,:\, \phi \circ A \circ \phi^{-1} = A \text{ for all } \phi \in \Phi_K\},$$

i.e. the set of all $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$.
Such $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$ will determine a subset $\mathcal{M}^{G,K}$ of G-invariant metrics and we call these metrics
$\underline{\mathrm{Ad}(K)\text{-invariant metrics on the homogeneous space } G/H}$.

Each of the above subsets is a special subset of the space of all G-invariant metrics.

The above subsets of $\mathcal{M}^G$ (as special inner products in $\mathfrak{g}$) can be depicted in the following diagram:

Algebraic description of certain invariant metrics



Algebraic description of certain invariant metrics

Since the complete knowledge of $N_G(H)$ is not often possible, our aim is to choose appropriate subsets K with $H \subset K \subset N_G(H)$.

Application

Let $M = G/H$ be a homogeneous space. Choose K to be a closed subgroup of the compact Lie group G with $H \subset K \subset G$ and such that $K = L \times H$, where L is a subgroup of G. It can be shown that $K \subset N_G(H)$.

We will apply this to the Stiefel manifolds $G/H = \mathrm{SO}(k_1 + k_2 + k_3)/\,\mathrm{SO}(k_3)$ where $K = L \times H = (\mathrm{SO}(k_1) \times \mathrm{SO}(k_2)) \ \times \ SO(k_3)$ and consider $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times SO(k_3)$-invariant metrics.

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\,\mathrm{SO}(k_3)$

We first consider the homogeneous space
$G/K = \mathrm{SO}(k_1+k_2+k_3)/(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$, where the embedding of K in G is diagonal.
The isotropy representation of G/K has the form

$$\chi \cong \chi_{12} \oplus \chi_{13} \oplus \chi_{23},$$

and the tangent space $\mathfrak{m}$ of G/K decomposes into three $\mathrm{Ad}(K)$-submodules

$$\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}.$$

In fact, $\mathfrak{m}$ is given by $\mathfrak{k}^{\perp}$ in $\mathfrak{g} = \mathfrak{so}(k_1+k_2+k_3)$ with respect to $-B$. If we denote by $M(p,q)$ the set of all $p\times q$ matrices, then we see that $\mathfrak{m}$ is given by

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\,\mathrm{SO}(k_3)$

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & A_{12} & A_{13} \\ -{}^tA_{12} & 0 & A_{23} \\ -{}^tA_{13} & -{}^tA_{23} & 0 \end{pmatrix} \;\middle|\; A_{12} \in M(k_1,k_2), A_{13} \in M(k_1,k_3), A_{23} \in M(k_2,k_3) \right\}$$

and we have

$$\mathfrak{m}_{12} = \begin{pmatrix} 0 & A_{12} & 0 \\ -{}^tA_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{m}_{13} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ -{}^tA_{13} & 0 & 0 \end{pmatrix}, \quad \mathfrak{m}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & -{}^tA_{23} & 0 \end{pmatrix}.$$

Note that the action of $\mathrm{Ad}(k)$ $(k \in K)$ on $\mathfrak{m}$ is given by

$$\mathrm{Ad}(k) \begin{pmatrix} 0 & A_{12} & A_{13} \\ -{}^tA_{12} & 0 & A_{23} \\ -{}^tA_{13} & -{}^tA_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^th_1 A_{12} h_2 & {}^th_1 A_{13} h_3 \\ -{}^th_2 {}^tA_{12} h_1 & 0 & {}^th_2 A_{23} h_3 \\ -{}^th_3 {}^tA_{13} h_1 & -{}^th_3 {}^tA_{23} h_2 & 0 \end{pmatrix},$$

where $\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \in K$. Thus the irreducible submodules $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$ and $\mathfrak{m}_{23}$ are mutually non equivalent.

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\,\mathrm{SO}(k_3)$

For the tangent space $\mathfrak{p}$ of the Stiefel manifold $G/H = \mathrm{SO}(k_1+k_2+k_3)/\,\mathrm{SO}(k_3)$, we consider the decomposition

$$\mathfrak{p} = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23} \tag{4}$$

where the corresponding $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-submodules are non equivalent .

We consider G-invariant metrics on the Stiefel manifold G/H determined by the $\mathrm{Ad}(\mathrm{SO}(k_1)\times\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-invariant scalar products on $\mathfrak{p}$ given by

$$\begin{aligned} \langle\ ,\ \rangle &= x_1\,(-B)|_{\mathfrak{so}(k_1)} + x_2\,(-B)|_{\mathfrak{so}(k_2)} \\ &\quad + x_{12}\,(-B)|_{\mathfrak{m}_{12}} + x_{13}\,(-B)|_{\mathfrak{m}_{13}} + x_{23}\,(-B)|_{\mathfrak{m}_{23}} \end{aligned} \tag{5}$$

for $k_1 \geq 2$, $k_2 \geq 2$ and $k_3 \geq 1$.

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\,\mathrm{SO}(k_3)$

We set in the decomposition (4) $\mathfrak{so}(k_1) = \mathfrak{m}_1, \mathfrak{so}(k_2) = \mathfrak{m}_2$. Then it is possible to show that the following relations hold:

$$
\begin{array}{lll}
[\mathfrak{m}_1, \mathfrak{m}_1] = \mathfrak{m}_1, & [\mathfrak{m}_2, \mathfrak{m}_2] = \mathfrak{m}_2, & [\mathfrak{m}_1, \mathfrak{m}_{12}] = \mathfrak{m}_{12}, \\
[\mathfrak{m}_1, \mathfrak{m}_{13}] = \mathfrak{m}_{13}, & [\mathfrak{m}_2, \mathfrak{m}_{12}] = \mathfrak{m}_{12}, & [\mathfrak{m}_2, \mathfrak{m}_{23}] = \mathfrak{m}_{23}, \\
[\mathfrak{m}_{12}, \mathfrak{m}_{12}] \subset \mathfrak{m}_1 \oplus \mathfrak{m}_2, & [\mathfrak{m}_{13}, \mathfrak{m}_{13}] \subset \mathfrak{m}_1 \oplus \mathfrak{h}, & [\mathfrak{m}_{23}, \mathfrak{m}_{23}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \\
[\mathfrak{m}_{12}, \mathfrak{m}_{23}] \subset \mathfrak{m}_{13}, & [\mathfrak{m}_{13}, \mathfrak{m}_{23}] \subset \mathfrak{m}_{12}, & [\mathfrak{m}_{12}, \mathfrak{m}_{13}] \subset \mathfrak{m}_{23}.
\end{array}
$$

Thus we see that the only non zero triplets (up to permutation of indices) are

$$
\begin{bmatrix} 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 22 \end{bmatrix}, \begin{bmatrix} (12) \\ 1(12) \end{bmatrix}, \begin{bmatrix} (13) \\ 1(13) \end{bmatrix}, \begin{bmatrix} (12) \\ 2(12) \end{bmatrix}, \begin{bmatrix} (23) \\ 2(23) \end{bmatrix}, \begin{bmatrix} (13) \\ (12)(23) \end{bmatrix},
$$

where $\begin{bmatrix} i \\ ii \end{bmatrix}$ is non zero only for $k_1, k_2 \geq 3$.

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\,\mathrm{SO}(k_3)$

The components of the Ricci tensor r for the left-invariant metric $\langle\ ,\ \rangle$ on G/H defined by *(5), are given as follows:*

Lemma

$$
\begin{aligned}
r_1 &= \frac{k_1-2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2\frac{x_1}{{x_{12}}^2} + k_3\frac{x_1}{{x_{13}}^2}\right),\\
r_2 &= \frac{k_2-2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1\frac{x_2}{{x_{12}}^2} + k_3\frac{x_2}{{x_{23}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right)\\
&\quad -\frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{12}}^2} + (k_2-1)\frac{x_2}{{x_{12}}^2}\right),
\end{aligned}
$$

The Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^{k_1+k_2+k_3} = \mathrm{SO}(k_1+k_2+k_3)/\mathrm{SO}(k_3)$

Lemma

$$\begin{aligned} r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\ &\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{13}}^2}\right) \\ r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) \\ &\quad - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2}\right), \end{aligned}$$

where $n = k_1 + k_2 + k_3$.

The Stiefel manifolds $V_{1+k_2}\mathbb{R}^{1+k_2+k_3} = \mathrm{SO}(1+k_2+k_3)/\mathrm{SO}(k_3)$

For $k_1 = 1$ and $k_2 \geq 2$, we have the Stiefel manifold $G/H = \mathrm{SO}(1+k_2+k_3)/\mathrm{SO}(k_3)$ with corresponding decomposition

$$\mathfrak{p} = \mathfrak{so}(k_2) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}.$$

We consider G-invariant metrics on G/H determined by the $\mathrm{Ad}(\mathrm{SO}(k_2)\times\mathrm{SO}(k_3))$-invariant scalar products on $\mathfrak{p}$ given by

$$\langle\ ,\ \rangle = x_2\,(-B)|_{\mathfrak{so}(k_2)} + x_{12}\,(-B)|_{\mathfrak{m}_{12}} + x_{13}\,(-B)|_{\mathfrak{m}_{13}} + x_{23}\,(-B)|_{\mathfrak{m}_{23}}. \quad (6)$$

For simplicity we use the notation

$$\langle\ ,\ \rangle = \begin{pmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_2 & x_{23} \\ x_{13} & x_{23} & * \end{pmatrix}.$$

The Stiefel manifolds $V_{1+k_2}\mathbb{R}^{1+k_2+k_3} = \mathrm{SO}(1+k_2+k_3)/\mathrm{SO}(k_3)$

Lemma

The components of the Ricci tensor r for the left-invariant metric $\langle\ ,\ \rangle$ on G/H defined by *(6), are given as follows*

$$r_2 = \frac{k_2-2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(\frac{x_2}{{x_{12}}^2} + k_3\frac{x_2}{{x_{23}}^2}\right),$$

$$r_{12} = \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{12}}^2}\right),$$

$$r_{23} = \frac{1}{2x_{23}} + \frac{1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2}\right),$$

$$r_{13} = \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right),$$

where $n = 1+k_2+k_3$.

The Stiefel manifolds $V_{1+k_2}\mathbb{R}^{1+k_2+k_3} = \mathrm{SO}(1+k_2+k_3)/\mathrm{SO}(k_3)$

We consider the system of equations

$$r_2 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23}. \tag{7}$$

Then finding Einstein metrics of the form (6) reduces to finding positive solutions of system (7), and we normalize our equations by putting $x_{23} = 1$.

For simplicity, we consider the case when $k_2 = 3$ and $k_3 = n - 4$ $(n \geq 6)$ that is the Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$ and consider $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(n-4))$-invariant metrics of the form (6).

The Stiefel manifolds $V_4\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-4)$

Now the system (7) reduces to the system of equations:

$$\left.\begin{array}{l} f_1 = -(n-4){x_{12}}^3x_2 + (n-4){x_{12}}^2x_{13}x_2^2 + (n-4)x_{12}{x_{13}}^2x_2 \\ -2(n-2)x_{12}x_{13}x_2 + (n-4)x_{12}x_2 + {x_{12}}^2x_{13} + 3x_{13}x_2^2 = 0, \\ f_2 = (n-3){x_{12}}^3 - 2(n-2){x_{12}}^2x_{13} - (n-5)x_{12}{x_{13}}^2 \\ +2(n-2)x_{12}x_{13} + (3-n)x_{12} + 2{x_{12}}^2x_{13}x_2 - 2x_{13}x_2 = 0, \\ f_3 = (n-2)x_{12}x_{13} - (n-2)x_{12} + {x_{12}}^2 - x_{12}x_{13}x_2 \\ -2{x_{13}}^2 + 2 = 0. \end{array}\right\} \quad (8)$$

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

We consider a polynomial ring $R = \mathbb{Q}[z, x_2, x_{12}, x_{13}]$ and an ideal I generated by $\{f_1,\, f_2,\, f_3,\, z\, x_2\, x_{12}\, x_{13} - 1\}$ to find non-zero solutions of equations (8). We take a lexicographic order $>$ with $z > x_2 > x_{12} > x_{13}$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $(x_{13}-1)\, h_1(x_{13})$, where $h_1(x_{13})$ is a polynomial of x_{13} with degree 10 given by

$$
\begin{aligned}
& h_1(x_{13}) \\
& = (n-1)^3(5n-11)^2 \left(n^3 - 10n^2 + 33n - 35\right) \left(n^3 - 6n^2 + 9n - 3\right) {x_{13}}^{10} \\
& -2(n-1)^2(5n-11) \left(17n^8 - 356n^7 + 3221n^6 - 16396n^5 + 51159n^4 \right. \\
& \left. -99720n^3 + 117862n^2 - 76568n + 20649\right) {x_{13}}^9 \\
& +(n-1) \left(4n^{11} + 389n^{10} - 11430n^9 + 136940n^8 - 946084n^7 + 4220820n^6 \right. \\
& -12735744n^5 + 26330445n^4 - 36830352n^3 + 33361745n^2 - 17678114n \\
& \left. +4164053\right) {x_{13}}^8
\end{aligned}
$$

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-4)$

$$
\begin{aligned}
&-4\left(8n^{12} - 38n^{11} - 2320n^{10} + 43360n^9 - 379590n^8 + 2055155n^7\right.\\
&-7507061n^6 + 19112638n^5 - 34063584n^4 + 41706995n^3 - 33417851n^2\\
&\left.+15765962n - 3316050\right) {x_{13}}^7\\
&+\left(112n^{12} - 2718n^{11} + 27906n^{10} - 149523n^9 + 354855n^8 + 588726n^7\right.\\
&-7694150n^6 + 29295831n^5 - 65164167n^4 + 92342878n^3 - 82220114n^2\\
&\left.+41992646n - 9373722\right) {x_{13}}^6\\
&-2\left(112n^{12} - 3338n^{11} + 45506n^{10} - 376557n^9 + 2113393n^8 - 8496684n^7\right.\\
&+25132832n^6 - 55172371n^5 + 89317711n^4 - 104159676n^3 + 83190848n^2\\
&\left.-40884390n + 9337014\right) {x_{13}}^5\\
&+\left(280n^{12} - 8710n^{11} + 123662n^{10} - 1060617n^9 + 6124653n^8\right.\\
&-25086974n^7 + 74662934n^6 - 162341127n^5 + 255246159n^4\\
&\left.-282268554n^3 + 208035522n^2 - 91729890n + 18337990\right) {x_{13}}^4
\end{aligned}
$$

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-4)$

$$
\begin{aligned}
&-4\left(56n^{12} - 1710n^{11} + 23600n^{10} - 194131n^{9} + 1056185n^{8} - 3982619n^{7}\right.\\
&+10582237n^{6} - 19666327n^{5} + 24629929n^{4} - 18903391n^{3} + 6556083n^{2}\\
&\left.+985682n - 1096346\right) {x_{13}}^{3}\\
&+(n-1)\left(112n^{11} - 3115n^{10} + 38156n^{9} - 268869n^{8} + 1189262n^{7}\right.\\
&-3348224n^{6} + 5627178n^{5} - 3967104n^{4} - 3831854n^{3} + 11143963n^{2}\\
&\left.-9643014n + 3094229\right) {x_{13}}^{2}\\
&-2(n-5)(n-3)(n-1)^2(n+1)\left(16n^{7} - 275n^{6} + 1868n^{5} - 6039n^{4}\right.\\
&\left.+7372n^{3} + 7943n^{2} - 31120n + 23163\right) x_{13}\\
&+(n-5)^2(n-3)^2(n-1)^3(n+1)^2\left(4n^{3} - 23n^{2} - 10n + 161\right).
\end{aligned}
$$

Note that, if the equation $h_1(x_{13}) = 0$ has real solutions, then these are positive.

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

If $x_{13} = 1$ we see that $f_3 = x_{12}(x_{12} - x_2) = 0$ and thus the system of equations (8) reduces the system of equations

$$x_{12} = x_2, \quad (n-1){x_2}^2 - 2(n-2)x_2 + 2 = 0.$$

Thus we obtain two solutions for the system of equations (8) :

$$x_{12} = x_2 = (n-2-\sqrt{n^2-6n+6})/(n-1), \; x_{13} = x_{23} = 1 \quad \text{and}$$

$$x_{12} = x_2 = (n-2+\sqrt{n^2-6n+6})/(n-1), \; x_{13} = x_{23} = 1.$$

These are known as the Jensen's Einstein metrics on Stiefel manifolds.

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

Consider the case when $x_{13} \neq 1$. Then we have $h_1(x_{13}) = 0$ and we claim that the equation $h_1(x_{13}) = 0$ has at least two positive roots.

- At first we consider the value $h_1(x_{13})$ at $x_{13} = 1$. We have

$$h_1(1) = -800\left(6n^5 - 88n^4 + 476n^3 - 1175n^2 + 1274n - 490\right)$$
$$= -800\Big(6(n-6)^5 + 92(n-6)^4 + 524(n-6)^3 + 1345(n-6)^2$$
$$+1430(n-6) + 278\Big).$$

  Thus we see that $h_1(1) < 0$ for $n \geq 6$.

- Secondly we consider the value $h_1(x_{13})$ at $x_{13} = 0$. Then we have

$$h_1(0) = (n-5)^2(n-3)^2(n-1)^3(n+1)^2\left(4n^3 - 23n^2 - 10n + 161\right)$$
$$= (n-5)^2(n-3)^2(n-1)^3(n+1)^2\times$$
$$\Big(4(n-5)^3 + 37(n-5)^2 + 60(n-5) + 36\Big).$$

  Thus we see that $h_1(0) > 0$ for $n \geq 6$.

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

Consider the case when $x_{13} \neq 1$. Then we have $h_1(x_{13}) = 0$ and we claim that the equation $h_1(x_{13}) = 0$ has at least two positive roots.

- At first we consider the value $h_1(x_{13})$ at $x_{13} = 1$. We have

$$h_1(1) = -800\left(6n^5 - 88n^4 + 476n^3 - 1175n^2 + 1274n - 490\right)$$
$$= -800\Big(6(n-6)^5 + 92(n-6)^4 + 524(n-6)^3 + 1345(n-6)^2$$
$$+1430(n-6) + 278\Big).$$

  Thus we see that $h_1(1) < 0$ for $n \geq 6$.

- Secondly we consider the value $h_1(x_{13})$ at $x_{13} = 0$. Then we have

$$h_1(0) = (n-5)^2(n-3)^2(n-1)^3(n+1)^2\left(4n^3 - 23n^2 - 10n + 161\right)$$
$$= (n-5)^2(n-3)^2(n-1)^3(n+1)^2\times$$
$$\Big(4(n-5)^3 + 37(n-5)^2 + 60(n-5) + 36\Big).$$

  Thus we see that $h_1(0) > 0$ for $n \geq 6$.

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

- Thirdly we consider the value $h_1(x_{13})$ at $x_{13} = 2$. We have

$$
\begin{aligned}
&h_1(2) = (n-5)\Big(4n^{11} + 313n^{10} - 2902n^9 - 11175n^8 + 334728n^7\\
&-2555222n^6 + 10151316n^5 - 22397134n^4 + 25374596n^3\\
&-8599331n^2 - 8372942n + 6279733\Big)\\
&= (n-5)\Big(4(n-5)^{11} + 533(n-5)^{10} + 18248(n-5)^9 + 292860(n-5)^8\\
&+2795928(n-5)^7 + 17723008(n-5)^6 + 77831856(n-5)^5\\
&+236200016(n-5)^4 + 477068416(n-5)^3 + 593616384(n-5)^2\\
&+389736448(n-5) + 89941248\Big).
\end{aligned}
$$

Thus we see that $h_1(0) > 0$, $h_1(1) < 0$ and $h_1(2) > 0$ for $n \geq 6$. Hence, we obtain two solutions $x_{13} = \alpha_{13},\ \beta_{13}$ of the equation $h_1(x_{13}) = 0$ between $0 < \alpha_{13} < 1$ and $1 < \beta_{13} < 2$.
More precisely, for $n \geq 9$, we have

$$1 - 2/n - 6/n^2 < \alpha_{13} < 1 - 2/n - 7/(2n^2),$$

$$1 + 50/(63n^2) < \beta_{13} < 1 + 3/n^2.$$

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

We consider a polynomial ring $R = \mathbb{Q}[z, x_2, x_{12}, x_{13}]$ and an ideal J generated by $\{f_1,\ f_2,\ f_3,\ z\, x_2\, x_{12}\, x_{13}\, (x_{13}-1) - 1\}$ and take a lexicographic order $>$ with $z > x_2 > x_{12} > x_{13}$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal J contains the polynomials $h_1(x_{13})$ and

$$8(n-3)(n-2)^3(n-1)^2 a(n)\, x_{12} - w_{12}(x_{13}),$$

Also, for the same ideal J and the lexicographic order $>$ with $z > x_{12} > x_2 > x_{13}$ for monomials on R, we see that a Gröbner basis for J contains the polynomial

$$8(n-5)(n-2)^3(n-1)^3(n+1)\left(4n^3 - 23n^2 - 10n + 161\right) a(n) x_2 - w_2(x_{13}).$$

where $a(n)$ is a polynomial of n of degree 43 with integer coefficients and $w_{12}(x_{13})$, $w_2(x_{13})$ are polynomials of x_{13} (and n) with integer coefficients.

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\,\mathrm{SO}(n-4)$

It is easy to check that $a(n) > 0$ for $n \geq 6$. Thus for the positive values $x_{13} = \alpha_{13},\ \beta_{13}$ found above we obtain real values $x_2 = \alpha_2,\ \beta_2$ and $x_{12} = \alpha_{12},\ \beta_{12}$ as solutions of the system of equations (8).

- We claim that $\alpha_2, \beta_2, \alpha_{12}, \beta_{12}$ are positive .
  Consider the ideal J generated by $\{f_1,\ f_2,\ f_3,\ z\,x_2\,x_{12}\,x_{13}\,(x_{13}-1)-1\}$ and take a lexicographic order $>$ with $z > x_2 > x_{13} > x_{12}$ for a monomial ordering on R. Then a Gröbner basis for the ideal J contains the polynomial $h_2(x_{12})$ of x_{12} of the form

$$h_2(x_{12}) = \sum_{k=0}^{10} b_k(n){x_{12}}^k$$

  where, for $n \geq 6$, $b_k(n)$ are positive for even k and negative for odd. Thus if the equation $h_2(x_{12}) = 0$ has real solutions, then these are positive.

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

- For the same ideal J take a lexicographic order $>$ with $z > x_{12} > x_{13} > x_2$ for a monomial ordering on R. Then a Gröbner basis for the ideal J contains the polynomial $h_3(x_2)$ of x_2 of the form

$$h_3(x_2) = \sum_{k=0}^{10} c_k(n){x_2}^k$$

where, for $n \geq 6$, $c_k(n)$ are positive for even k and negative for odd. Thus if the equation $h_3(x_2) = 0$ has real solutions, then these are positive. In particular, the solutions $x_2 = \alpha_2, \beta_2$ are positive. Notice that the positive solutions $\{x_2 = \alpha_2, x_{12} = \alpha_{12}, x_{13} = \alpha_{13}, x_{23} = 1\}$ and $\{x_2 = \beta_2, x_{12} = \beta_{12}, x_{13} = \beta_{13}, x_{23} = 1\}$ satisfy $\alpha_{13}, \beta_{13} \neq 1$, thus these solutions are different from Jensen's metrics.

The Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

We see that, for $n \geq 16$,

$$1-\frac{2}{n}-\frac{6}{n^2} < \alpha_{13} < 1-\frac{2}{n}-\frac{7}{2n^2}, \qquad 1+\frac{50}{63n^2} < \beta_{13} < 1+\frac{3}{n^2},$$

$$2-\frac{2}{n}-\frac{6}{n^2} < \alpha_{12} < 2-\frac{4}{n}-\frac{31}{4n^2}, \qquad \frac{5}{3n}+\frac{815}{162n^2} < \beta_{12} < \frac{5}{3n}+\frac{10}{n^2},$$

$$\frac{1}{2n}+\frac{13}{8n^2} < \alpha_2 < \frac{1}{2n}+\frac{11}{5n^2}, \qquad \frac{5}{9n}+\frac{23}{20n^2} < \beta_2 < \frac{5}{9n}+\frac{10}{n^2}.$$

Comparison of the metrics on $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

Jensen's metrics on Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

$$\langle\ ,\ \rangle = \begin{pmatrix} 0 & a & 1 \\ a & a & 1 \\ 1 & 1 & * \end{pmatrix},\ \mathrm{Ad}(\mathrm{SO}(4)) \times \mathrm{SO}(n-4)\text{-invariant.}$$

Our Einstein metrics

$$\langle\ ,\ \rangle = \begin{pmatrix} 0 & \beta & \gamma \\ \beta & \alpha & 1 \\ \gamma & 1 & * \end{pmatrix},\ \mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(n-4))\text{-invariant}$$

($\alpha, \beta, \gamma \neq 1$ are all different).

For the Stiefel manifolds $V_\ell\mathbb{R}^{k+k+\ell} = \mathrm{SO}(2k+\ell)/\mathrm{SO}(\ell)$ ($\ell > k \geq 3$) Einstein metrics of Arvanitoyeorgos, Dzhepko and Nikonorov

$$\langle\ ,\ \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & * \end{pmatrix} \quad (\ \alpha, \beta \text{ are different }).$$

The Stiefel manifolds $V_{1+k_2}\mathbb{R}^{1+k_2+k_3} = \mathrm{SO}(1+k_2+k_3)/\mathrm{SO}(k_3)$

For general Stiefel manifolds $V_{1+k_2}\mathbb{R}^{1+k_2+k_3}$ ($k_2 \geq 3$ and $k_3 \geq 2$), the system of equations (7) reduces to the system of polynomial equations of three variables $\{x_2, x_{12}, x_{13}\}$ similar to $\{f_1 = 0, f_2 = 0, f_3 = 0\}$.

Again we compute a Gröbner basis and we obtain a polynomial of x_{13} with degree 10 similar to $h_1(x_{13})$ and $x_{13} - 1$.

Then we can show that at least two positive solutions for the polynomial and hence, we can show that there exist at least two solutions of the system of equations (7).

For $x_{13} = 1$ we obtain Jensen's Einstein metrics.

The Lie groups $G = \mathrm{SO}(n)$ $(n = k_1 + k_2 + k_3)$

- For the tangent space $\mathfrak{so}(k_1 + k_2 + k_3)$ of the Lie group $G = \mathrm{SO}(k_1 + k_2 + k_3)$, we consider the decomposition

$$\mathfrak{so}(k_1 + k_2 + k_3) = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}, \qquad (9)$$

where the corresponding $\mathrm{Ad}(K)$-submodules are non equivalent. By taking into account the diffeomorphism

$$G/\{e\} \cong (G \times \mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))/\mathrm{diag}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)),$$

those left-invariant metrics on G which are $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))$-invariant are given by

$$\begin{aligned} \langle\ ,\ \rangle \;=\; & x_1\,(-B)|_{\mathfrak{so}(k_1)} + x_2\,(-B)|_{\mathfrak{so}(k_2)} + x_3\,(-B)|_{\mathfrak{so}(k_3)} \\ & + x_{12}\,(-B)|_{\mathfrak{m}_{12}} + x_{13}\,(-B)|_{\mathfrak{m}_{13}} + x_{23}\,(-B)|_{\mathfrak{m}_{23}} \end{aligned} \qquad (10)$$

for $k_1 \geq 2$, $k_2 \geq 2$ and $k_3 \geq 2$.

The Lie groups $G = \mathrm{SO}(n)$ $(n = k_1 + k_2 + k_3)$

Lemma 3

The components of the Ricci tensor r for the left-invariant metric $\langle\ ,\ \rangle$ on G defined by (10), are given as follows

$$
\begin{aligned}
r_1 &= \frac{k_1 - 2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2 \frac{x_1}{{x_{12}}^2} + k_3 \frac{x_1}{{x_{13}}^2}\right), \\
r_2 &= \frac{k_2 - 2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1 \frac{x_2}{{x_{12}}^2} + k_3 \frac{x_2}{{x_{23}}^2}\right), \\
r_3 &= \frac{k_3 - 2}{4(n-2)x_3} + \frac{1}{4(n-2)}\left(k_1 \frac{x_3}{{x_{13}}^2} + k_2 \frac{x_3}{{x_{23}}^2}\right),
\end{aligned}
$$

The Lie groups $G = \mathrm{SO}(n)$ $(n = k_1 + k_2 + k_3)$

Lemma 4

$$
\begin{aligned}
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{12}}^2} + (k_2-1)\frac{x_2}{{x_{12}}^2}\right), \\
r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{13}}^2} + (k_3-1)\frac{x_3}{{x_{13}}^2}\right), \\
r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2} + (k_3-1)\frac{x_3}{{x_{23}}^2}\right).
\end{aligned}
$$

The Lie groups $G = \mathrm{SO}(n)$ $(n = 3 + 3 + k_3)$

We consider the system of equations

$$r_1 = r_2, \quad r_2 = r_3, \quad r_3 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23}. \tag{11}$$

Then finding Einstein metrics of the form (10) reduces to finding positive solutions of system (11).
We put $k_1 = k_2 = 3$, so $k_3 = n - 6$, and consider our equations by putting

$$x_{13} = x_{23} = 1, \quad x_2 = x_1.$$

Then the system of equations (11) reduces to the system of equations:

$$\left.\begin{array}{rcl} g_1 & = & n{x_1}^2{x_{12}}^2x_3 - nx_1{x_{12}}^2 - 6{x_1}^2{x_{12}}^2x_3 + 3{x_1}^2x_3 \\ & & -6x_1{x_{12}}^2{x_3}^2 + 8x_1{x_{12}}^2 + {x_{12}}^2x_3 = 0, \\ g_2 & = & -nx_12^3x_3 + n{x_{12}}^2 + 4x_1x_3 + 6{x_{12}}^3x_3 \\ & & +6{x_{12}}^2{x_3}^2 - 8{x_{12}}^2 - 8x_{12}x_3 = 0, \\ g_3 & = & nx_12^3 + n{x_{12}}^2x_3 - 2n{x_{12}}^2 + 2x_1{x_{12}}^2 - 4x_1 \\ & & -3{x_{12}}^3 - 7{x_{12}}^2x_3 + 4{x_{12}}^2 + 8x_{12} = 0. \end{array}\right\} \tag{12}$$

The Lie groups $G = \mathrm{SO}(n)$ $(n = 3 + 3 + k_3)$

By computing a Gröbner basis for $\{g_1, g_2, g_3\}$, we see that there exist at least two positive solutions of the system for $n \geq 10$ of the form

$$\langle \ , \ \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & \gamma \end{pmatrix} \quad (\alpha, \beta \text{ are different, } \beta \neq 1).$$

We can see that these metrics are not naturally reductive.

- For $G = \mathrm{SO}(7), \mathrm{SO}(8), \mathrm{SO}(9)$, we consider separately and we see that $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3))$-invariant Einstein metrics on $\mathrm{SO}(7)$, $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{SO}(2))$-invariant Einstein metrics on $\mathrm{SO}(8)$ and $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{SO}(3))$-invariant Einstein metrics on $\mathrm{SO}(9)$ which are not naturally reductive.

References

1. D. V. Alekseevsky, I. Dotti and C. Ferraris: *Homogeneous Ricci positive* 5*-manifolds*, Pacific. J. Math. 175 (1) (1996) 1–12.
2. A. Arvanitoyeorgos: *Homogeneous Einstein metrics on Stiefel manifolds*, Comment. Math. Univ. Carolinae 37 (3) (1996) 627–634.
3. A. Arvanitoyeorgos, I. Chrysikos and Y. Sakane: *Homogeneous Einstein metrics on generalized flag manifolds with five isotropy summands*, Intern. J. Math. 24 (10) (2013) 1350077 (52 pages).

4. A. Arvanitoyeorgos, V.V. Dzhepko and Yu. G. Nikonorov: *Invariant Einstein metrics on some homogeneous spaces of classical Lie groups*, Canad. J. Math. 61 (6) (2009) 1201–1213.

5. A. Arvanitoyeorgos, V.V. Dzhepko and Yu. G. Nikonorov: *Invariant Einstein metrics on certain Stiefel manifolds*, Differential Geometry and its Applications, Proc. Conf., in Honor of Leonard Euler, Olomouc, August 2007, World Sci. 2008, 35–44.

References

6. A. Arvanitoyeorgos, V.V. Dzhepko and Yu. G. Nikonorov: *Invariant Einstein metrics on quaternionic Stiefel manifolds*, Bull. Greek Math. Soc. 53 (2007) 1–14.

7. A. Arvanitoyeorgos, K. Mori and Y. Sakane: *Einstein metrics on compact Lie groups which are not naturally reductive*, Geom. Dedicata 160 (1) (2012) 261–285.

8. A. Arvanitoyeorgos, Y. Sakane and M. Statha: *New homogeneous Einstein metrics on Stiefel manifolds*, Differential Geom. Appl. (in press).

9. A. Back and W.Y. Hsiang: *Equivariant geometry and Kervaire spheres*, Trans. Amer. Math. Soc. 304 (1) (1987) 207–227.

10. Z. Chen and K. Liang: *Non-naturally reductive Einstein metrics on the compact simple Lie group* F_4, arXive: 21-11-13.

11. J. E. D' Atri and W. Ziller: *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Memoirs Amer. Math. Soc. 19 (215) (1979).

12. W. G. Gibbons, H. Lü and C. N. Pope: *Einstein metrics on group manifolds and cosets*, J. Geom. Phys. 61 (2011) 947–960.

References

13. G. Jensen: *Einstein metrics on principal fiber bundles*, J. Differential Geom. 8 (1973) 599–614.

14. M. Kerr: *New examples of homogeneous Einstein metrics*, Michigan J. Math. 45 (1) (1998) 115–134.

15. S. Kobayashi: *Topology of positively pinched Kähler manifolds*, Tôhoku Math. J. 15 (1963) 121–139.

16. A. H. Mujtaba: *Homogeneous Einstein metrics on* $SU(n)$, J. Geom. Phys. 62 (2012) 976–980.

17. J-S. Park and Y. Sakane: *Invariant Einstein metrics on certain homogeneous spaces*, Tokyo J. Math. 20 (1) (1997) 51–61.

18. C. N. Pope: *Homogeneous Einstein metrics on SO(n)*, arXive: 2010.

19. A. Sagle: *Some homogeneous Einstein manifolds*, Nagoya Math. J. 39 (1970) 81–106.

20. M. Wang and W. Ziller: *Existence and non-excistence of homogeneous Einstein metrics*, Invent. Math. 84 (1986) 177–194.