Maurizio Parton - Paolo Piccinni

$\mathrm{Spin}(9)$, Rosenfeld planes and canonical differential forms in octonionic geometry

MP, Paolo Piccinni.
$\mathrm{Spin}(9)$ and almost complex structures on 16-dimensional manifolds.
Annals of Global Analysis and Geometry, 41 (2012), 321–345.

MP, Paolo Piccinni.
Spheres with more than 7 vector fields: all the fault of $\mathrm{Spin}(9)$.
Linear Algebra and its Applications, 438 (2013), 1113–1131.

Liviu Ornea, MP, Paolo Piccinni, Victor Vuletescu.
$\mathrm{Spin}(9)$ geometry of the octonionic Hopf fibration.
Transformation Groups, 18 (2013), 845–864.

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Canonical Differential Forms, Rosenfeld Planes, and a Matryoshka in Octonionic Geometry.
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Further references

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Thomas Friedrich.
Weak $\mathrm{Spin}(9)$-Structures on 16-dimensional Riemannian Manifolds.
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Clifford structures on Riemannian manifolds.
Advances in Mathematics, 228 (2011), 940–967.

Main aim of the talk

Convince you that $\mathrm{Spin}(9)$ is beautiful.

Method

- Introduce $\mathrm{Spin}(9)$ little brother: $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$
- Show you that $\mathrm{Spin}(9)$ is involved in many curious phenomena

Relatives

Construction of relevant differential forms associated with the groups

$$\mathrm{Spin}(9), \mathrm{Spin}(10), \mathrm{Spin}(12), \mathrm{Spin}(16)$$

appearing as structure and holonomy group in the exceptional symmetric spaces

FII $\quad$ $\mathrm{EIII}, \mathrm{EVI}, \mathrm{EVIII}$

Cayley plane → FII

Rosenfeld planes → $\mathrm{EIII}, \mathrm{EVI}, \mathrm{EVIII}$

Berger's list and $\mathrm{Spin}(9)$ refutation

Holonomy of simply connected, irreducible, nonsymmetric?

$\mathrm{U}(n)$

$\mathrm{SO}(n)$

$\mathrm{SU}(n)$

$\mathrm{Spin}(9)$

$\mathrm{Spin}(7)$

$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$

G_2

$\mathrm{Sp}(n)$

Berger's list and $\mathrm{Spin}(9)$ refutation

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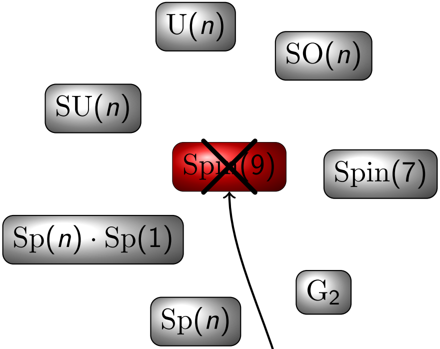

$\mathrm{U}(n)$

$\mathrm{SO}(n)$

$\mathrm{SU}(n)$

$\cancel{\mathrm{Spin}(9)}$

$\mathrm{Spin}(7)$

$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$

G_2

$\mathrm{Sp}(n)$

Simply connected, complete, holonomy $\mathrm{Spin}(9)$

$$\Leftrightarrow$$

$$\mathbb{O}P^2 = \frac{\mathrm{F}_4}{\mathrm{Spin}(9)}(s > 0), \quad \mathbb{R}^{16}(\text{flat}), \quad \mathbb{O}H^2 = \frac{\mathrm{F}_{4(-20)}}{\mathrm{Spin}(9)}(s < 0)$$

[Alekseevsky, Funct. Anal. Prilozhen 1968].

First $\mathrm{Spin}(9)$ definition

Definition

$\mathrm{Spin}(9)$ is the Lie group which has been excluded from a list.

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Definition

$\mathrm{Spin}(9)$ is the Lie group which has been excluded from a list.

USELESS

What is $\mathrm{Spin}(9)$?

Definition

$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].

What is Spin(9)?

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- $\Lambda^8(\mathbb{R}^{16}) \overset{\mathrm{Spin}(9)}{=} \Lambda^8_1 + \dots$ [Friedrich, Asian Journ. Math 2001].
- $\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of any element of Λ^8_1

  [Brown-Gray, Diff. Geom. in honor of K. Yano 1972].

Definition

$\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of the 8-form

$$\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$$

▸ Details ▸ More details

[Berger, Ann. Éc. Norm. Sup. 1972].

1. What you should expect in this talk
2. Spin(9)
   - Why and what
   - Quaternionic analogy
   - Curiosities about $\mathrm{Spin}(9)$
3. After $\mathrm{Spin}(9)$: Rosenfeld projective planes
4. Applications
5. Conclusion

The closest relative: the quaternionic group $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$

Analogy 1

$\mathrm{Spin}(9)$ and $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ are the symmetry groups of the Hopf fibrations:

$$S^{15} \longrightarrow \mathbb{O}P^1 \qquad\qquad S^7 \longrightarrow \mathbb{H}P^1$$

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Analogy 1

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Analogy 2

- $\mathrm{Spin}(9)$ is the stabilizer in $\mathrm{SO}(16)$ of the 8-form

$$\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$$

- $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ is the stabilizer in $\mathrm{SO}(8)$ of the 4-form

$$\Omega = \int_{\mathbb{H}P^1} p_l^* \nu_l \, dl$$

Two alternative constructions for the quaternionic form Ω

Sum of squares: classical

$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$, where $\omega_I, \omega_J, \omega_K$ are orthogonal local Kähler forms

Two alternative constructions for the quaternionic form Ω

Sum of squares: classical

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Sum of squares: involutions

$$\Omega = \tau_2(\Theta), \qquad \text{where}$$

- $\tau_2(\Theta) = \sum_{\alpha<\beta} \theta_{\alpha\beta}^2$
- $\Theta = (\theta_{\alpha\beta})$ matrix of Kähler forms of $J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta$
- $\mathcal{I}_1, \dots, \mathcal{I}_5$ self-adjoint anti-commuting involutions in $\mathbb{R}^8$:

$$\mathcal{I}_1, \dots, \mathcal{I}_5 \in \mathrm{SO}(8), \quad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \mathcal{I}_\alpha^2 = \mathrm{Id}, \quad \mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$$

▸ Explicit $\mathcal{I}_\alpha$

The five involutions of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ as 8 × 8 matrices

$$\mathcal{I}_1 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right)$$

$$\mathcal{I}_2 = \left(\begin{array}{c|c} 0 & -R_i^{\mathbb{H}} \\ \hline R_i^{\mathbb{H}} & 0 \end{array}\right)$$

$$\mathcal{I}_5 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right)$$

◂ Go back

$$\mathcal{I}_3 = \left(\begin{array}{c|c} 0 & -R_j^{\mathbb{H}} \\ \hline R_j^{\mathbb{H}} & 0 \end{array}\right)$$

$$\mathcal{I}_4 = \left(\begin{array}{c|c} 0 & -R_k^{\mathbb{H}} \\ \hline R_k^{\mathbb{H}} & 0 \end{array}\right)$$

Nine involutions for $\mathrm{Spin}(9)$

Analogy 3

$$\Phi_{\mathrm{Spin}(9)} = \tau_4(\Theta), \qquad \text{where}$$

- $t^9 + \cancel{\tau_2(\Theta)}t^7 + \tau_4(\Theta)t^5 + \cancel{\tau_6(\Theta)}t^3 + \tau_8(\Theta)t$ characteristic polynomial of Θ
- $\Theta = (\theta_{\alpha\beta})$ matrix of Kähler forms of $J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta$
- $\mathcal{I}_1, \dots, \mathcal{I}_9$ self-adjoint anti-commuting involutions in $\mathbb{R}^{16}$:

$$\mathcal{I}_1, \dots, \mathcal{I}_9 \in \mathrm{SO}(16), \quad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \mathcal{I}_\alpha^2 = \mathrm{Id}, \quad \mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$$

▸ Explicit $\mathcal{I}_\alpha$

[MP-Piccinni, Ann. Glob. Anal. Geom. 2012].

The nine involutions of $\mathrm{Spin}(9)$ as 16×16 matrices

$$\mathcal{I}_3 = \left(\begin{array}{c|c} 0 & -R_j \\ \hline R_j & 0 \end{array}\right)$$

$$\mathcal{I}_2 = \left(\begin{array}{c|c} 0 & -R_i \\ \hline R_i & 0 \end{array}\right)$$

$$\mathcal{I}_1 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right)$$

$$\mathcal{I}_4 = \left(\begin{array}{c|c} 0 & -R_k \\ \hline R_k & 0 \end{array}\right)$$

$$\mathcal{I}_9 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right)$$

◂ Go back

$$\mathcal{I}_5 = \left(\begin{array}{c|c} 0 & -R_e \\ \hline R_e & 0 \end{array}\right)$$

$$\mathcal{I}_8 = \left(\begin{array}{c|c} 0 & -R_h \\ \hline R_h & 0 \end{array}\right)$$

$$\mathcal{I}_6 = \left(\begin{array}{c|c} 0 & -R_f \\ \hline R_f & 0 \end{array}\right)$$

$$\mathcal{I}_7 = \left(\begin{array}{c|c} 0 & -R_g \\ \hline R_g & 0 \end{array}\right)$$

$\mathrm{Spin}(9)$ and $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ as structure groups

Analogy 4

- A $\mathrm{Spin}(9)$-structure on M^{16} is a rank 9 vector subbundle

$$\mathsf{span}\{\mathcal{I}_1, \dots, \mathcal{I}_9\} \subset \mathrm{End}(M)$$

- A $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$-structure on M^8 is a rank 5 vector subbundle

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$\mathrm{Spin}(9)$ and $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ as structure groups

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Due to the dual role Sp-Spin of $\mathrm{Sp}(1) = \mathrm{Spin}(3)$ and $\mathrm{Sp}(2) = \mathrm{Spin}(5)$

$\{J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{1\leq\alpha<\beta\leq 9}$ generates $\mathfrak{spin}(9)$

- There are 36 Kähler forms $\theta_{\alpha\beta}$ for $1 \leq \alpha < \beta \leq 9$
- There are 84 Kähler forms $\theta_{\alpha\beta\gamma}$ for $1 \leq \alpha < \beta \leq 9$

Remark

$$\mathfrak{so}(16) = \Lambda^2(\mathbb{R}^{16}) = \Lambda^2_{36} \oplus \Lambda^2_{84} = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$$

generated by $\theta_{\alpha\beta}$

generated by $\theta_{\alpha\beta\gamma}$

Matryoshka-like structure

Nestedness

The family of complex structures $\{J_{\alpha\beta}\}_{1\leq\alpha<\beta\leq 9}$ is compatible with the inclusion of Lie algebras

$$\mathfrak{spin}(7)_\Delta \subset \mathfrak{spin}(8) \subset \mathfrak{spin}(9)$$

in the sense that

$$\begin{aligned}\mathfrak{spin}(7)_\Delta &= \mathrm{span}\{J_{\alpha\beta}\}_{2\leq\alpha<\beta\leq 8}\\ &\subseteq \mathrm{span}\{J_{\alpha\beta}\}_{1\leq\alpha<\beta\leq 8} = \mathfrak{spin}(8)\\ &\subseteq \mathrm{span}\{J_{\alpha\beta}\}_{1\leq\alpha<\beta\leq 9} = \mathfrak{spin}(9)\end{aligned}$$

Spheres with more than 7 vector fields: Blame $\mathrm{Spin}(9)$!

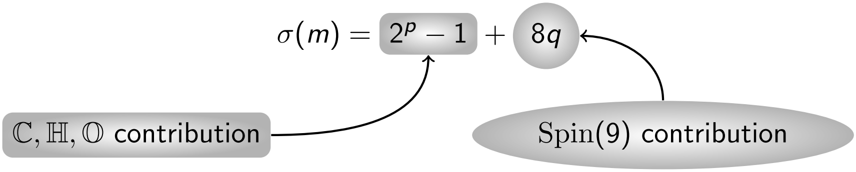
- Spheres $S^{m-1} \subset \mathbb{R}^m$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1$, 2 or 3 in

$$m = (2k+1)2^p$$

- In the general case

$$m = (2k+1)2^p 16^q \qquad \text{with } q \geq 0 \quad \text{and} \quad p = 0, 1, 2, 3$$

the maximum number of vector fields is



$$\sigma(m) = \boxed{2^p - 1} + \boxed{8q}$$

$\mathbb{C}, \mathbb{H}, \mathbb{O}$ contribution

$\mathrm{Spin}(9)$ contribution

No S^1-subfibration

Similarly to the quaternionic Hopf fibration, one would expect several S^1-subfibrations for the octonionic Hopf fibration on $S^{15} \subset \mathbb{O}^2$.

Theorem

Any global vector field on S^{15} which is tangent to the fibers of the octonionic Hopf fibration $S^{15} \to S^8$ has at least one zero.

`[Ornea-MP-Piccinni-Vuletescu, Transformation Groups, 2013]`

`[Loo-Verjovsky, Topology, 1992]`

1. What you should expect in this talk
2. $\mathrm{Spin}(9)$
3. After $\mathrm{Spin}(9)$: Rosenfeld projective planes
   - Why and what
   - $\mathrm{Spin}(10)$
   - $\mathrm{Spin}(12)$ and $\mathrm{Spin}(16)$
4. Applications
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Rosenfeld projective planes

- What are the Rosenfeld projective planes?

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Cayley plane

Cayley plane: $\mathbb{O}P^2 = \dfrac{\mathrm{F}_4}{\mathrm{Spin}(9)} = \mathrm{FII}, \quad \dim \mathrm{FII} = 16$

Rosenfeld projective planes

- What are the Rosenfeld projective planes?

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Cayley plane: $\mathbb{O}P^2 = \dfrac{\mathrm{F}_4}{\mathrm{Spin}(9)} = \mathrm{FII}, \quad \dim \mathrm{FII} = 16$

$\mathrm{EIII}, \quad \dim \mathrm{EIII} = 32$

$\mathrm{EVI}, \quad \dim \mathrm{EVI} = 64$

$\mathrm{EVIII}, \quad \dim \mathrm{EVIII} = 128$

Rosenfeld projective planes

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Cayley plane: $\mathbb{O}P^2 = \dfrac{\mathrm{F}_4}{\mathrm{Spin}(9)} = \mathrm{FII}$, $\dim \mathrm{FII} = 16$

Rosenfeld projective planes

$$(\mathbb{C} \otimes \mathbb{O})P^2 = \frac{\mathrm{E}_6}{\mathrm{Spin}(10) \cdot \mathrm{U}(1)} = \mathrm{EIII}, \quad \dim \mathrm{EIII} = 32$$

$$(\mathbb{H} \otimes \mathbb{O})P^2 = \frac{\mathrm{E}_7}{\mathrm{Spin}(12) \cdot \mathrm{Sp}(1)} = \mathrm{EVI}, \quad \dim \mathrm{EVI} = 64$$

$$(\mathbb{O} \otimes \mathbb{O})P^2 = \frac{\mathrm{E}_8}{\mathrm{Spin}(16)^+} = \mathrm{EVIII}, \quad \dim \mathrm{EVIII} = 128$$

Rosenfeld projective planes

- Focus on EIII

$$(\mathbb{C} \otimes \mathbb{O})P^2 = \frac{\mathrm{E}_6}{\mathrm{Spin}(10) \cdot \mathrm{U}(1)} = \mathrm{EIII}, \quad \dim \mathrm{EIII} = 32$$

What is $\mathrm{Spin}(n)$?

Roughly. . .

- $\mathrm{SO}(2)$: multiplication by a unitary complex number in $\mathbb{R}^2 = \mathbb{C}$
- $\mathrm{SO}(3)$: conjugation by a unitary quaternion in $\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$
- $\mathrm{SO}(4)$: conjugation by 2 unitary quaternions in $\mathbb{R}^4 = \mathbb{H}$

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...we can say that...

The action of the above Lie groups $\mathrm{SO}(n)$ on $\mathbb{R}^n$, for $n = 2, \ldots, 4$, can be described in terms of multiplication in some algebra embedding $\mathbb{R}^n$ as a subspace.

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Motivation for Clifford algebras

$\mathrm{Spin}(n)$ is the generalization of the above fact to every n. The algebra embedding the group $\mathrm{Spin}(n)$ and the vector space $\mathbb{R}^n$ is called the Clifford algebra.

▸ Go to MFD of $\mathrm{Spin}(n)$

A basis for $\mathfrak{spin}(10)$

- $\mathfrak{spin}(10) = \mathfrak{lie}\{\mathfrak{spin}(9), \mathfrak{u}(1)\} \subset \mathfrak{su}(16)$, where $\mathfrak{u}(1)$ is generated by

$$\begin{pmatrix} i\,\mathrm{Id}_8 & 0 \\ 0 & -i\,\mathrm{Id}_8 \end{pmatrix}$$

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$$= \overset{\boxed{\stackrel{\text{def}}{=} \mathcal{I}_0}}{\begin{pmatrix} i\,\mathrm{Id}_8 & 0 \\ 0 & i\,\mathrm{Id}_8 \end{pmatrix}} \cdot \underset{\boxed{= \mathcal{I}_9}}{\begin{pmatrix} \mathrm{Id}_8 & 0 \\ 0 & -\,\mathrm{Id}_8 \end{pmatrix}}$$

$\mathcal{I}_0$ is a complex structure, not an involution. It acts on the first factor of $\mathbb{C} \otimes \mathbb{O}^2$ by complex multiplication.

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Proposition

$$\mathfrak{spin}(10) = \mathrm{span}\{J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{0 \leq \alpha < \beta \leq 9}$$

Canonical $\mathrm{Spin}(10)$-form

- Denote by J^N the basis $\{J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{0\leq\alpha<\beta\leq 9}$ of $\mathfrak{spin}(10)$
- Denote by Θ^N its associated skew-symmetric 10×10 matrix of Kähler forms:

$$\Theta^N = (\theta_{\alpha\beta})_{0\leq\alpha<\beta\leq 9}$$

Canonical $\mathrm{Spin}(10)$ form: 8-form in $\mathbb{R}^{32}$

The fourth coefficient $\tau_4(\Theta^N)$ of the characteristic polynomial of Θ^N is a canonical 8-form associated with the representation $\mathrm{Spin}(10) \subset \mathrm{SU}(16)$:

$$\Phi_{\mathrm{Spin}(10)} = \sum_{0\leq\alpha_1<\alpha_2<\alpha_3<\alpha_4\leq 9} (\theta_{\alpha_1\alpha_2} \wedge \theta_{\alpha_3\alpha_4} - \theta_{\alpha_1\alpha_3} \wedge \theta_{\alpha_2\alpha_4} + \theta_{\alpha_1\alpha_4} \wedge \theta_{\alpha_2\alpha_3})^2$$

$\Phi_{\mathrm{Spin}(10)}$ generates $H^8(\mathrm{EIII})$

Cohomology of EIII

$$H^*(\ (\mathbb{C}\otimes\mathbb{O})P^2\ ) = \frac{\mathbb{R}[d_2, d_8]}{\text{(suitable relations)}} \qquad d_2\ \in H^2,\ d_8\ \in H^8$$

$\Phi_{\mathrm{Spin}(10)}$ generates $H^8(\mathrm{EIII})$

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I'm the Kähler form

I'm Hermitian symmetric

$\Phi_{\mathrm{Spin}(10)}$ generates $H^8(\mathrm{EIII})$

The d_8-Lemma

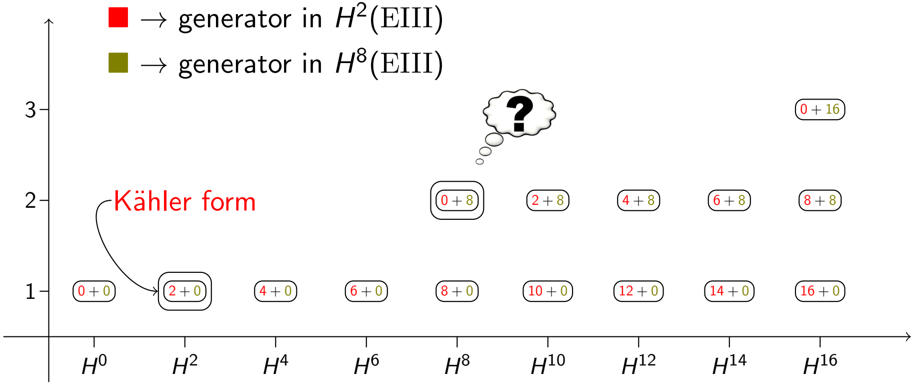
$$H^*(\ (\mathbb{C}\otimes\mathbb{O})P^2\ ) = \frac{\mathbb{R}[d_2, d_8]}{\text{(suitable relations)}} \qquad d_2\ \in H^2,\ \ d_8\ \in H^8$$

I'm the Kähler form

I'm Hermitian symmetric

What am I?

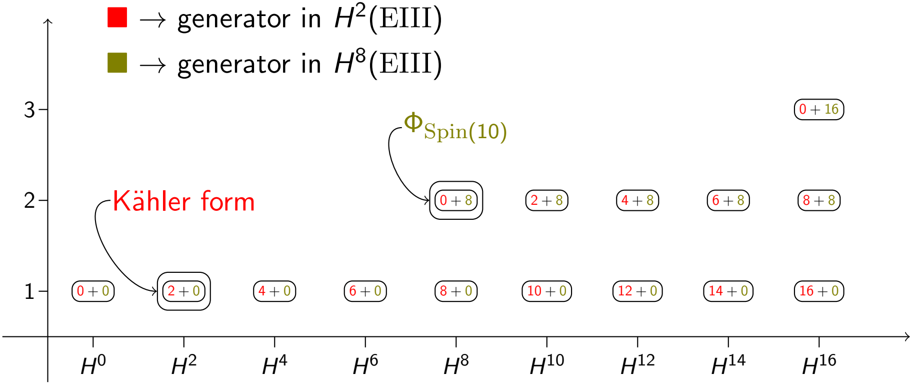
$\Phi_{\mathrm{Spin}(10)}$ generates $H^8(\mathrm{EIII})$



🟥 $\rightarrow$ generator in $H^2(\mathrm{EIII})$

🟩 $\rightarrow$ generator in $H^8(\mathrm{EIII})$

$\Phi_{\mathrm{Spin}(10)}$ generates $H^8(\mathrm{EIII})$



🟥 → generator in $H^2(\mathrm{EIII})$

🟩 → generator in $H^8(\mathrm{EIII})$

Time check

How many minutes?

▸ Go on with EVI, EVIII

▸ Skip to applications

EVI: cohomology ring

The cohomology of

$$\mathrm{EVI} = (\mathbb{H}\otimes\mathbb{O})P^2 = \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)}$$

is given by

$$H^*(\ (\mathbb{H}\otimes\mathbb{O})P^2\ ) = \frac{\mathbb{R}[d_4, d_8, d_{12}]}{\text{(suitable relations)}} \qquad d_4\ ,\ d_8\ ,\ d_{12}\ \in H^4, H^8, H^{12}$$

?

?

I'm the quaternion-Kähler form

I'm quaternion-Kähler

A basis for $\mathfrak{spin}(12) \subset \mathfrak{sp}(16)$

- Consider the matrices

$$\begin{pmatrix} i\,\mathrm{Id}_8 & 0 \\ 0 & -i\,\mathrm{Id}_8 \end{pmatrix}, \quad \begin{pmatrix} j\,\mathrm{Id}_8 & 0 \\ 0 & -j\,\mathrm{Id}_8 \end{pmatrix}, \quad \begin{pmatrix} k\,\mathrm{Id}_8 & 0 \\ 0 & -k\,\mathrm{Id}_8 \end{pmatrix}$$

  where i, j, k act as quaternionic multiplication on $\mathbb{H} \otimes \mathbb{O}^2$

- Denote by $\mathcal{I}_0, \mathcal{I}_{-1}, \mathcal{I}_{-2}$ the i, j, k multiplication

Proposition

$$\mathfrak{spin}(12) = \mathrm{span}\{J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{-2 \leq \alpha < \beta \leq 9}$$

Canonical $\mathrm{Spin}(12)$ forms: 8 and 12-form in $\mathbb{R}^{64}$

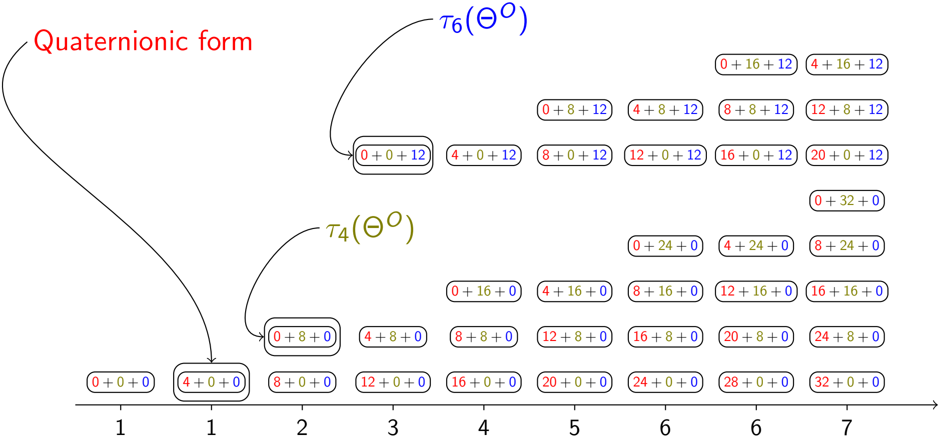
If Θ^O is the matrix of Kähler forms of $\{J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}$, then $\tau_4(\Theta^O)$ and $\tau_6(\Theta^O)$ are a canonical 8-form and a canonical 12-form on $\mathbb{R}^{64} = \mathbb{H} \otimes \mathbb{O}^2$ associated to $\mathrm{Spin}(12)$.

EVI: cohomology ring

🟥 → generator in $H^4(\mathrm{EVI})$

🟫 → generator in $H^8(\mathrm{EVI})$

🟦 → generator in $H^{12}(\mathrm{EVI})$



Quaternionic form

$\tau_6(\Theta^O)$

$\tau_4(\Theta^O)$

								$0+8+24$
						$0+0+24$	$4+0+24$	$8+0+24$
							$0+16+12$	$4+16+12$
					$0+8+12$	$4+8+12$	$8+8+12$	$12+8+12$
			$0+0+12$	$4+0+12$	$8+0+12$	$12+0+12$	$16+0+12$	$20+0+12$
								$0+32+0$
						$0+24+0$	$4+24+0$	$8+24+0$
				$0+16+0$	$4+16+0$	$8+16+0$	$12+16+0$	$16+16+0$
		$0+8+0$	$4+8+0$	$8+8+0$	$12+8+0$	$16+8+0$	$20+8+0$	$24+8+0$
$0+0+0$	$4+0+0$	$8+0+0$	$12+0+0$	$16+0+0$	$20+0+0$	$24+0+0$	$28+0+0$	$32+0+0$
1	1	2	3	4	5	6	6	7

EVIII: cohomology ring

The cohomology of

$$\mathrm{EVIII} = (\mathbb{O} \otimes \mathbb{O})P^2 = \frac{\mathrm{E}_8}{\mathrm{Spin}(16)^+}$$

is given by

$$H^*(\ (\mathbb{O} \otimes \mathbb{O})P^2\ ) = \frac{\mathbb{R}[d_8, d_{12}, d_{16}, d_{20}]}{\text{(suitable relations)}}\ \overset{?}{d_8}, \overset{?}{d_{12}}, \overset{?}{d_{16}}, \overset{?}{d_{20}} \in H^8, H^{12}, H^{16}, H^{20}$$

A basis for $\mathfrak{spin}(16) \subset \mathfrak{so}(128)$

- Denote by $\mathcal{I}_0, \mathcal{I}_{-1}, \mathcal{I}_{-2}, \mathcal{I}_{-3}, \mathcal{I}_{-4}, \mathcal{I}_{-5}, \mathcal{I}_{-6}$ the i, j, k, e, f, g, h multiplication by the octonion units on the first factor of $\mathbb{O} \otimes \mathbb{O}^2$

Proposition

$$\mathfrak{spin}(16) = \text{span}\{J_{\alpha\circ\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{-6\leq\alpha<\beta\leq 9}$$

Canonical $\mathrm{Spin}(16)$ forms: 8, 12, 16 and 20-form in $\mathbb{R}^{128}$

If Θ^R is the matrix of Kähler forms of $\{J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}$, then $\tau_4(\Theta^R)$, $\tau_6(\Theta^R)$, $\tau_8(\Theta^R)$ and $\tau_{10}(\Theta^R)$ are canonical forms associated with the standard $\mathrm{Spin}(16)$ structure on $\mathbb{R}^{128} = \mathbb{O} \otimes \mathbb{O}^2$.

EVIII: cohomology ring

The cohomology of

$$\mathrm{EVIII} = (\mathbb{O} \otimes \mathbb{O})P^2 = \frac{\mathrm{E}_8}{\mathrm{Spin}(16)^+}$$

I'm $\tau_{10}(\Theta^R)$

I'm $\tau_4(\Theta^R)$

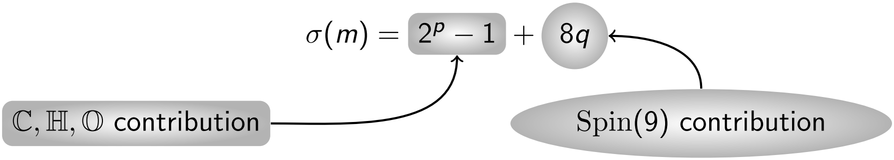
is given by

$$H^*(\ (\mathbb{O} \otimes \mathbb{O})P^2\ ) = \frac{\mathbb{R}[d_8, d_{12}, d_{16}, d_{20}]}{\text{(suitable relations)}} \quad d_8, d_{12}, d_{16}, d_{20} \in H^8, H^{12}, H^{16}, H^{20}$$

I'm $\tau_8(\Theta^R)$

I'm $\tau_6(\Theta^R)$

More than 7 vector fields on spheres? $\mathrm{Spin}(9)$'s fault...



$$\sigma(m) = \boxed{2^p - 1} + \boxed{8q}$$

$\mathbb{C}, \mathbb{H}, \mathbb{O}$ contribution

$\mathrm{Spin}(9)$ contribution

Maximal system of $\sigma(m) = 8, 9, 11, 15$ vector fields on $S^{15}, S^{31}, S^{63}, S^{127}$

Sphere	$\sigma(m)$	Vector fields	Structures involved
S^{15} $(p = 0, q = 1)$	$0 + 8$	$J_{19}, \dots, J_{89}$	$\mathrm{Spin}(9)$
S^{31} $(p = 1, q = 1)$	$1 + 8$	$i\cdot,\quad J_{19}, \dots, J_{89}$	$\mathbb{C} + \mathrm{Spin}(9)$
S^{63} $(p = 2, q = 1)$	$3 + 8$	$i\cdot, j\cdot, k\cdot, J_{19}, \dots, J_{89}$	$\mathbb{H} + \mathrm{Spin}(9)$
S^{127} $(p = 3, q = 1)$	$7 + 8$	$i\cdot, j\cdot, k\cdot, e\cdot, f\cdot, g\cdot, h\cdot,\quad J_{19}, \dots, J_{89}$	$\mathbb{O} + \mathrm{Spin}(9)$

but $\mathrm{Spin}(10)$, $\mathrm{Spin}(12)$ and $\mathrm{Spin}(16)$ are co-conspirators

$$\sigma(m) = \begin{cases} \boxed{0 + 8q} \longleftarrow \mathrm{Spin}(9) \text{ contribution} \\ \boxed{1 + 8q} \longleftarrow \mathrm{Spin}(10) \text{ contribution} \\ \boxed{3 + 8q} \longleftarrow \mathrm{Spin}(12) \text{ contribution} \\ \boxed{7 + 8q} \longleftarrow \mathrm{Spin}(16) \text{ contribution} \end{cases}$$

Maximal system of $\sigma(m) = 8, 9, 11, 15$ vector fields on $S^{15}, S^{31}, S^{63}, S^{127}$

Sphere	$\sigma(m)$	Vector fields	Structures involved
S^{15} $(p = 0, q = 1)$	$0 + 8$	$J_{19}, \dots, J_{89}$	$\mathrm{Spin}(9)$
S^{31} $(p = 1, q = 1)$	$1 + 8$	$J_{09}, J_{19}, \dots, J_{89}$	$\mathrm{Spin}(10)$
S^{63} $(p = 2, q = 1)$	$3 + 8$	$J_{(-2)9}, J_{(-1)9}, J_{09}, J_{19}, \dots, J_{89}$	$\mathrm{Spin}(12)$
S^{127} $(p = 3, q = 1)$	$7 + 8$	$J_{(-6)9}, \dots, J_{(-1)9}, J_{09}, J_{19}, \dots, J_{89}$	$\mathrm{Spin}(16)$

The dolls. . .

- Recall that the $\mathrm{Spin}(9)$ family

$$J^I = \{J_{\alpha\beta}\}_{1\leq\alpha<\beta\leq 9}$$

contains the $\mathrm{Spin}(8)$ and $\mathrm{Spin}(7)_\Delta$ subfamilies

$$J^P = \{J_{\alpha\beta}\}_{1\leq\alpha<\beta\leq 8}, \quad J^S = \{J_{\alpha\beta}\}_{2\leq\alpha<\beta\leq 8}$$

- Using the first factor multiplication in $\mathbb{C}\otimes\mathbb{O}^2$, $\mathbb{H}\otimes\mathbb{O}^2$, $\mathbb{O}\otimes\mathbb{O}^2$ we obtain the larger families

$$J^N = \{J_{\alpha\beta}\}_{0\leq\alpha<\beta\leq 9}, J^O = \{J_{\alpha\beta}\}_{-2\leq\alpha<\beta\leq 9}, J^R = \{J_{\alpha\beta}\}_{-6\leq\alpha<\beta\leq 9}$$

. . . and their Matryoshka

The Lie group inclusions

$$\mathrm{Spin}(7)_\Delta \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9) \subset \mathrm{Spin}(10) \subset \mathrm{Spin}(12) \subset \mathrm{Spin}(16)$$

are preserved by the spinor inclusions

$$J^S \subset J^P \subset J^I \subset J^N \subset J^O \subset J^R$$

. . . and their Matryoshka

The Lie group inclusions

$$\mathrm{Spin}(7)_\Delta \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9) \subset \mathrm{Spin}(10) \subset \mathrm{Spin}(12) \subset \mathrm{Spin}(16)$$

are preserved by the spinor inclusions

$$J^S \subset J^P \subset J^I \subset J^N \subset J^O \subset J^R = J^{\mathrm{SPINOR}}$$

INOR details

$$\mathfrak{so}(16) = \Lambda^2(\mathbb{R}^{16}) = \boxed{\Lambda^2_{36}} \oplus \Lambda^2_{84} = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$$

generated by $J^I = \{\mathcal{I}_\alpha \circ \mathcal{I}_\beta\}$

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$$\mathfrak{so}(16) = \Lambda^2(\mathbb{R}^{16}) = \boxed{\Lambda^2_{36}} \oplus \Lambda^2_{84} = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$$

generated by $J^I = \{\mathcal{I}_\alpha \circ \mathcal{I}_\beta\}$

Add $\mathcal{I}_0$ to obtain $J^N = \{\mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{0 \leq \alpha < \beta \leq 9}$ and

$$\operatorname{span} J^N = \mathfrak{spin}(10) \subset \mathfrak{so}(32)$$

Add $\mathcal{I}_{-1}, \mathcal{I}_{-2}$ to obtain $J^O = \{\mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{-2 \leq \alpha < \beta \leq 9}$ and

$$\operatorname{span} J^O = \mathfrak{spin}(12) \subset \mathfrak{so}(64)$$

Add $\mathcal{I}_{-6}, \ldots, \mathcal{I}_{-3}$ to obtain $J^R = \{\mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{-6 \leq \alpha < \beta \leq 9}$ and

$$\operatorname{span} J^R = \mathfrak{spin}(16) \subset \mathfrak{so}(128)$$

INOR details

$$\boxed{\mathfrak{so}(16)} = \Lambda^2(\mathbb{R}^{16}) = \boxed{\Lambda^2_{36}} \oplus \Lambda^2_{84} = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$$

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$$\operatorname{span} J^O = \mathfrak{spin}(12) \subset \mathfrak{so}(64)$$

Add $\mathcal{I}_{-6}, \ldots, \mathcal{I}_{-3}$ to obtain $J^R = \{\mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{-6 \leq \alpha < \beta \leq 9}$ and

$$\operatorname{span} J^R = \boxed{\mathfrak{spin}(16)} \subset \mathfrak{so}(128)$$

Even Clifford structures

A rank r even Clifford structure on a Riemannian manifold M^n is:

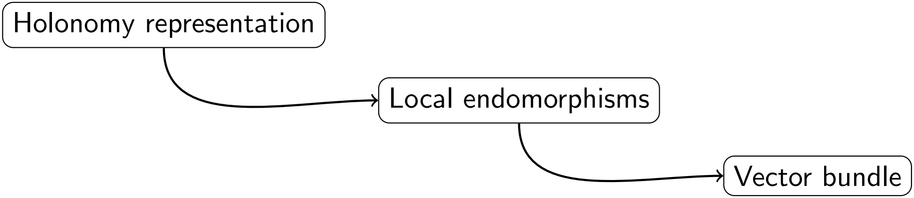
- a rank r oriented Euclidean vector bundle E over M
- an algebra bundle morphism $\phi : \mathrm{Cl}_0(E) \to \mathrm{End}(TM)$ such that $\phi(\Lambda^2 E) \subset \mathrm{End}^-(TM)$

Classification

Rank of E	M^n	n
$\dots$	$\dots$	$\dots$
9	$\mathbb{O}P^2 = \mathrm{FII} = \frac{\mathrm{F}_4}{\mathrm{Spin}(9)}$	16
10	$(\mathbb{C} \otimes \mathbb{O})P^2 = \mathrm{EIII} = \frac{\mathrm{E}_6}{\mathrm{Spin}(10) \cdot \mathrm{U}(1)}$	32
12	$(\mathbb{H} \otimes \mathbb{O})P^2 = \mathrm{EVI} = \frac{\mathrm{E}_7}{\mathrm{Spin}(12) \cdot \mathrm{Sp}(1)}$	64
16	$(\mathbb{O} \otimes \mathbb{O})P^2 = \mathrm{EVIII} = \frac{\mathrm{E}_8}{\mathrm{Spin}(16)^+}$	128

[Moroianu-Semmelmann, Adv. Math. 2011]

Explicit even Clifford structures on Rosenfeld planes

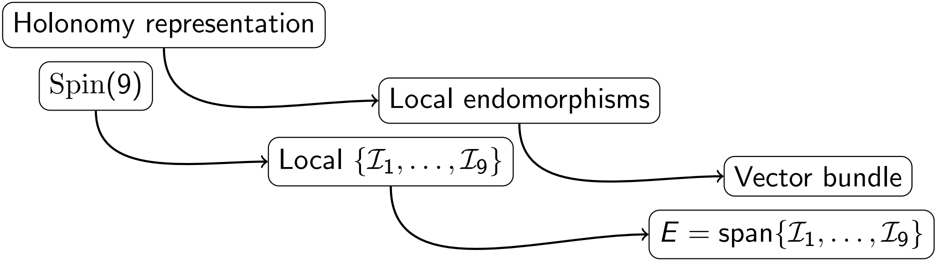


Holonomy representation

Local endomorphisms

Vector bundle

Explicit even Clifford structures on Rosenfeld planes



Holonomy representation

Spin(9)

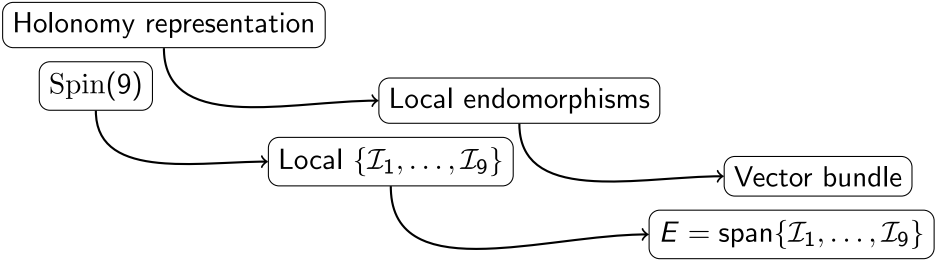
Local endomorphisms

Local $\{\mathcal{I}_1, \dots, \mathcal{I}_9\}$

Vector bundle

$E = \text{span}\{\mathcal{I}_1, \dots, \mathcal{I}_9\}$

Explicit even Clifford structures on Rosenfeld planes



Holonomy representation

Spin(9)

Local endomorphisms

Local $\{\mathcal{I}_1, \dots, \mathcal{I}_9\}$

Vector bundle

$E = \mathrm{span}\{\mathcal{I}_1, \dots, \mathcal{I}_9\}$

Rank of E	E	ϕ	M^n	n
9	$\mathrm{span}\{\mathcal{I}_1, \dots, \mathcal{I}_9\}$	$\mathcal{I}_\alpha \wedge \mathcal{I}_\beta \mapsto J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta$	$\mathbb{O}P^2$	16
10	$\mathrm{span}\{\mathcal{I}_0, \dots, \mathcal{I}_9\}$	$\mathcal{I}_\alpha \wedge \mathcal{I}_\beta \mapsto J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta$	$(\mathbb{C} \otimes \mathbb{O})P^2$	32
12	$\mathrm{span}\{\mathcal{I}_{-2}, \dots, \mathcal{I}_9\}$	$\mathcal{I}_\alpha \wedge \mathcal{I}_\beta \mapsto J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta$	$(\mathbb{H} \otimes \mathbb{O})P^2$	64
16	$\mathrm{span}\{\mathcal{I}_{-6}, \dots, \mathcal{I}_9\}$	$\mathcal{I}_\alpha \wedge \mathcal{I}_\beta \mapsto J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta$	$(\mathbb{O} \otimes \mathbb{O})P^2$	128

Wrapping up

$\mathrm{Spin}(9)$ is the octonionic version of the quaternionic group $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$.

$\mathrm{Spin}(9)$ is the reason why there are more than 7 vector fields on spheres.

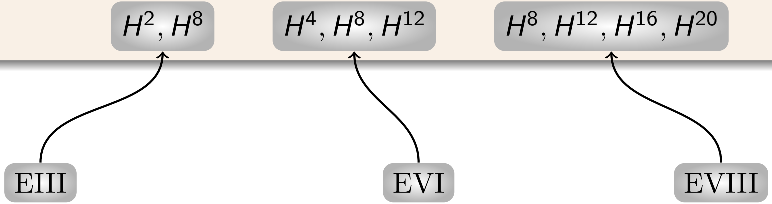
The $\mathrm{Spin}(9)$ form on $\mathbb{R}^{16} = \mathbb{O}^2$ can be extended to $\mathbb{C} \otimes \mathbb{O}^2$, $\mathbb{H} \otimes \mathbb{O}^2$ and $\mathbb{O} \otimes \mathbb{O}^2$, to obtain explicit generators for the cohomology of particular symmetric spaces called Rosenfeld planes.

Between the $\mathrm{Spin}(n)$ groups, $\mathrm{Spin}(10)$, $\mathrm{Spin}(12)$ and $\mathrm{Spin}(16)$ are more equal than others to $\mathrm{Spin}(9)$.


That's all Folks!

Homology interpretation

Interpretation in terms of the homology of the Rosenfeld planes of the canonical differential forms living in



H^2, H^8 $\quad$ H^4, H^8, H^{12} $\quad$ $H^8, H^{12}, H^{16}, H^{20}$

EIII $\quad$ EVI $\quad$ EVIII

Extending

Following

$$\{\mathcal{I}_\alpha\}_{1\leq\alpha\leq 9} \in \mathrm{End}(\mathbb{O}^2) \to \{\mathcal{I}_\alpha\}_{0\leq\alpha\leq 9} \in \mathrm{End}(\mathbb{C}\otimes\mathbb{O}^2) \to \ldots$$

what happens extending the Pauli matrices

$$\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \mathrm{End}(\mathbb{C}^2) \to \{\mathcal{I}_0, \ldots, \mathcal{I}_3\} \in \mathrm{End}(\mathbb{C}\otimes\mathbb{C}^2)$$

and the

$$\mathcal{I}_1, \ldots, \mathcal{I}_5 \in \mathrm{End}(\mathbb{H}^2) \to \{\mathcal{I}_0, \ldots, \mathcal{I}_5\} \in \mathrm{End}(\mathbb{C}\otimes\mathbb{H}^2) \to \ldots$$

Subordinated structures

Can we write formulas as in

$$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$$

for our $\mathrm{Spin}(9), \ldots, \mathrm{Spin}(16)$ canonical 8-forms in terms of compatible quaternonic structures?

Minimal formal definition of $\mathrm{Spin}(n)$

Definition

- $\mathrm{Cl}(n)$ = Clifford algebra = algebra generated by vectors $v \in \mathbb{R}^n$ such that
$$v \cdot v = -\|v\|^2 \cdot 1$$
- α = canonical involution of $\mathrm{Cl}(n)$:
$$\alpha(v) = -v \qquad \text{for vectors } v \in \mathbb{R}^n$$
- $\mathrm{Cl}_0(n)$ = +1-eigenspace of α.
- $\|\cdot\|$ = norm of $\mathrm{Cl}(n)$ = extension of $\|\cdot\|$ to $\mathrm{Cl}(n)$.
- $$\mathrm{Spin}(n) = \{x \in \mathrm{Cl}_0(n) | x\mathbb{R}^n x^{-1} \subset \mathbb{R}^n \text{and} \|x\| = 1\}$$

◂ Go back

Details for $\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$

- $\nu_l =$ volume form on the octonionic lines $l \overset{\text{def}}{=} \{(x, mx)\}$ or $l \overset{\text{def}}{=} \{(0, y)\}$ in $\mathbb{O}^2$.
- $p_l : \mathbb{O}^2 \to l =$ projection on l.
- $p_l^* \nu_l =$ 8-form in $\mathbb{O}^2 = \mathbb{R}^{16}$.
- The integral over $\mathbb{O}P^1$ can be computed over $\mathbb{O}$ with polar coordinates.
- The formula arise from distinguished 8-planes in the $\mathrm{Spin}(9)$-geometry → (forthcoming) calibrations.

◂ Go back

Berger and calibrations

Curiosity

Berger appears to know about the fact that $\Phi_{\mathrm{Spin}(9)}$ is a calibration on $\mathbb{O}P^2$ already in 1970 [Berger, L'Enseignement Math. 1970] and more explicitly in 1972 [Berger, Ann. Éc. Norm. Sup. 1972, Theorem 6.3], very early with respect to the forthcoming calibration theory.

Time check

ADATTARE E SPOSTARE

Do we have at least ... minutes left?

▸ Yes, go ahead as planned

▸ No, skip quaternionic analogy

First attempt

Remark

Since $\mathrm{Spin}(10) \subset \mathrm{SU}(16)$, we would like to imitate $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, and look for $\mathcal{I}_0, \dots, \mathcal{I}_9$ self-adjoint, anti-commuting involutions in $\mathbb{C}^{16}$.

First attempt

Remark

Since $\mathrm{Spin}(10) \subset \mathrm{SU}(16)$, we would like to imitate $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, and look for $\mathcal{I}_0, \ldots, \mathcal{I}_9$ self-adjoint, anti-commuting involutions in $\mathbb{C}^{16}$.

Would-be proposition

$\mathrm{Spin}(10) \subset \mathrm{SU}(16)$ is generated by 10 self-adjoint, anti-commuting involutions $\mathcal{I}_0, \ldots, \mathcal{I}_9$.

First attempt

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Since $\mathrm{Spin}(10) \subset \mathrm{SU}(16)$, we would like to imitate $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, and look for $\mathcal{I}_0, \ldots, \mathcal{I}_9$ self-adjoint, anti-commuting involutions in $\mathbb{C}^{16}$.

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[FALSE]

First attempt

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Would-be proposition

[FALSE]

$\mathrm{Spin}(10) \subset \mathrm{SU}(16)$ is generated by 10 self-adjoint, anti-commuting involutions $\mathcal{I}_0, \dots, \mathcal{I}_9$.

Proposition

$\mathbb{C}^{16}$ with its standard Hermitian scalar product does not admit any family of 10 self-adjoint, anti-commuting involutions $\mathcal{I}_0, \dots, \mathcal{I}_9$.

351 terms of $\Phi_{\mathrm{Spin}(9)}$

Go back

12345678		-14	123456	1′2′	2	123456	3′4′	-2	123456	5′6′	-2	123456	7′8′	-2
123457	1′3′	2	123457	2′4′	2	123457	5′7′	-2	123457	6′8′	2	123458	1′4′	2
123458	2′3′	-2	123458	5′8′	-2	123458	6′7′	-2	123467	1′4′	-2	123467	2′3′	2
123467	5′8′	-2	123467	6′7′	-2	123468	1′3′	2	123468	2′4′	2	123468	5′7′	2
123468	6′8′	-2	123478	1′2′	-2	123478	3′4′	2	123478	5′6′	-2	123478	7′8′	-2
1234	1′2′3′4′	-2	1234	5′6′7′8′	-2	123567	1′5′	-2	123567	2′6′	-2	123567	3′7′	-2
123567	4′8′	2	123568	1′6′	-2	123568	2′5′	2	123568	3′8′	-2	123568	4′7′	-2
123578	1′7′	-2	123578	2′8′	2	123578	3′5′	2	123578	4′6′	2	1235	1′2′3′5′	-1
1235	1′2′4′6′	-1	1235	1′3′4′7′	-1	1235	1′5′6′7′	-1	1235	2′3′4′8′	1	1235	2′5′6′8′	1
1235	3′5′7′8′	1	1235	4′6′7′8′	1	123678	1′8′	-2	123678	2′7′	-2	123678	3′6′	2
123678	4′5′	-2	1236	1′2′3′6′	-1	1236	1′2′4′5′	1	1236	1′3′4′8′	-1	1236	1′5′6′8′	-1
1236	2′3′4′7′	-1	1236	2′5′6′7′	-1	1236	3′6′7′8′	1	1236	4′5′7′8′	-1	1237	1′2′3′7′	-1
1237	1′2′4′8′	1	1237	1′3′4′5′	1	1237	1′5′7′8′	-1	1237	2′3′4′6′	1	1237	2′6′7′8′	-1
1237	3′5′6′7′	-1	1237	4′5′6′8′	1	1238	1′2′3′8′	-1	1238	1′2′4′7′	-1	1238	1′3′4′6′	1
1238	1′6′7′8′	-1	1238	2′3′4′5′	-1	1238	2′5′7′8′	1	1238	3′5′6′8′	-1	1238	4′5′6′7′	-1
124567	1′6′	2	124567	2′5′	-2	124567	3′8′	-2	124567	4′7′	-2	124568	1′5′	-2
124568	2′6′	-2	124568	3′7′	2	124568	4′8′	-2	124578	1′8′	-2	124578	2′7′	-2
124578	3′6′	-2	124578	4′5′	2	1245	1′2′3′6′	1	1245	1′2′4′5′	-1	1245	1′3′4′8′	-1
1245	1′5′6′8′	-1	1245	2′3′4′7′	-1	1245	2′5′6′7′	-1	1245	3′6′7′8′	-1	1245	4′5′7′8′	1
124678	1′7′	2	124678	2′8′	-2	124678	3′5′	2	124678	4′6′	2	1246	1′2′3′5′	-1
1246	1′2′4′6′	-1	1246	1′3′4′7′	1	1246	1′5′6′7′	1	1246	2′3′4′8′	-1	1246	2′5′6′8′	-1
1246	3′5′7′8′	1	1246	4′6′7′8′	1	1247	1′2′3′8′	-1	1247	1′2′4′7′	-1	1247	1′3′4′6′	-1
1247	1′6′7′8′	1	1247	2′3′4′5′	1	1247	2′5′7′8′	-1	1247	3′5′6′8′	-1	1247	4′5′6′7′	-1
1248	1′2′3′7′	1	1248	1′2′4′8′	-1	1248	1′3′4′5′	1	1248	1′5′7′8′	-1	1248	2′3′4′6′	1
1248	2′6′7′8′	-1	1248	3′5′6′7′	1	1248	4′5′6′8′	-1	125678	1′2′	-2	125678	3′4′	-2
125678	5′6′	2	125678	7′8′	-2	1256	1′2′5′6′	-2	1256	3′4′7′8′	-2	1257	1′2′5′7′	-1
1257	1′2′6′8′	1	1257	1′3′5′6′	-1	1257	1′3′7′8′	1	1257	2′4′5′6′	-1	1257	2′4′7′8′	1
1257	3′4′5′7′	-1	1257	3′4′6′8′	1	1258	1′2′5′8′	-1	1258	1′2′6′7′	-1	1258	1′4′5′6′	-1
1258	1′4′7′8′	1	1258	2′3′5′6′	1	1258	2′3′7′8′	-1	1258	3′4′5′8′	-1	1258	3′4′6′7′	-1
1267	1′2′5′8′	-1	1267	1′2′6′7′	-1	1267	1′4′5′6′	1	1267	1′4′7′8′	-1	1267	2′3′5′6′	-1
1267	2′3′7′8′	1	1267	3′4′5′8′	-1	1267	3′4′6′7′	-1	1268	1′2′5′7′	1	1268	1′2′6′8′	-1
1268	1′3′5′6′	-1	1268	1′3′7′8′	1	1268	2′4′5′6′	-1	1268	2′4′7′8′	1	1268	3′4′5′7′	1
1268	3′4′6′8′	-1	1278	1′2′7′8′	-2	1278	3′4′5′6′	-2	12	1′2′3′4′5′6′	2	12	1′2′3′4′7′8′	-2
12	1′2′5′6′7′8′	-2	12	3′4′5′6′7′8′	-2	134567	1′7′	2	134567	2′8′	2	134567	3′5′	-2
134567	4′6′	2	134568	1′8′	2	134568	2′7′	-2	134568	3′6′	-2	134568	4′5′	-2
134578	1′5′	-2	134578	2′6′	2	134578	3′7′	-2	134578	4′8′	-2	1345	1′2′3′7′	1
1345	1′2′4′8′	1	1345	1′3′4′5′	-1	1345	1′5′7′8′	-1	1345	2′3′4′6′	1	1345	2′6′7′8′	1
1345	3′5′6′7′	-1	1345	4′5′6′8′	-1	134678	1′6′	-2	134678	2′5′	-2	134678	3′8′	-2
134678	4′7′	2	1346	1′2′3′8′	1	1346	1′2′4′7′	-1	1346	1′3′4′6′	-1	1346	1′6′7′8′	-1
1346	2′3′4′5′	-1	1346	2′5′7′8′	-1	1346	3′5′6′8′	-1	1346	4′5′6′7′	1	1347	1′2′3′5′	-1
1347	1′2′4′6′	1	1347	1′3′4′7′	-1	1347	1′5′6′7′	1	1347	2′3′4′8′	-1	1347	2′5′6′8′	1
1347	3′5′7′8′	-1	1347	4′6′7′8′	1	1348	1′2′3′6′	-1	1348	1′2′4′5′	-1	1348	1′3′4′8′	-1
1348	1′5′6′8′	1	1348	2′3′4′7′	1	1348	2′5′6′7′	-1	1348	3′6′7′8′	-1	1348	4′5′7′8′	-1
135678	1′3′	-2	135678	2′4′	2	135678	5′7′	2	135678	6′8′	2	1356	1′2′5′7′	-1
1356	1′2′6′8′	-1	1356	1′3′5′6′	-1	1356	1′3′7′8′	-1	1356	2′4′5′6′	1	1356	2′4′7′8′	1
1356	3′4′5′7′	1	1356	3′4′6′8′	1	1357	1′3′5′7′	-2	1357	2′4′6′8′	-2	1358	1′3′5′8′	-1
1358	1′3′6′7′	-1	1358	1′4′5′7′	-1	1358	1′4′6′8′	-1	1358	2′3′5′7′	1	1358	2′3′6′8′	1
1358	2′4′5′8′	1	1358	2′4′6′7′	1	1367	1′3′5′8′	-1	1367	1′3′6′7′	-1	1367	1′4′5′7′	1
1367	1′4′6′8′	1	1367	2′3′5′7′	-1	1367	2′3′6′8′	-1	1367	2′4′5′8′	1	1367	2′4′6′7′	1
1368	1′3′6′8′	-2	1368	2′4′5′7′	-2	1378	1′2′5′7′	1	1378	1′2′6′8′	1	1378	1′3′5′6′	-1
1378	1′3′7′8′	-1	1378	2′4′5′6′	1	1378	2′4′7′8′	1	1378	3′4′5′7′	-1	1378	3′4′6′8′	-1
13	1′2′3′4′5′7′	2	13	1′2′3′4′6′8′	2	13	1′3′5′6′7′8′	-2	13	2′4′5′6′7′8′	2	145678	1′4′	-2
145678	2′3′	-2	145678	5′8′	2	145678	6′7′	-2	1456	1′2′5′8′	-1	1456	1′2′6′7′	1
1456	1′4′5′6′	-1	1456	1′4′7′8′	-1	1456	2′3′5′6′	-1	1456	2′3′7′8′	-1	1456	3′4′5′8′	1
1456	3′4′6′7′	-1	1457	1′3′5′8′	-1	1457	1′3′6′7′	1	1457	1′4′5′7′	-1	1457	1′4′6′8′	1
1457	2′3′5′7′	-1	1457	2′3′6′8′	1	1457	2′4′5′8′	-1	1457	2′4′6′7′	1	1458	1′4′5′8′	-2
1458	2′3′6′7′	-2	1467	1′4′6′7′	-2	1467	2′3′5′8′	-2	1468	1′3′5′8′	-1	1468	1′3′6′7′	1
1468	1′4′5′7′	1	1468	1′4′6′8′	-1	1468	2′3′5′7′	1	1468	2′3′6′8′	-1	1468	2′4′5′8′	-1
1468	2′4′6′7′	1	1478	1′2′5′8′	1	1478	1′2′6′7′	-1	1478	1′4′5′6′	-1	1478	1′4′7′8′	-1
1478	2′3′5′6′	-1	1478	2′3′7′8′	-1	1478	3′4′5′8′	-1	1478	3′4′6′7′	1	14	1′2′3′4′5′8′	2
14	1′2′3′4′6′7′	-2	14	1′4′5′6′7′8′	-2	14	2′3′5′6′7′8′	-2	1567	1′2′3′5′	-1	1567	1′2′4′6′	1
1567	1′3′4′7′	1	1567	1′5′6′7′	-1	1567	2′3′4′8′	1	1567	2′5′6′8′	-1	1567	3′5′7′8′	-1
1567	4′6′7′8′	1	1568	1′2′3′6′	-1	1568	1′2′4′5′	-1	1568	1′3′4′8′	1	1568	1′5′6′8′	-1
1568	2′3′4′7′	-1	1568	2′5′6′7′	1	1568	3′6′7′8′	-1	1568	4′5′7′8′	-1	1578	1′2′3′7′	-1
1578	1′2′4′8′	-1	1578	1′3′4′5′	-1	1578	1′5′7′8′	-1	1578	2′3′4′6′	1	1578	2′6′7′8′	1
1578	3′5′6′7′	1	1578	4′5′6′8′	1	15	1′2′3′5′6′7′	-2	15	1′2′4′5′6′8′	-2	15	1′3′4′5′7′8′	-2
15	2′3′4′6′7′8′	2	1678	1′2′3′8′	-1	1678	1′2′4′7′	1	1678	1′3′4′6′	-1	1678	1′6′7′8′	-1
1678	2′3′4′5′	-1	1678	2′5′7′8′	-1	1678	3′5′6′8′	1	1678	4′5′6′7′	-1	16	1′2′3′5′6′8′	-2
16	1′2′4′5′6′7′	2	16	1′3′4′6′7′8′	-2	16	2′3′4′5′7′8′	-2	17	1′2′3′5′7′8′	-2	17	1′2′4′6′7′8′	2
17	1′3′4′5′6′7′	2	17	2′3′4′5′6′8′	2	18	1′2′3′6′7′8′	-2	18	1′2′4′5′7′8′	-2	18	1′3′4′5′6′8′	2
18	2′3′4′5′6′7′	-2												