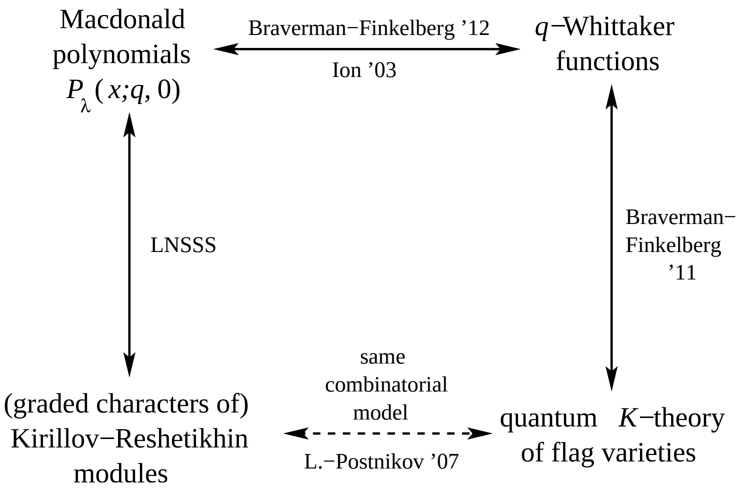
Specialized Macdonald polynomials, quantum K-theory, and Kirillov-Reshetikhin modules

Cristian Lenart

Max-Planck-Institut für Mathematik and
State University of New York at Albany

47th Sophus Lie Seminar
Castle Rauischholzhausen, May 2014


Macdonald polynomials
$P_\lambda(x;q,0)$
Braverman−Finkelberg ’12
Ion ’03
q−Whittaker functions
LNSSS
Braverman−Finkelberg ’11
(graded characters of) Kirillov−Reshetikhin modules
same combinatorial model
L.−Postnikov ’07
quantum K−theory of flag varieties

Macdonald polynomials

λ: dominant weight for classical subsystem of untwisted affine root system.

Macdonald polynomials

λ: dominant weight for classical subsystem of untwisted affine root system.

$P_\lambda(x; q, t)$: Weyl group invariant polynomials, orthogonal, generalizing the corresponding irreducible characters $= P_\lambda(x; 0, 0)$.

Macdonald polynomials

λ: dominant weight for classical subsystem of untwisted affine root system.

$P_\lambda(x; q, t)$: Weyl group invariant polynomials, orthogonal, generalizing the corresponding irreducible characters $= P_\lambda(x; 0, 0)$.

Defined in the DAHA setup, as common eigenfunctions of the Cherednik operators Y_μ.

Braverman-Finkelberg q-Whittaker functions

$\Psi_\lambda(x;q)$: eigenfunctions of the quantum difference Toda integrable system (Etingof, Sevostyanov).

Braverman-Finkelberg q-Whittaker functions

$\Psi_\lambda(x;q)$: eigenfunctions of the quantum difference Toda integrable system (Etingof, Sevostyanov).

Theorem (Braverman-Finkelberg, Ion)

We have

$$P_\lambda(x;q,t=0) = \widehat{\Psi}_\lambda(x;q)\,.$$

Schubert calculus

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

Schubert calculus

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

$H^*(G/B)$ and $K(G/B)$ have bases of Schubert classes;

Schubert calculus

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

$H^*(G/B)$ and $K(G/B)$ have bases of Schubert classes; for K-theory, they are the classes $[\mathcal{O}_w] = [\mathcal{O}_{X_w}]$ of structure sheaves of X_w.

Schubert calculus

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

$H^*(G/B)$ and $K(G/B)$ have bases of Schubert classes; for K-theory, they are the classes $[\mathcal{O}_w] = [\mathcal{O}_{X_w}]$ of structure sheaves of X_w.

The quantum cohomology algebra $QH^*(G/B)$ still has the Schubert basis, but over $\mathbb{C}[q_1, \ldots, q_r]$.

Schubert calculus

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

$H^*(G/B)$ and $K(G/B)$ have bases of Schubert classes; for K-theory, they are the classes $[\mathcal{O}_w] = [\mathcal{O}_{X_w}]$ of structure sheaves of X_w.

The quantum cohomology algebra $QH^*(G/B)$ still has the Schubert basis, but over $\mathbb{C}[q_1, \dots, q_r]$.

The structure constants (for multiplying Schubert classes) are the 3-point Gromov-Witten (GW) invariants.

Schubert calculus

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

$H^*(G/B)$ and $K(G/B)$ have bases of Schubert classes; for K-theory, they are the classes $[\mathcal{O}_w] = [\mathcal{O}_{X_w}]$ of structure sheaves of X_w.

The quantum cohomology algebra $QH^*(G/B)$ still has the Schubert basis, but over $\mathbb{C}[q_1, \ldots, q_r]$.

The structure constants (for multiplying Schubert classes) are the 3-point Gromov-Witten (GW) invariants.

A k-point GW invariant (of degree d) counts curves of degree d passing through k given Schubert varieties.

Quantum K-theory

Givental and Lee defined K-theoretic GW invariants by applying the K-theory Euler characteristic when the space of curves (through given Schubert varieties) is infinite.

Quantum K-theory

Givental and Lee defined K-theoretic GW invariants by applying the K-theory Euler characteristic when the space of curves (through given Schubert varieties) is infinite.

The structure constants for the quantum K-theory $QK(G/B)$ are defined based on the 2- and 3-point invariants (complex formula).

Quantum K-theory

Givental and Lee defined K-theoretic GW invariants by applying the K-theory Euler characteristic when the space of curves (through given Schubert varieties) is infinite.

The structure constants for the quantum K-theory $QK(G/B)$ are defined based on the 2- and 3-point invariants (complex formula).

The K-theoretic J-function is the generating function of 1-point K-theoretic GW invariants.

Quantum K-theory

Givental and Lee defined K-theoretic GW invariants by applying the K-theory Euler characteristic when the space of curves (through given Schubert varieties) is infinite.

The structure constants for the quantum K-theory $QK(G/B)$ are defined based on the 2- and 3-point invariants (complex formula).

The K-theoretic J-function is the generating function of 1-point K-theoretic GW invariants.

Theorem (Braverman-Finkelberg)

In simply-laced types, the q-Whittaker function $\Psi_\lambda(x;q)$ (viewed as a function of λ) coincides with the K-theoretic J-function.

Kirillov-Reshetikhin (KR) modules

$W^{r,s}$: finite dimensional modules for $\widehat{\mathfrak{g}}$ ($r \in I$, $s \geq 1$).

Kirillov-Reshetikhin (KR) modules

$W^{r,s}$: finite dimensional modules for $\widehat{\mathfrak{g}}$ ($r \in I$, $s \geq 1$).

Let $\mathbf{p} = (p_1, p_2, \ldots)$ be a composition, and

$$W^{\otimes \mathbf{p}} = W^{p_1,1} \otimes W^{p_2,1} \otimes \ldots, \qquad \lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

Kirillov-Reshetikhin (KR) modules

$W^{r,s}$: finite dimensional modules for $\widehat{\mathfrak{g}}$ ($r \in I$, $s \geq 1$).

Let $\mathbf{p} = (p_1, p_2, \ldots)$ be a composition, and

$$W^{\otimes \mathbf{p}} = W^{p_1,1} \otimes W^{p_2,1} \otimes \ldots , \qquad \lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

$X_\lambda(x; q)$: the (graded) character of $W^{\otimes \mathbf{p}}$.

Kirillov-Reshetikhin (KR) modules

$W^{r,s}$: finite dimensional modules for $\widehat{\mathfrak{g}}$ ($r \in I$, $s \geq 1$).

Let $\mathbf{p} = (p_1, p_2, \ldots)$ be a composition, and

$$W^{\otimes \mathbf{p}} = W^{p_1,1} \otimes W^{p_2,1} \otimes \ldots, \qquad \lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

$X_\lambda(x;q)$: the (graded) character of $W^{\otimes \mathbf{p}}$.

Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

For all untwisted affine root systems $A_{n-1}^{(1)} - G_2^{(1)}$, we have

$$P_\lambda(x;q,0) = X_\lambda(x;q)\,.$$

The underlying combinatorics

The quantum alcove model describes all the mentioned structures:

The underlying combinatorics

The quantum alcove model describes all the mentioned structures:

- the specialized Macdonald polynomials $P_\lambda(x; q, 0)$ and the q-Whittaker functions (Ram-Yip formula),

The underlying combinatorics

The quantum alcove model describes all the mentioned structures:

- the specialized Macdonald polynomials $P_\lambda(x; q, 0)$ and the q-Whittaker functions (Ram-Yip formula),
- the quantum K-theory of G/B (conjecture by L.-Postnikov),

The underlying combinatorics

The quantum alcove model describes all the mentioned structures:

- the specialized Macdonald polynomials $P_\lambda(x; q, 0)$ and the q-Whittaker functions (Ram-Yip formula),
- the quantum K-theory of G/B (conjecture by L.-Postnikov),
- the tensor products of one-column KR modules (LNSSS).

The underlying combinatorics

The quantum alcove model describes all the mentioned structures:

- the specialized Macdonald polynomials $P_\lambda(x;q,0)$ and the q-Whittaker functions (Ram-Yip formula),
- the quantum K-theory of G/B (conjecture by L.-Postnikov),
- the tensor products of one-column KR modules (LNSSS).

The model is uniform for all Lie types $A_{n-1} - G_2$.

Finite root systems $\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$

Reflections s_α, $\alpha \in \Phi$.

Finite root systems $\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$

Reflections s_α, $\alpha \in \Phi$.

The Weyl group $W = \langle s_\alpha \ : \ \alpha \in \Phi^+ \rangle$.

Finite root systems $\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$

Reflections s_α, $\alpha \in \Phi$.

The Weyl group $W = \langle s_\alpha \, : \, \alpha \in \Phi^+ \rangle$.

The quantum Bruhat graph $\mathrm{QBG}(W)$ on W is the directed graph with labeled edges

$$w \stackrel{\alpha}{\longrightarrow} ws_\alpha \, ,$$

where

$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of the Bruhat order)}\, , \qquad \text{or}$$

$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1 \qquad (\mathrm{ht}(\alpha^\vee) = \langle \rho, \alpha^\vee \rangle)\, .$$

Finite root systems $\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$

Reflections s_α, $\alpha \in \Phi$.

The Weyl group $W = \langle s_\alpha \ : \ \alpha \in \Phi^+ \rangle$.

The quantum Bruhat graph $\mathrm{QBG}(W)$ on W is the directed graph with labeled edges

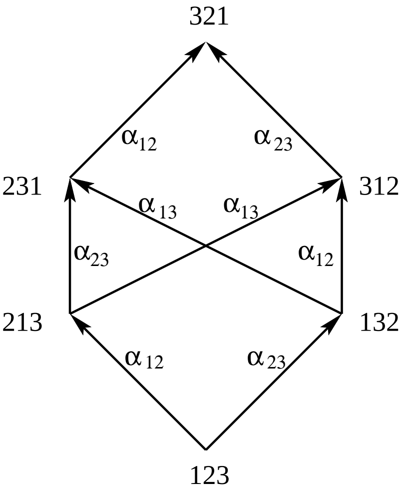
$$w \xrightarrow{\alpha} ws_\alpha \,,$$

where

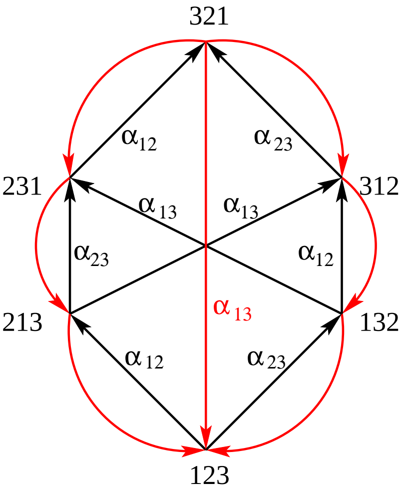
$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of the Bruhat order)}\,, \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1 \qquad (\mathrm{ht}(\alpha^\vee) = \langle \rho, \alpha^\vee \rangle)\,.$$

Comes from the multiplication of Schubert classes in the quantum cohomology of flag varieties $QH^*(G/B)$ (Fulton and Woodward).

Bruhat graph for S_3:



Quantum Bruhat graph for S_3:



The quantum alcove model

Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \ldots, \beta_m)\,.$$

The quantum alcove model

Given a dominant weight λ, we associate with it a sequence of roots, called a λ-chain:

$$\Gamma = (\beta_1, \ldots, \beta_m)\,.$$

Fact. The construction of a λ-chain is based on a reduced decomposition of the affine Weyl group element corresponding to $A_\circ - \lambda$. This gives a sequence of alcoves from $A_\circ$ to $A_\circ - \lambda$.

The quantum alcove model (cont.)

Given $\Gamma = (\beta_1, \ldots, \beta_m)$, let $r_i := s_{\beta_i}$.

The quantum alcove model (cont.)

Given $\Gamma = (\beta_1, \ldots, \beta_m)$, let $r_i := s_{\beta_i}$.

The objects of the model: subsets of positions in Γ

$$J = (j_1 < \ldots < j_s) \subseteq \{1, \ldots, m\}.$$

The quantum alcove model (cont.)

Given $\Gamma = (\beta_1, \ldots, \beta_m)$, let $r_i := s_{\beta_i}$.

The objects of the model: subsets of positions in Γ

$$J = (j_1 < \ldots < j_s) \subseteq \{1, \ldots, m\}\,.$$

For $w \in W$ and J, construct the chain $\pi(w, J)$ of elements in W:

$$w_0 = w, \quad \ldots, \quad w_i := w r_{j_1} \ldots r_{j_i}, \quad \ldots, \quad w_s = \mathrm{end}(w, J)\,.$$

The quantum alcove model (cont.)

Given $\Gamma = (\beta_1, \ldots, \beta_m)$, let $r_i := s_{\beta_i}$.

The objects of the model: subsets of positions in Γ

$$J = (j_1 < \ldots < j_s) \subseteq \{1, \ldots, m\} .$$

For $w \in W$ and J, construct the chain $\pi(w, J)$ of elements in W:

$$w_0 = w, \quad \ldots, \quad w_i := w r_{j_1} \ldots r_{j_i}, \quad \ldots, \quad w_s = \mathrm{end}(w, J) .$$

Important structures:

$$\mathcal{A}_q(\Gamma, w) := \{J \,:\, \pi(w, J) \text{ path in QBG}(W)\} ,$$
$$\mathcal{A}_{\lessdot}(\Gamma, w) := \{J :\, \pi(w, J) \text{ saturated chain in } (W, <)\} .$$

The quantum alcove model (cont.)

Given $\Gamma = (\beta_1, \ldots, \beta_m)$, let $r_i := s_{\beta_i}$.

The objects of the model: subsets of positions in Γ

$$J = (j_1 < \ldots < j_s) \subseteq \{1, \ldots, m\}\,.$$

For $w \in W$ and J, construct the chain $\pi(w, J)$ of elements in W:

$$w_0 = w, \quad \ldots, \quad w_i := w r_{j_1} \ldots r_{j_i}, \quad \ldots, \quad w_s = \mathrm{end}(w, J)\,.$$

Important structures:

$$\mathcal{A}_q(\Gamma, w) := \{J \,:\, \pi(w, J) \text{ path in QBG}(W)\}\,,$$
$$\mathcal{A}_{\lessdot}(\Gamma, w) := \{J :\, \pi(w, J) \text{ saturated chain in } (W, <)\}\,.$$

Let $\mathcal{A}_q(\Gamma) := \mathcal{A}_q(\Gamma, 1_W)$ and $\mathcal{A}_{\lessdot}(\Gamma) := \mathcal{A}_{\lessdot}(\Gamma, 1_W)$.

Macdonald polynomials: the Ram-Yip formula

Given a dominant weight λ, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Macdonald polynomials: the Ram-Yip formula

Given a dominant weight λ, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Given $J \in \mathcal{A}_q(\Gamma)$, we associate with it

- a weight $\mathrm{weight}(J)$,

Macdonald polynomials: the Ram-Yip formula

Given a dominant weight λ, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Given $J \in \mathcal{A}_q(\Gamma)$, we associate with it

- a weight $\mathrm{weight}(J)$,
- a statistic $\mathrm{height}(J)$, which "measures" the down steps $w_{i-1} > w_i$ in the path $\pi(w, J)$ in $\mathrm{QBG}(W)$.

Macdonald polynomials: the Ram-Yip formula

Given a dominant weight λ, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Given $J \in \mathcal{A}_q(\Gamma)$, we associate with it

- a weight $\mathrm{weight}(J)$,
- a statistic $\mathrm{height}(J)$, which "measures" the down steps $w_{i-1} > w_i$ in the path $\pi(w, J)$ in $\mathrm{QBG}(W)$.

Theorem (Ram-Yip, L.)

$$P_\lambda(X; q, 0) = \sum_{J \in \mathcal{A}_q(\Gamma)} q^{\mathrm{height}(J)} \, x^{\mathrm{weight}(J)} \, .$$

Macdonald polynomials: the Ram-Yip formula

Given a dominant weight λ, consider a λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$.

Given $J \in \mathcal{A}_q(\Gamma)$, we associate with it

- a weight $\mathrm{weight}(J)$,
- a statistic $\mathrm{height}(J)$, which "measures" the down steps $w_{i-1} > w_i$ in the path $\pi(w, J)$ in $\mathrm{QBG}(W)$.

Theorem (Ram-Yip, L.)

$$P_\lambda(X; q, 0) = \sum_{J \in \mathcal{A}_q(\Gamma)} q^{\mathrm{height}(J)} \, x^{\mathrm{weight}(J)} \, .$$

Remark. For $q = 0$, we retrieve the alcove model (L. and Postnikov, cf. Gaussent and Littelmann, Littelmann):

$$P_\lambda(X; 0, 0) = \mathrm{ch}(V_\lambda) = \sum_{J \in \mathcal{A}_{\lessdot}(\Gamma)} x^{\mathrm{weight}(J)} \, .$$

$K(G/B)$ and $QK(G/B)$: Chevalley formulas

Recall: $K(G/B)$ and $QK(G/B)$ have bases of Schubert classes $[\mathcal{O}_{X_w}] = [\mathcal{O}_w]$, $w \in W$.

$K(G/B)$ and $QK(G/B)$: Chevalley formulas

Recall: $K(G/B)$ and $QK(G/B)$ have bases of Schubert classes $[\mathcal{O}_{X_w}] = [\mathcal{O}_w]$, $w \in W$.

Let $\Gamma_{\mathrm{rev}} =$ reverse of an ω_k-chain (ω_k a fundamental weight).

$K(G/B)$ and $QK(G/B)$: Chevalley formulas

Recall: $K(G/B)$ and $QK(G/B)$ have bases of Schubert classes $[\mathcal{O}_{X_w}] = [\mathcal{O}_w]$, $w \in W$.

Let $\Gamma_{\rm rev}$ = reverse of an ω_k-chain (ω_k a fundamental weight).

Theorem (L.-Postnikov, L.-Shimozono)

In $K(G/B)$ (finite-type or Kac-Moody), we have

$$[\mathcal{O}_w] \cdot [\mathcal{O}_{s_k}] = \sum_{J \in \mathcal{A}_{\lessdot}(\Gamma_{\rm rev}, w) \setminus \{\emptyset\}} (-1)^{|J|-1} [\mathcal{O}_{{\rm end}(w,J)}] \,.$$

$K(G/B)$ and $QK(G/B)$: Chevalley formulas

Recall: $K(G/B)$ and $QK(G/B)$ have bases of Schubert classes $[\mathcal{O}_{X_w}] = [\mathcal{O}_w]$, $w \in W$.

Let $\Gamma_{\rm rev}$ = reverse of an ω_k-chain (ω_k a fundamental weight).

Theorem (L.-Postnikov, L.-Shimozono)

In $K(G/B)$ (finite-type or Kac-Moody), we have

$$[\mathcal{O}_w] \cdot [\mathcal{O}_{s_k}] = \sum_{J \in \mathcal{A}_{\lessdot}(\Gamma_{\rm rev}, w) \setminus \{\emptyset\}} (-1)^{|J|-1} [\mathcal{O}_{{\rm end}(w,J)}] \,.$$

Conjecture (L.-Postnikov)

In $QK(G/B)$ (finite-type), we have:

$$[\mathcal{O}_w] * [\mathcal{O}_{s_k}] \stackrel{?}{=} \sum_{J \in \mathcal{A}_q(\Gamma_{\rm rev}, w) \setminus \{\emptyset\}} (-1)^{|J|-1} \, q_1^* \ldots q_r^* \, [\mathcal{O}_{{\rm end}(w,J)}] \,.$$

Evidence for the conjectured formula in $QK(G/B)$

(L.-Maeno) Based on some relations in $QK(SL_n/B)$ discovered by Kirillov-Maeno, we constructed polynomials $\mathfrak{G}_w(x;q)$, called quantum Grothendieck polynomials.

Evidence for the conjectured formula in $QK(G/B)$

(L.-Maeno) Based on some relations in $QK(SL_n/B)$ discovered by Kirillov-Maeno, we constructed polynomials $\mathfrak{G}_w(x;q)$, called quantum Grothendieck polynomials.

- They multiply as in the conjectured Chevalley formula.

Evidence for the conjectured formula in $QK(G/B)$

(L.-Maeno) Based on some relations in $QK(SL_n/B)$ discovered by Kirillov-Maeno, we constructed polynomials $\mathfrak{G}_w(x;q)$, called quantum Grothendieck polynomials.

- They multiply as in the conjectured Chevalley formula.
- They are conjectured to represent Schubert classes $[\mathcal{O}_w]$ in $QK(SL_n/B)$.

Kirillov-Reshetikhin (KR) modules/crystals

Recall the KR modules, as modules for $U_q(\widehat{\mathfrak{g}})$: $W^{r,s}$ and

$$W^{\otimes \mathbf{p}} = W^{p_1,1} \otimes W^{p_2,1} \otimes \cdots .$$

Kirillov-Reshetikhin (KR) modules/crystals

Recall the KR modules, as modules for $U_q(\widehat{\mathfrak{g}})$: $W^{r,s}$ and

$$W^{\otimes \mathbf{p}} = W^{p_1,1} \otimes W^{p_2,1} \otimes \ldots .$$

Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots): $\widetilde{f}_0, \ldots, \widetilde{f}_r$.

Kirillov-Reshetikhin (KR) modules/crystals

Recall the KR modules, as modules for $U_q(\widehat{\mathfrak{g}})$: $W^{r,s}$ and

$$W^{\otimes \mathbf{p}} = W^{p_1,1} \otimes W^{p_2,1} \otimes \ldots .$$

Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots): $\widetilde{f}_0, \ldots, \widetilde{f}_r$.

Fact. $W^{\otimes \mathbf{p}}$ has a basis (crystal basis) $B = B^{\otimes \mathbf{p}}$ such that in the limit $q \to 0$ we have

$$\widetilde{f}_i : B \to B \sqcup \{0\}, \qquad \widetilde{f}_i\, b = b' \iff b \xrightarrow{i} b'.$$

Kirillov-Reshetikhin (KR) modules/crystals

Recall the KR modules, as modules for $U_q(\widehat{\mathfrak{g}})$: $W^{r,s}$ and

$$W^{\otimes \mathbf{p}} = W^{p_1,1} \otimes W^{p_2,1} \otimes \dots .$$

Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots): $\widetilde{f}_0 , \dots , \widetilde{f}_r$.

Fact. $W^{\otimes \mathbf{p}}$ has a basis (crystal basis) $B = B^{\otimes \mathbf{p}}$ such that in the limit $q \to 0$ we have

$$\widetilde{f}_i \; : \; B \to B \sqcup \{0\} , \qquad \widetilde{f}_i \, b = b' \iff b \xrightarrow{i} b' .$$

So $B^{\otimes \mathbf{p}}$ is a colored directed graph (connected).

Models for KR crystals

Fact. In the classical types $A - D$ there are tableau models (the usual column-strict fillings in type $A_{n-1}^{(1)}$, but more involved in the other types, particularly for $B_n^{(1)}$ and $D_n^{(1)}$).

Models for KR crystals

Fact. In the classical types $A-D$ there are tableau models (the usual column-strict fillings in type $A_{n-1}^{(1)}$, but more involved in the other types, particularly for $B_n^{(1)}$ and $D_n^{(1)}$).

Goal. Uniform model for all types $A_{n-1}^{(1)} - G_2^{(1)}$, based on the quantum alcove model.

The quantum alcove model for KR crystals

Given $\mathbf{p} = (p_1, p_2, \ldots)$ and an arbitrary Lie type, let

$$\lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

The quantum alcove model for KR crystals

Given $\mathbf{p} = (p_1, p_2, \ldots)$ and an arbitrary Lie type, let

$$\lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

Let Γ be a λ-chain, and consider $\mathcal{A}_q(\Gamma)$.

The quantum alcove model for KR crystals

Given $\mathbf{p} = (p_1, p_2, \ldots)$ and an arbitrary Lie type, let

$$\lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

Let Γ be a λ-chain, and consider $\mathcal{A}_q(\Gamma)$.

Construction. (L. and Lubovsky, generalization of L.-Postnikov, Gaussent-Littelmann) *Crystal operators* $\widetilde{f}_1, \ldots, \widetilde{f}_r$ *and* $\widetilde{f}_0$ *on* $\mathcal{A}_q(\Gamma)$*.*

The quantum alcove model for KR crystals

Given $\mathbf{p} = (p_1, p_2, \ldots)$ and an arbitrary Lie type, let

$$\lambda = \omega_{p_1} + \omega_{p_2} + \ldots .$$

Let Γ be a λ-chain, and consider $\mathcal{A}_q(\Gamma)$.

Construction. (L. and Lubovsky, generalization of L.-Postnikov, Gaussent-Littelmann) *Crystal operators* $\widetilde{f}_1, \ldots, \widetilde{f}_r$ *and* $\widetilde{f}_0$ *on* $\mathcal{A}_q(\Gamma)$*.*

Main Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

The (combinatorial) crystal $\mathcal{A}_q(\Gamma)$ *is isomorphic to the tensor product of KR crystals* $B^{\otimes \mathbf{p}}$*.*

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

More precisely, $D_B : B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

- it is constant on classical components (0-arrows removed);

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

More precisely, $D_B : B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

- it is constant on classical components (0-arrows removed);
- it decreases by 1 along certain 0-arrows.

The energy function

It originates in the theory of exactly solvable lattice models.

The energy function defines a grading on the classical components (no 0-arrows) of $B = B^{\otimes \mathbf{p}}$ (Schilling and Tingley).

More precisely, $D_B : B \to \mathbb{Z}_{\geq 0}$ satisfies the following conditions:

- it is constant on classical components (0-arrows removed);
- it decreases by 1 along certain 0-arrows.

Goal. A more efficient uniform calculation, based only on the combinatorial data associated with a crystal vertex.

The energy via the quantum alcove model

Consider $J = \{j_1 < j_2 < \ldots < j_s\}$ in $\mathcal{A}_q(\Gamma)$ for $\Gamma = (\beta_1, \ldots, \beta_m)$, i.e., we have a path in the quantum Bruhat graph

$$1_W = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \,.$$

The energy via the quantum alcove model

Consider $J = \{j_1 < j_2 < \ldots < j_s\}$ in $\mathcal{A}_q(\Gamma)$ for $\Gamma = (\beta_1, \ldots, \beta_m)$, i.e., we have a path in the quantum Bruhat graph

$$1_W = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \,.$$

Recall that $\mathrm{height}(J)$ measures the down steps in the above path.

The energy via the quantum alcove model

Consider $J = \{j_1 < j_2 < \ldots < j_s\}$ in $\mathcal{A}_q(\Gamma)$ for $\Gamma = (\beta_1, \ldots, \beta_m)$, i.e., we have a path in the quantum Bruhat graph

$$1_W = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s \,.$$

Recall that $\mathrm{height}(J)$ measures the down steps in the above path.

Theorem (L.-Naito-Sagaki-Schilling-Shimozono)

Given $J \in \mathcal{A}_q(\Gamma)$, which is identified with $B^{\otimes \mathbf{p}}$, we have

$$D_B(J) = -\mathrm{height}(J)\,.$$

The combinatorial R-matrix via the quantum alcove model

This is the (unique) affine crystal isomorphism which commutes factors in the tensor product of KR crystals $B^{\otimes \mathbf{p}}$

The combinatorial R-matrix via the quantum alcove model

This is the (unique) affine crystal isomorphism which commutes factors in the tensor product of KR crystals $B^{\otimes \mathbf{p}}$ (the swap $a \otimes b \mapsto b \otimes a$ is *not* a crystal isomorphism!).

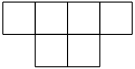
The combinatorial R-matrix via the quantum alcove model

This is the (unique) affine crystal isomorphism which commutes factors in the tensor product of KR crystals $B^{\otimes \mathbf{p}}$ (the swap $a \otimes b \mapsto b \otimes a$ is *not* a crystal isomorphism!).

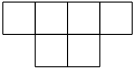
Theorem (L.-Lubovsky)

We give a uniform realization, based on the quantum alcove model, of the combinatorial R-matrix.

Example in type A_2.

$\mathbf{p} = (1, 2, 2, 1) =$  $; \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4, 2, 0).$

Example in type A_2.

$\mathbf{p} = (1,2,2,1) =$  ; $\quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0)$.

A λ-chain as a concatenation of ω_1-, ω_2-, ω_2-, and ω_1-chains:

$$\Gamma = (\ \ (1,2),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (2,3),\ \ (1,3)\ \mid\ (1,2),\ \ (1,3)\ \ ).$$

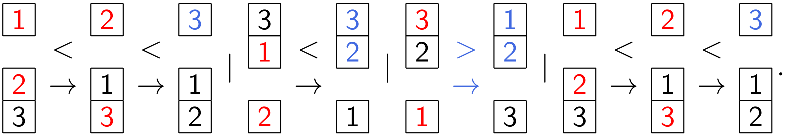
Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$\left(\ \underline{(1,2)},\ \ \underline{(1,3)}\ \mid\ \underline{(2,3)},\ \ (1,3)\ \mid\ (2,3),\ \ \underline{(1,3)}\ \mid\ \underline{(1,2)},\ \ \underline{(1,3)}\ \right).$$

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$\big(\ \underline{(1,2)},\ \underline{(1,3)}\ \mid\ \underline{(2,3)},\ (1,3)\ \mid\ (2,3),\ \underline{(1,3)}\ \mid\ \underline{(1,2)},\ \underline{(1,3)}\ \big).$$

Claim: J is admissible. Indeed, the corresponding path in the quantum Bruhat graph is



Example. Let $J = \{1,2,3,6,7,8\}$.

$$\left(\ \underline{(1,2)},\ \underline{(1,3)}\ \middle|\ \underline{(2,3)},\ (1,3)\ \middle|\ (2,3),\ \underline{(1,3)}\ \middle|\ \underline{(1,2)},\ \underline{(1,3)}\ \right).$$

Claim: J is admissible. Indeed, the corresponding path in the quantum Bruhat graph is

$$\begin{array}{c}\begin{array}{|c|}\hline 1\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\end{array}\ \begin{array}{c}<\\ \rightarrow\end{array}\ \begin{array}{c}\begin{array}{|c|}\hline 2\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\end{array}\ \begin{array}{c}<\\ \rightarrow\end{array}\ \begin{array}{c}\begin{array}{|c|}\hline 3\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{array}\ \Big|\ \begin{array}{c}\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 2\\\hline\end{array}\end{array}\ \begin{array}{c}<\\ \rightarrow\end{array}\ \begin{array}{c}\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 1\\\hline\end{array}\end{array}\ \Big|\ \begin{array}{c}\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 1\\\hline\end{array}\end{array}\ \begin{array}{c}>\\ \rightarrow\end{array}\ \begin{array}{c}\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 3\\\hline\end{array}\end{array}\ \Big|\ \begin{array}{c}\begin{array}{|c|}\hline 1\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\end{array}\ \begin{array}{c}<\\ \rightarrow\end{array}\ \begin{array}{c}\begin{array}{|c|}\hline 2\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\end{array}\ \begin{array}{c}<\\ \rightarrow\end{array}\ \begin{array}{c}\begin{array}{|c|}\hline 3\\\hline\end{array}\\ \\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{array}.$$

The corresponding element in $B^{\otimes \mathbf{p}} = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{1,1}$ represented via column-strict fillings:

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline\end{array}.$$

The energy calculation

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.

The energy calculation

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.

We considered the λ-chain Γ and $J = \{1, 2, 3, 6, 7, 8\} \in \mathcal{A}(\Gamma)$:

$$\begin{array}{rcccccccccccc} \Gamma = & (& \underline{(1,2)}, & \underline{(1,3)} & \mid & \underline{(2,3)}, & (1,3) & \mid & (2,3), & \underline{(1,3)} & \mid & \underline{(1,2)}, & \underline{(1,3)} \;) \, , \\ (h_i) = & (& 2, & 4 & \mid & 2, & 3 & \mid & 1, & 2 & \mid & 1, & 1 \;) \, . \end{array}$$

The energy calculation

Example. Consider the running example: $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1$ in type A_2.

We considered the λ-chain Γ and $J = \{1, 2, 3, 6, 7, 8\} \in \mathcal{A}(\Gamma)$:

$$\begin{array}{rcccccccccc} \Gamma = & (\underline{(1,2)}, & \underline{(1,3)} & | & \underline{(2,3)}, & (1,3) & | & (2,3), & \underline{(1,3)} & | & \underline{(1,2)}, & \underline{(1,3)} \;), \\ (h_i) = & (\quad 2, & 4 & | & 2, & 3 & | & 1, & 2 & | & 1, & 1 \;). \end{array}$$

We have

$$\mathrm{height}(J) = 2\,.$$