# Homogeneous Almost Complex and Related Structures in dim 6

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based on joint work with
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Description of symmetries of geometric structures is a central question in Lie theory. For structures of finite type the symmetry algebra is finite-dimensional and the symmetry transformations form a Lie group. In particular, for Cartan geometries of type $(G, H)$ the symmetry group has dimension $\leq \dim G$ and this bound is attained by the flat model.

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Almost complex structures have infinite type, and it is indeed possible that an almost complex manifold $(M, J)$ has infinite-dimensional Lie algebra $\mathfrak{sym}(M, J)$. An obvious example is $(\mathbb{C}^n, i)$, but this effect is also possible for non-integrable almost complex structure.

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**Example.** Consider $\mathbb{C}^2(z, w)$ with the almost complex structure given by

$$J\partial_z = i\partial_z + w\partial_{\bar{w}}, \quad J\partial_w = i\partial_w.$$

This structure is non-integrable $N_J(\partial_z, \partial_w) = -2i\partial_{\bar{w}}$, and it has the following infinite transformation pseudogroup of symmetries:

$$(z, w) \mapsto (e^{2ir}z + c, e^{-ir}(w + \zeta(z))),$$

where $r \in \mathbb{R}$, $c \in \mathbb{C}$ and $\zeta_{\bar{z}} = \frac{i}{2}\bar{\zeta}$.

# Maximal symmetry of non-integrable ACS

If $J$ on $M^{2n}$ is integrable, then the local symmetry pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M, J)$ is parametrized by $n$ functions of $n$ variables.

**Theorem (BK 2012)**

*If $N_J \neq 0$, then the local transformations from the pseudogroup $G$ depend on at most $\sigma$ functions of $(n-1)$ arguments, where $\sigma = n-1$ for $n = 2, 3$ and $\sigma = n-2$ for $n > 3$.*

Thus even in the almost complex case, the symmetry transformation groups can be in general very large (generically though $G = \{e\}$).

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**Theorem (BK 2012)**

*Suppose that the almost complex structure $J$ is non-degenerate in the sense that the Nijenhuis tensor as the map $N_J : \Lambda^2 TM \to TM$ is epimorphic for $n > 2$ and that its image $\Pi = \mathrm{Im}(N_J) \subset TM$ is a non-integrable vector distribution for $n = 2$. Then $\dim G < \infty$.*

# Nondegenerate ACS and their symmetries

We recall the following characterization of automorphisms of ACM:

**Theorem (W.Boothby, S.Kobayashi, H.Wang 1963)**

*Let $(M, J)$ be a compact almost complex manifold. Then the automorphism group $\mathrm{Aut}(M, J)$ is a (finite-dimensional) Lie group.*

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The result from the previous page yields a sufficient criterion for the symmetry group to be finite-dimensional.

**Theorem (BK 2012)**

*Let $(M, J)$ be a connected almost complex manifold such that at one point the structure $J$ is non-degenerate. Then $\mathrm{Aut}(M, J)$ is a Lie group.*

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2. $Sp(2)/U(2)$, $Sp(1,1)/U(1,1)$;
3. $SL(2,\mathbb{C}) = SL(2,\mathbb{C})^2/SL(2,\mathbb{C})_{\text{diag}}$;
4. $\mathbb{S}^6 = G_2/SU(3)$, $\mathbb{S}^{2,4} = G_2^*/SU(1,2)$.

The first two lines are Hermitian symmetric spaces; all almost complex structures except those from the last line are integrable.

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**Example (Calabi structure on $\mathbb{S}^6$)**

Let $\mathbb{S}^6 \subset \mathtt{Im}(\mathbb{O})$. The tangent space $T_x\mathbb{S}^6 = x^\perp$ is invariant under octonionic $\times$-multiplication by $x$, and $x^2 = -1$. This defines $J$. There exists a complex basis $e_1, e_2, e_3$ of $T_x\mathbb{S}^6$ such that $N_J(e_i, e_j) = \epsilon_{ijk} e_k$ as in the standard cross-product on $\mathbb{R}^3 = so(3)$; the operation is extended by complex anti-linearity, and $J$ is non-degenerate. The Calabi structure $J$ is strictly nearly Kähler.

# Formulation of the problem

Irreducible isotropy is a strong condition, we would like to relax it. Our working assumption is that the group $H$ in $M = G/H$ is semi-simple. It is possible to further relax this assumption by claiming that $H$ is reductive, but the list of possible structures becomes rather long, and so we restrict to semi-simple isotropy.

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In addition to classification of the general homogeneous almost complex structures, we are especially interested in complex (and further integrable specifications: Kähler, SNK, SKT etc) and non-degenerate (NDG). The latter means that the map

$$N_J : \Lambda^2_{\mathbb{C}} T_x M \to T_x M$$

between two complex vector spaces of $\dim_{\mathbb{C}} = 3$ is an isomorphism of real vector spaces (anti-isomorphism of complex vector spaces).

# Possible semi-simple isotropy subalgebras

A homogeneous space $M = G/H$ admits a $G$-invariant ACS iff the isotropy representation of the Lie algebra $\mathfrak{h}$ of $H$ is complex. Thus $\mathfrak{h}$ has a faithful representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m}, \mathbb{C})$, $\mathfrak{m} \simeq \mathbb{C}^3$.

## Lemma (BK, D.Alekseevsky, H.Winther 2013)

*Semi-simple subalgebras of $\mathfrak{gl}(3, \mathbb{C})$ (up to conjugation) are these:*

- $\mathfrak{h} = \mathfrak{su}(2)$ *or* $\mathfrak{su}(1,1)$, *representations* $V + \mathbb{C}$, $\mathfrak{ad}^{\mathbb{C}}$;
- $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$, *representations* $V + \mathbb{C}$, $\mathfrak{ad}$;
- $\mathfrak{h} = \mathfrak{sl}_3(\mathbb{R})$, *representation* $V^{\mathbb{C}}$;
- $\mathfrak{h} = \mathfrak{su}(3)$ *or* $\mathfrak{su}(1,2)$ *or* $\mathfrak{sl}_3(\mathbb{C})$, *representation* $V$.

*Here* $V$ *is the standard rep,* $\mathfrak{ad}$ *is the adjoint rep and* $\mathbb{C}$ *is a trivial complex representation.*

Next we need to recover $\mathfrak{g}$ from $\mathfrak{h}$ and the isotropy representation as ecoded into

$$0 \to \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{m} \to 0.$$

**Theorem (BK, D.Alekseevsky, H.Winther 2013)**

*The only homogeneous almost complex structures on $M = G/H$ with semi-simple isotropy group $H$ are (up to covering and quotient by discrete central subgroup) the following:*

- ($\mathrm{I}_1$) *the octonionic almost complex structure on* $\mathbb{S}^6 = G_2^c/SU(3)$;
- ($\mathrm{I}_2$) *split-octonionic almost complex structure on* $\mathbb{S}^{2,4} = G_2^*/SU(1,2)$;
- ($\mathrm{II}_1$) *4-parametric family on* $U(3)/SU(2)$, $U(1,2)/SU(2)$;
- ($\mathrm{II}_2$) *2-parametric family on* $U(1,2)/SU(1,1)$, $GL(3)/SU(1,1)$;
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### Corollary (BK, D.Alekseevsky, H.Winther 2013)

*The only compact ACM with ss-isotropy are* $\mathbb{S}^6, \mathbb{S}^1 \times \mathbb{S}^5, \mathbb{S}^3 \times \mathbb{S}^3$.

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**Theorem (BK, D.Alekseevsky, H.Winther 2013)**

*We have* $\mathrm{Aut}(M, J) \supset G$ *always. If $J$ is NDG, then* $\mathrm{Aut}(M, J) = G$.

# Table 1 (out of 5) of ACS of type III

Isotropy $\mathfrak{h} = \mathfrak{su}(2) = \mathtt{Im}(\mathbb{H})$, representation $\mathbb{H} \oplus \mathbb{C}$; $\mathfrak{h}$ acts from the left.
Data: $\mathrm{e} \in \mathbb{C}$ fixed, $x, y \in \mathbb{H}$ arbitrary, $q, p, b \in \mathtt{Im}\mathbb{H}$, $b \in \mathbb{R}q$, $q^2 = -1$.

Hermitian structure: $J(x, \mathrm{e}) = (xq, i\mathrm{e})$; $g(\xi, \eta) = \omega(\xi, J\eta)$;
$\omega = \omega_{\mathbb{H}} + \omega_{\mathbb{C}}$, $\omega_{\mathbb{H}}(x, y) = \mathtt{Re}(xb\bar{y})$, $\omega_{\mathbb{C}}(\mathrm{e}, i\mathrm{e}) = 1$.

Below the brackerts for $[\mathfrak{h}, \mathfrak{h}]$ and $[\mathfrak{h}, \mathfrak{m}]$ are suppressed.

| Lie algebra structure of $\mathfrak{g}$ | conditions & notes |
|---|---|
| $[x, y] = \alpha\mathtt{Re}(xi\bar{y})\mathrm{e}$<br>$[x, i\mathrm{e}] = x(\frac{\varepsilon}{2} + ri)$<br>$[\mathrm{e}, i\mathrm{e}] = \varepsilon\mathrm{e}$ | $\varepsilon \in \{0, 1\}$, $\alpha \neq 0$<br>$J$ - NDG unless $r = 0$ or $q = \pm i$<br>$d\omega = 0$ iff $\varepsilon = 1, b = \alpha i$ |
| $[x, y] = (\mathtt{Re}(xi\bar{y}) + i\mathtt{Re}(xp\bar{y}))\mathrm{e}$ | $J$ - degenerate; $d\omega \neq 0$ |
| $[x, \mathrm{e}] = \alpha x$<br>$[x, i\mathrm{e}] = x(\beta + ri)$<br>$[\mathrm{e}, i\mathrm{e}] = \varepsilon\mathrm{e}$ | $(\varepsilon, \alpha) \in \{(0, 0), (0, 1), (1, 0)\}$<br>$J$ – degenerate<br>$d\omega = 0$ iff $\alpha = \beta = 0, b \in \mathbb{R}i$ |
| $[x, y] = \mathtt{Re}(xi\bar{y})\mathrm{e} + \varepsilon \cdot \mathfrak{ad}_{\mathtt{Im}(x\bar{y})}$<br>$[x, \mathrm{e}] = 3\varepsilon xi$ | $d\omega \neq 0$; $J$ – NDG unless $q = \pm i$<br>$\varepsilon = -1 \Rightarrow \mathfrak{g} = \mathfrak{u}(3)$<br>$\varepsilon = 1 \Rightarrow \mathfrak{g} = \mathfrak{u}(1, 2)$ |

# Outline of the method

Our starting point is a semi-simple Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(3, \mathbb{C}) = \mathfrak{gl}(\mathfrak{m}, \mathbb{C})$, and the goal is to introduce a Lie algebra structure on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Since $\mathfrak{h}$ is semi-simple the representation is completely reducible, and $\mathfrak{m}$ can be chosen as an invariant complement.

Thus the bracket $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ is the subalgebra structure, and the bracket $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ is the representation, so we only need the brackets

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The Jacobi identity morphism $\mathrm{Jac} : \Lambda^3\mathfrak{g} \to \mathfrak{g}$ is trivial for 3 arguments from $\mathfrak{h}$ because $\mathfrak{h}$ is a Lie algebra, for 2 arguments from $\mathfrak{h}$ and 1 from $\mathfrak{m}$ because $\rho$ is a representation and for 1 argument from $\mathfrak{h}$ and 2 from $\mathfrak{m}$ iff the map $\Psi$ is equivariant. This latter is given by a trivial component in the decomposition of the $\mathfrak{h}$-module $\Lambda^2\mathfrak{m}^* \otimes \mathfrak{g}$.

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Finally the last component of the Jacobi map $\mathrm{Jac} : \Lambda^3\mathfrak{m}^* \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is linear-quadratic in $\Psi$ and is a constraint to resolve straighforwardly. This fixes the parameters in $\Psi$, determining $\mathfrak{g}$.

# Homogeneous almost Hermitian structures

Invariant almost Hermitian structures on $M = G/H$ are elements of the set

$$S^2_J\mathfrak{m}^* = \{g \in (S^2\mathfrak{m}^*)^{\mathfrak{h}} : g(J\xi, J\eta) = g(\xi,\eta), \det(g) \neq 0\}$$
$$\simeq \Lambda^2_J\mathfrak{m}^* = \{\omega \in (\Lambda^2\mathfrak{m}^*)^{\mathfrak{h}} : \omega(J\xi, J\eta) = \omega(\xi,\eta), \omega^3 \neq 0\}.$$

The Kähler form is defined by $\omega(\xi,\eta) = g(J\xi,\eta)$. With the exception of $\mathfrak{sl}(2,\mathbb{C})$-isotropy, every homogenous space in our classification possesses invariant almost (pseudo-)Hermitian structure $(g, J, \omega)$.

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These almost Hermitian structures possess a large variety of types. There are Hermitian, but there are also non-degenerate among them. In fact, all proper homogeneous spaces and 11 families of Lie groups admit NDG $J$.

Only 3 out of 16 (strict) Gray-Hervella classes are not realizable in our classification (semi-simple isotropy): $\mathcal{W}_2$, $\mathcal{W}_{1,4}$, $\mathcal{W}_{2,4}$.

# Homogeneous Kähler structures

**Theorem (BK, D.Alekseevsky, H.Winther 2013)**

*Non-flat (pseudo-)Kähler homogeneous 6D manifolds with semi-simple isotropy are $M = G/H$ with $H = SU(2)$ or $H = SU(1,1)$, isotropy representation $\mathfrak{m} = V + \mathbb{C}$, where $V = \mathbb{H}$ or resp. $V = \mathbb{H}_s$.*
*The reductive complement $\mathfrak{m}$ is a Lie algebra given by the following relations ($x, y \in V,\ \mathrm{e}, i\mathrm{e} \in \mathbb{C}$; $\omega_V(x,y) = \mathtt{Re}(xi\bar{y})$, $\omega_{\mathbb{C}}(\mathrm{e}, i\mathrm{e}) = 1$)*

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1) $$[x,y] = \alpha\,\mathtt{Re}(xi\bar{y})\mathrm{e},\ [x,i\mathrm{e}] = x(\tfrac{1}{2} + ri),\ [\mathrm{e},i\mathrm{e}] = \mathrm{e}.$$

*Here $\omega = \alpha \cdot \omega_V + \omega_{\mathbb{C}}$, $J(x,\mathrm{e}) = (xi, i\mathrm{e})$. Metric $g(\xi,\eta) = \omega(\xi, J\eta)$ can have signature $(6,0)$ or $(4,2)$, and it is Einstein with the cosmological constant $-4$, and is not conformally flat.*

2) $$[x,i\mathrm{e}] = rxi,\ [\mathrm{e},i\mathrm{e}] = \mathrm{e}.$$

*Here $\omega = \omega_V + b \cdot \omega_{\mathbb{C}}$, $J(x,\mathrm{e}) = (xi, i\mathrm{e})$, Metric $g(\xi,\eta) = \omega(\xi, J\eta)$ can have signature $(6,0)$ or $(4,2)$, and it is not Einstein or conformally flat.*

# Strictly nearly Kähler structures

A Hermitian structure $(g, J, \omega)$ is called nearly Kähler if

$$\nabla\omega \in \Omega^3 M,$$

and strictly nearly Kähler (SNK) if $d\omega \neq 0$. SNK structures are always NDG. They are Einstein, and can be considered as Calabi-Yau manifolds with torsion. R.Bryant and M.Verbitsky separately discovered Hitchin-type Lagrangians in 6D with the critical points being always either complex or SNK.

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J.Butruille classified homogeneous SNK in 6D, which up to covering/factor gives a list of 4 structures:

$$\mathbb{S}^6,\ \mathbb{S}^3 \times \mathbb{S}^3,\ \mathbb{C}P^3,\ \mathbb{F}(1,2) = SU(3)/(U(1) \times U(1)).$$

In the pseudo-Kähler case, i.e. signature $(4,2)$, the SNK structures from our list are: $\mathbb{S}^{2,4}$, $SL(2) \times SL(2)$ and a left-invariant structure on a solvable Lie group, not having analogs in the definite signature.

# Strong Kähler with torsion structures

Strong Kähler with torsion (SKT) is another type of Hermitian structure. We mention them although they are not specific to dim 6. They satisfy

$$\partial\bar{\partial}\omega = 0 \quad \Leftrightarrow \quad d_J^2\omega = 0,$$

where $d_J = d \circ J$.
SKT structures are related to generalized Kähler structures, which have in turn found applications to super-symmetric $\sigma$-models.

There are 2 SKT structures, both supported on the Lie group $\mathbb{R}^4 \rtimes S_2$, one of each signature. Some other structures satisfy the following generalization of the SKT-property: $d_J^3\omega = 0$.

However the standard almost Hermitian structure $(g, J, \omega)$ on $\mathbb{S}^6$ has $d_J^3\omega \neq 0$, while $d_J^4\omega = 0$.

# Example: Deformation of Calabi-Eckmann structure

The manifold $M = U(3)/SU(2) \simeq \mathbb{S}^1 \times \mathbb{S}^5$ carries the Calabi-Eckmann structure defined via the splitting:

$$TM^6 = T\mathbb{S}^1 \oplus T\mathbb{S}^1_{\text{Hopf}} \oplus CR^4$$

Picking any complex structure on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1_{\text{Hopf}}$ and the CR-structure on $CR^4$ gives the (2D family of) complex structure(s) $J_{CE}$.

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We found a 2D family of deformations of the CR-structure invariant under $U(3)$; it is given by $\mathbb{S}^2 \subset \mathfrak{su}(2) \simeq \mathrm{Im}\mathbb{H}$ – the space of parameters with $\pm J_{CE}$ at the poles. Except for the poles, all $J$ are NDG.

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We found a 2D family of deformations of the CR-structure invariant under $U(3)$; it is given by $\mathbb{S}^2 \subset \mathfrak{su}(2) \simeq \mathrm{Im}\mathbb{H}$ – the space of parameters with $\pm J_{CE}$ at the poles. Except for the poles, all $J$ are NDG.

The critical points of restriction of the mentioned Bryant-Verbitsky functional to the space $\mathbb{S}^2$ of symmetric $J$ are: the poles and also the equator. The structures $J_{\text{equator}}$ are not SNK, but are distinguished by the property $d_J^3\omega = 0$. For all other $J \in \mathbb{S}^2$ we have: $d_J^4\omega = 0$.

# Example: Deformation of Calabi-Eckmann structure

The manifold $M = U(3)/SU(2) \simeq \mathbb{S}^1 \times \mathbb{S}^5$ carries the Calabi-Eckmann structure defined via the splitting:

$$TM^6 = T\mathbb{S}^1 \oplus T\mathbb{S}^1_{\text{Hopf}} \oplus CR^4$$

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Similarly $U(2,1)/SU(2) \simeq \mathbb{T}^2 \times B^4$, $U(2,1)/SU(1,1) \simeq \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{C}$, and $GL(3)/SU(1,1) \simeq \mathbb{R}^* \times \mathbb{R}_+ \times T\mathbb{S}^2$ are considered.

# Symmetries of NDG ACS

As was mentioned, non-degeneracy of ACS $J$ (in any dimension) implies that $\mathrm{Aut}(M, J)$ is a Lie group. In 6D we can give a sharp result.

**Theorem (BK 2013)**

*Let the almost complex structure $J$ on $M^6$ be non-degenerate. Then* $\dim \mathrm{Aut}(M, J) \leq 14$ *and the equality is attained only when* $M = \mathbb{S}^6$ *and* $J$ *is* $G_2^c$*-invariant or* $M = \mathbb{S}^{2,4}$ *and* $J$ *is* $G_2^n$*-invariant.*

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Thus the most symmetric NDG structure $J$ is non-unique, there are 2 such. Let us notice that also in general Cartan geometries the maximal symmetry is not attained at one structure, but for parabolic geometries the maximal symmetric structure is unique.

It was observed that often there is a significant gap between the symmetry dimension of the maximal model and the so called sub-maximal model (next in symmetry dimension). This first gap is of much interest.

# Known results on the gap phenomenon from parabolic geometries up to 2012

| *Geometry* | *Max* | *Submax* | *Citation* |
|---|---|---|---|
| scalar 2nd order ODE mod point | 8 | 3 | Tresse (1896) |
| projective str 2D | 8 | 3 | Tresse (1896) |
| $(2,3,5)$-distributions | 14 | 7 | Cartan (1910) |
| projective str $\dim = n+1,\ n \geq 2$ | $n^2+4n+3$ | $n^2+4$ | Egorov (1951) |
| scalar 3rd order ODE mod contact | 10 | 5 | Wafo Soh, Qu Mahomed (2002) |
| conformal $(2,2)$ str | 15 | 9 | Kruglikov (2012) |
| pair of 2nd order ODE | 15 | 9 | Casey, Dunajski, Tod (2012) |

# Sample of results on gaps from BK & D.The 2013

| Geometry | Max | Submax |
|---|---|---|
| Sign $(p,q)$ conf geom $n=p+q,\ \ p,q\geq 2$ | $\binom{n+2}{2}$ | $\binom{n-1}{2}+6$ |
| Systems 2nd ord ODE in $m\geq 2$ dep vars | $(m+2)^2-1$ | $m^2+5$ |
| Generic $m$-distributions on $\binom{m+1}{2}$-dim manifolds | $\binom{2m+1}{2}$ | $\begin{cases} \frac{m(3m-7)}{2}+10, & m\geq 4; \\ 11, & m=3 \end{cases}$ |
| Lagrangian contact str | $m^2+2m$ | $(m-1)^2+4,\ m\geq 3$ |
| Contact projective str | $m(2m+1)$ | $\begin{cases} 2m^2-5m+8, & m\geq 3; \\ 5, & m=2 \end{cases}$ |
| Contact path geometries | $m(2m+1)$ | $2m^2-5m+9,\ m\geq 3$ |
| Exotic parabolic contact structure of type $E_8/P_8$ | 248 | 147 |

For nondegenerate ACS in 6D we also observed the gap phenomenon

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*Let almost complex structure $J$ on $M^6$ be nondegenerate.*
*If* $\dim \mathrm{Aut}(M, J) \neq 14$, *then* $\dim \mathrm{Aut}(M, J) \leq 9$.
*Moreover if $J$ admits 9 symmetries, then the automorphism group acts locally transitively, so $(M, J)$ is locally homogeneous.*

Thus our classification of NDG $J$ on $M^6 = G/H$ yields a large family of sub-maximal symmetric almost complex structures in 6D.

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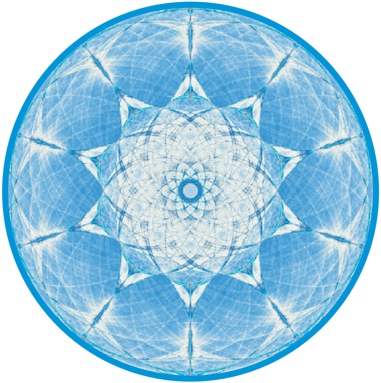
Thus our classification of NDG $J$ on $M^6 = G/H$ yields a large family of sub-maximal symmetric almost complex structures in 6D.

The technique for proving the gap results for parabolic geometries is not applicable for NDG ACS. Our apporach is based on the following:

- Liner classification of NDG Nijenhuis tensors $N_J$ in 6D (BK2002);
- Classification of subalgebras in $SU(3)$ and $SU(1,2)$;
- 1-jet determinacy of symmetries of NDG ACS (BK 2013);
- Reconstruction algorithm of $\mathfrak{g}$ via $\mathfrak{h}$ and its rep on $\mathfrak{m}$;
- Disqualifying intransitive structures from representaiton viewpoint.



the END