Positively curved GKM manifolds

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(joint work with Michael Wiemeler, arXiv:1402.2397)

47th Seminar Sophus Lie
May 31, 2014

Curvature

Let (M, g) be a Riemannian manifold, with its Levi-Civita connection ∇ and associated curvature tensor R.

For a two-dimensional subspace $\sigma \subset T_pM$, spanned by orthonormal vectors X and Y, the sectional curvature of σ is defined by

$$K(\sigma) = \langle R(X, Y)Y, X\rangle.$$

Unsolved problem: classify Riemannian manifolds with positive sectional curvature, i.e., $K(\sigma) > 0$ for all σ.

Known examples in even dimensions

In this talk we consider even-dimensional compact connected orientable Riemannian manifolds with positive sectional curvature.

The only known examples are:

1. Spheres S^{2n}
2. The projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$
3. The Wallach spaces $\mathrm{SU}(3)/T^2$, $\mathrm{Sp}(3)/\mathrm{Sp}(1)^3$, $\mathrm{F}_4/\mathrm{Spin}(8)$
4. Eschenburg's twisted flag manifold $\mathrm{SU}(3)//T^2$.

Note: these spaces are all highly symmetric!

Results assuming a large symmetry group

Examples of structural results:

- (Grove–Searle) If a torus T^k acts effectively and isometrically on a positively curved simply-connected compact Riemannian manifold of dimension n, then $k \leq [\frac{n+1}{2}]$, and equality can occur if and only if M is diffeomorphic to S^n or $\mathbb{C}P^{n/2}$.
- (Wilking) If $\dim \mathrm{Iso}(M) \geq 2n - 5$ (same assumptions on M), then M is homotopy equivalent to a compact rank-one symmetric space, or isometric to a homogeneous space of positive curvature.
- (Amann–Kennard) If n is even, and a torus T of dimension at least $\log_{4/3}(n)$ acts effectively and isometrically on M, then

$$\chi(M) \leq \sum b_{2i}(M^T) \leq \left(\frac{2}{n} + 1\right)^{1+\log_{4/3}\left(\frac{n}{2}+1\right)}$$

Torus actions of GKM type

An action of a torus T on an orientable differentiable manifold M satisfying $H^{\mathrm{odd}}(M, \mathbb{R}) = 0$ is called GKM$_k$ (named after a paper by Goresky–Kottwitz–MacPherson) if

1. The action has only finitely many fixed points
2. For each fixed point $p \in M^T$, any k weights of the isotropy representation are linearly independent.

If $k = 2$ then we simply say that the action is GKM.

Torus actions of GKM type

Geometric interpretation of the second condition: Let $p \in M^T$, and decompose

$$T_pM = \bigoplus_{\alpha} V_\alpha$$

into weight spaces, $\dim V_\alpha = 2$. Then for a subtorus $T' \subset T$ we have

$$T_pM^{T'} = \bigoplus_{\alpha:\, \alpha|_{\mathfrak{t}'}=0} V_\alpha.$$

Condition 2: If $\dim T' = \dim T - 1$, then $\dim M^{T'} \leq 2$.

An example

Consider the T^n-action on $\mathbb{C}P^n$ by

$$(e^{i\varphi_1}, \dots, e^{i\varphi_n}) \cdot [z_0 : \dots : z_n] = [z_0 : e^{i\varphi_1} z_1 : \dots e^{i\varphi_n} z_n].$$

Fixed points: $[1 : 0 : \dots : 0], \dots, [0 : \dots : 0 : 1]$. Components of $M^{T'}$, where $\dim T' = \dim T - 1$: either a fixed point or of the form

$$\{[0 : \dots : 0 : u : 0 : \dots : 0 : v : 0 : \dots : 0]\} = \mathbb{C}P^1 \cong S^2.$$

Each of these S^2 is T-invariant and contains two fixed points!

The GKM graph

In general: any two-dimensional component of a submanifold $M^{T'}$ as above is a two-sphere, and contains exactly two fixed points.

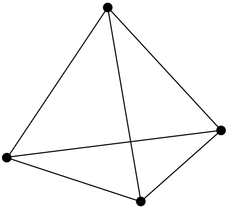
Thus: if $\dim M = 2n$, then for any fixed point p there are n invariant two-spheres containing p.

To any GKM action we can hence assign the GKM graph:

- Vertices: the fixed points.
- Edges: an edge connecting two fixed points for each T-invariant S^2 as above containing them.
- Labeling: An edge is labeled with the corresponding weight of the isotropy representation.

The GKM graph

GKM graph of the T^3-action on $\mathbb{C}P^3$:



More generally: any toric manifold satisfies the GKM condition, and the GKM graph is the one-skeleton of the momentum image.

GKM actions on the positively curved examples

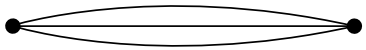
All the known examples of positively curved even-dimensional orientable manifolds admit an action of GKM type:

Guillemin–Holm–Zara: Let G/H be a homogeneous space of compact Lie groups with $\mathrm{rk}\, G = \mathrm{rk}\, H$, and let $T \subset H$ be a maximal torus. Then the T-action on G/H is GKM.

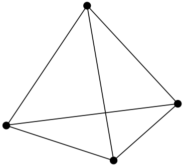
E.g.: weights of isotropy representation at eH: roots of G which are not roots of H. In particular: pairwise linearly independent.

Also Eschenburg's twisted flag admits a GKM action.

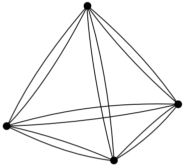
GKM actions on the positively curved examples



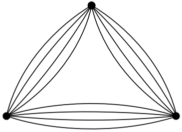
S^{2n}



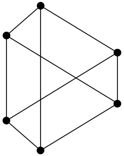
$\mathbb{C}P^n$



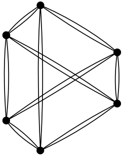
$\mathbb{H}P^n$



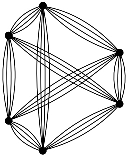
$\mathbb{O}P^2$



$\mathrm{SU}(3)/T^2$
and
$\mathrm{SU}(3)//T^2$



$\mathrm{Sp}(3)/\mathrm{Sp}(1)^3$



$F_4/\mathrm{Spin}(8)$

The GKM graph and cohomology

Fact: The GKM graph of a GKM action of a torus T on M determines the real cohomology ring $H^*(M)$.

Sketch: Consider equivariant cohomology $H^*_T(M)$. The condition $H^{\mathrm{odd}}(M) = 0$ implies that $H^*_T(M)$ is a free module over $H^*(BT)$.

Chang-Skjelbred-Lemma: Denote by $M_1 = \{p \in M \mid \dim Tp \leq 1\}$ the one-skeleton of the action. Then freeness of $H^*_T(M)$ implies that there is a short exact sequence

$$0 \longrightarrow H^*_T(M) \longrightarrow H^*_T(M^T) \longrightarrow H^*_T(M_1, M^T).$$

Thus the GKM graph determines $H^*_T(M)$. Freeness of $H^*_T(M)$ implies also $H^*(M) = H^*_T(M) \otimes_{H^*(BT)} \mathbb{R}$, hence $H^*_T(M)$ determines $H^*(M)$.

The main result

Theorem (—, Wiemeler)

Let M be a compact connected positively curved orientable Riemannian manifold.

1. *If M admits an isometric torus action of type GKM_4, then M has the real cohomology ring of S^{2n} or $\mathbb{C}P^n$.*
2. *If M admits an isometric torus action of type GKM_3, then M has the real cohomology ring of a compact rank one symmetric space.*

Idea of proof: determine all possible GKM graphs of a GKM_3-action on M and show that they are one of those described above. Easy in case 1, rather technical in case 2.

Main ingredient

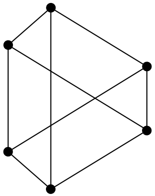
Given a GKM_3-action on M, then any component of $M^{T'}$, where $\dim T' = \dim T - 2$, is at most 4-dimensional.

It thus makes sense to speak about two-dimensional faces of the GKM graph: any two edges emanating from the same vertex determines a two-dimensional face, i.e., a subgraph corresponding to a four-dimensional submanifold.

These submanifolds are totally geodesic and admit an effective isometric T^2-action, hence the result of Grove and Searle applies: they are either S^4 or $\mathbb{C}P^2$. In particular: the two-dimensional faces have either 2 or 3 vertices!

Main ingredient

Note that this condition is violated for the GKM graphs of the Wallach spaces and the twisted flag:



Proof of case 1

Consider the GKM_4-case. If there are only 2 vertices, then the GKM graph is necessarily that of an action on a sphere.

Let v_1, v_2, v_3 be three vertices, and denote by K_{ij} the set of edges between v_i and v_j. Define a map

$$\phi : K_{12} \times K_{13} \longrightarrow K_{23}$$

sending two edges (e_1, e_2) to the third edge of the two-dimensional face determined by e_1 and e_2. If α_1 and α_2 are the weights of e_1 and e_2, then the weight of $\phi(e_1, e_2)$ is of the form $a\alpha_1 + b\alpha_2$. If $\phi(e_1', e_2') = \phi(e_1, e_2)$, then $a'\alpha_1' + b'\alpha_2' = a\alpha_1 + b\alpha_2$, a contradiction to the 4-independence of the weights.

Proof of case 1

Hence

$$\phi : K_{12} \times K_{13} \longrightarrow K_{23}$$

is injective, i.e., $|K_{12}| \cdot |K_{13}| \leq |K_{23}|$.

This implies: if $|K_{ij}| > 1$ for some i, j, then the other two K_{ij} must be empty. Because the graph is connected, this implies that the graph is necessarily a complete graph, i.e., the graph of a GKM action on $\mathbb{C}P^n$.

Generalizations

We can prove two generalizations of the main result:

- Integer coefficients: If M satisfies $H^{\mathrm{odd}}(M,\mathbb{Z}) = 0$ and admits an isometric GKM_3-action such that any two weights are coprime, then M has the integer cohomology of a compact rank one symmetric space.
- Non-orientable manifolds: If a non-orientable M admits an isometric GKM_3-action, then M has the real cohomology of a real projective space, i.e., $H^*(M) = H^0(M) = \mathbb{R}$.