Touching an Associoid

(Wolfgang Bertram, Seeking SLeeping beauty at Rauischholzhausen, May 31st, 2014)

1. *What is an associoid ?*

Motivation. A *torsor* (or: *pregroup, groud, heap,...*) is a set M together with a ternary product map $M^3 \to M$, $(x,y,z) \mapsto (xyz)$ satisfying the *identity of para-associativity*

$$\text{(PA)} \qquad (xy(zuv)) = (x(uzy)v) = ((xyz)uv)$$

and the *idempotent law*

$$\text{(IP)} \qquad (xxy) = y, \quad (wzz) = w\,.$$

Example. Every group $(G, e, \cdot)$ is a torsor with $(xyz) := xy^{-1}z$. And conversely, every torsor is of this form! So "torsors are for groups what affine spaces are for vector spaces".

Definition. A *semi-associoid* is a set M together with a *partially defined ternary product map*

$$M^3 \supset D \to M, \quad (x,y,z) \mapsto (xyz)$$

satisfying the identity (PA) (in the sense that, if one term is defined, then so are the other two, and equality holds); if it satisfies moreover (IP), then it is called an *associoid*.

The family of associoids. According to the nature of the domain $D \subset M^3$ of definition of the product map of the associoid, the following more or less classical associative objects (categories, in fact) are defined:

$$\begin{array}{ccc}
\textit{semi-associoid} & \supset & \textit{associoid} \\
\cup & & \cup \\
\textit{semi-pregroupoid} & \supset & \textit{pregroupoid} \\
 & & \cup \\
\cup & & \textit{(left or right) principal equivalence relation (prev)} \\
 & & \cup \\
\textit{semi-torsor (semi-pregroup)} & \supset & \textit{torsor (pregroup)}
\end{array}$$

Explanation: the domain D is defined in terms of *two equivalence relations* a, b *on* M, which are given by the fibers of two projections "target" and "domain":

$$M/a \;\leftarrow\; M \;\rightarrow\; M/b\,,$$

whereas for left or right prev's, just one of the two projections suffices to define D:

$$\begin{array}{rcl}
\textit{(semi-)pregroupoid} & : & D = M \times_a M \times_b M = (a \times M) \cap (M \times b) \\
 & & \cap \\
\textit{left (or right) prev} & : & D = a \times M \quad \text{(or: } M \times b) \\
 & & \cap \\
\textit{semi-torsor} & : & D = M^3 \text{(i.e., everywhere defined)}
\end{array}$$

Then the following *compatibility condition* is required:

$$\text{(C)} \qquad (x,y) \in a, (y,z) \in b \quad \Rightarrow \quad (z,(xyz)) \in a,\ (x,(xyz)) \in b.$$

Then the notion of *(left or right) prev* is an abstract-algebraic version of the one of *principal bundle*, stripped off the usual topological conditions. The notion of *pregroupoid* has been introduced by Anders Kock, and Peter Johnstone has observed that *a groupoid is the same thing as a pregroupoid together with some fixed bisection, called the set of units.*

2. The associative geometry of an associoid

(Bi)sections and local (bi)sections. Let $\mathcal{P}(\Omega)$ be the power set of Ω. For a fixed equivalence relation a on Ω, we say that $x \in \mathcal{P}(\Omega)$ is a *(local) section* of a, and we write $a \top x$ (resp $a \top^{loc} x$), if x contains exactly (resp. at most) one element from each equivalence class. If a, b are two equivalence relations, then $x \in \mathcal{P}(\Omega)$ is a *(local) bisection* if it is a (local) section both of a and of b. Spaces of (local) (bi) sections are denoted by

$$U_a, \qquad U_a^{loc}, \qquad U_{ab} = U_a \cap U_b, \qquad U_{ab}^{loc} = U_a^{loc} \cap U_b^{loc}.$$

Theorem. *Assume* $(\Omega, a, b, [\quad])$ *is a pregroupoid. Let* $\mathbf{a}$, $\mathbf{b}$ *the natural equivalence relations induced on* $\mathcal{P}(\Omega)$ *by* a, b. *Then*

(1) $\mathcal{P}(\Omega)$ *carries a natural semitorsor structure,*
(2) U_{ab} *carries a natural torsor structure,*
(3) $(\mathcal{P}(\Omega), \mathbf{a}, \mathbf{b})$ *carries a natural semi-pregroupoid structure,*
(4) U_{ab}^{loc} *carries a natural pregroupoid structure.*

The ternary maps defining these structures are all given by the following formula:

$$\boxed{(xyz)_{ab} := \left\{ \omega \in \Omega \,\middle|\, \begin{array}{c} \exists \xi \in x, \exists \eta \in y, \exists \zeta \in z : \\ \eta \sim_a \xi, \quad \eta \sim_b \zeta, \quad \omega = [\xi \eta \zeta] \end{array} \right\}}.$$

The transversal case: pair pregroupoid, and binary relations. We say that two equivalence relations are *transversal*, $a \top b$, if each equivalence class of a is a section of b, and vice versa. Lemma: this situation is set-theoretically isomorphic to a *direct product* $\Omega = \Omega_1 \times \Omega_2$, with a, b given by the fibers of the two projections. In this case $(\Omega, a, b, [\quad])$ is a purely set-theoretic object, called a *pair pregroupoid*, and we get the following special case of the preceding result: the structures described in the theorem correspond to

(1) $\mathcal{P}(\Omega) = \mathcal{R}(\Omega_2, \Omega_1)$ binary relations with semitorsor structure $X \circ Y^{-1} \circ Z$,
(2) $U_{ab} =$ the torsor of global bijections $\Omega_2 \to \Omega_1$ with $fg^{-1}h$,
(3) $(\mathcal{P}(\Omega), \mathbf{a}, \mathbf{b}) =$ semi-pregroupoid of relations keeping track of domain and image,
(4) $U_{ab}^{loc} =$ the pregroupoid of local bijections.

Case of two commuting principal equivalence relations is particularly interesting. For this, and full details on the things written above, see forthcoming arxiv preprint.

3. The group case. Let $\Omega = G$ be a group (not assumed commutative, but written additively) and A, B two subgroups of G. Define the *right equivalence relation* $x \sim_a y$ iff $A + x = A + y$ (so equivalence classes are right cosets of A), and the *left equivalence relation* $x \sim_b y$ iff $x + B = y + B$ (so equivalence classes are left cosets of B), and let $[xyz] = x - y + z$ (which is the usual torsor law of G, and thus is para-associative and idempotent). Then $(G, a, b, [\quad])$ is a pregroupoid which we denote by

$$G/B \quad \leftarrow \quad G \quad \to A\backslash G. \tag{1}$$

One should note that the structure of $(G, A, B, [\quad])$ is in fact much richer, and therefore the geometries U_{ab} have many other interesting features: we have called them "projective geometry of a group", see `http://arxiv.org/abs/1201.6201`. If, moreover, G is commutative, then we get an "associative geometry" in the sense of Kinyon and B., see *Journal of Lie Theory* 20 (2) (2010), 215-252 , arXiv : `http://arxiv.org/abs/0903.5441`.

In the remainder of the talk, I shall present, using the free software `geogebra`, dynamic images of the special case $\Omega = \mathbb{R}^3$, A, B of dimension 2, and taking *linear* subspaces instead of all of $\mathcal{P}(\Omega)$: so we are precisely in the case of a *real projective plane.*

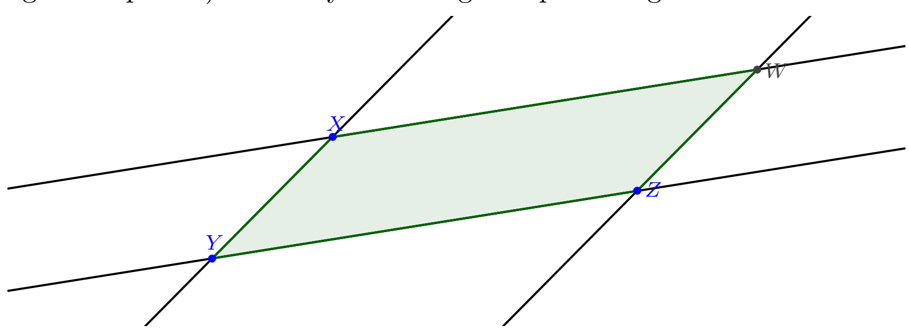
Some information on geogebra. Geogebra is free software available at the adress `http://www.geogebra.org/cms/fr/download`. On the web you find a lot of pages explaining everythig you can do with it, so it suffices to say here that it is really worth to try it out!

Here is another piece of good news (which I forgot to mention in my talk...): it is very easy to export the images into latex-files! Once you think your image is nice and you want to incorporate it into your latex-file, click "export to ps-tricks", then "generate code PSTricks", save the generated tex file and copy it into your own file. Before typesetting, add in the beginning of your file `\usepackage{pstricks-add}`, und use XeLaTeX for typesetting.

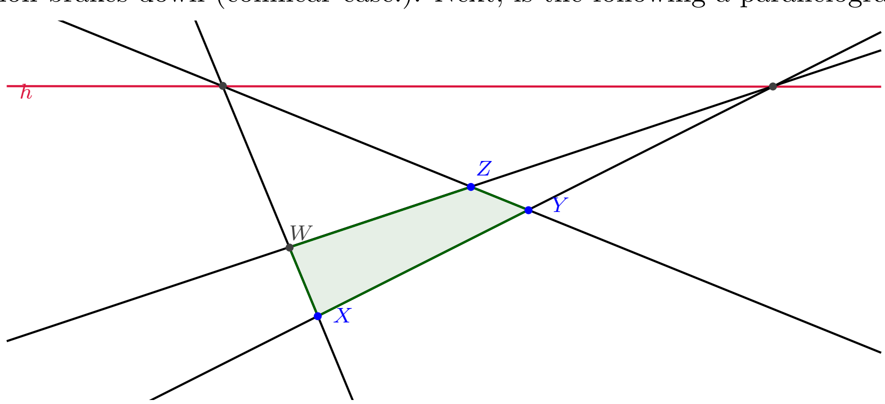
Commutative and Non-commutative Parallelogram Geometry: an Experimental Approach. The arxiv preprint having this title (`http://arxiv.org/abs/1305.6851`) is a sort of outcome when playing with the possibilites offered by geogebra. Meanwhile, there are some improvements on the mathematical side (if I have some more time, I will create a V2 of the above mentioned arxiv paper): as observed by A. Kock, identities (PA) and (IP) together are equivalent to (IP) combined with the following *Chasles relations* or *cancellation laws*:

$$\text{(Ch)} \qquad (xy(yuv)) = (xuv), \qquad (xyv) = (xyu)uv)$$

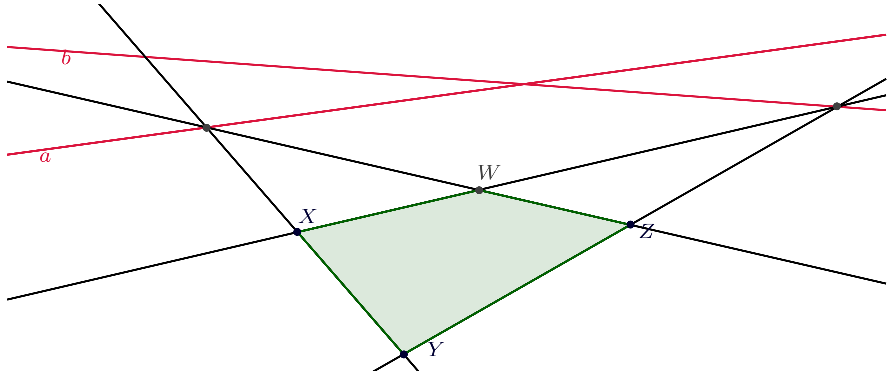
Geometrically, these correspond to *prism configurations*, which in turn correspond exactly to *Desargues theorem*. Thus the following figures, shown during the talk, illustrate in fact the situation for general Desarguesian projective spaces (see `http://arxiv.org/abs/1206.2222v1` for the case of Moufang planes, which are the most important class of non-Desarguesian planes). Start by observing this parallelogram:



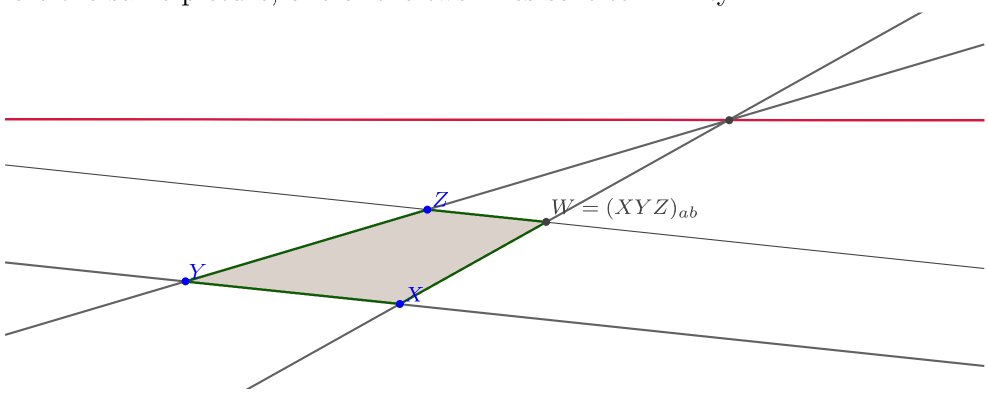
– it illustrates the torsor $\mathbb{R}^2$ with torsor law $W = (XYZ) = X - Y + Z$. Observe also that the software performs automatically the "continuous extension" in case the initial construction brakes down (collinear case!). Next, is the following a parallelogram?



In any case, it illustrates the formula. $W = (((X \vee Y) \wedge h) \vee Z) \wedge (((Z \vee Y) \wedge h) \vee X)$ What happens here of you replace the two h's appearing in the formula by two *different* lines, say a and b – so $W = (((X \vee Y) \wedge a) \vee Z) \wedge (((Z \vee Y) \wedge b) \vee X)$ - ?



And here the same picture, one of the two lines sent to infinity:



Two questions arise: why is this associative? and what happens in the collinear case? Both are answered using Kock's observation, which leads us to look at the following two kinds (left and right) of "non-commutative prisms" (there are four "free points", marked by letters, and then the figure of 6 points "closes up"):