Almost compact Clifford-Klein forms

Aleksy Tralle
(joint work with Maciej Bocheński)

University of Warmia and Mazury

Clifford-Klein forms

- G/H a homogeneous space of a connected semisimple real Lie group G with finite center.
- G/H admits an almost compact Clifford-Klein form, if there exists a discrete and not virtually abelian subgroup $\Gamma \subset G$ acting discontinously on G/H.
- G/H admits a compact Clifford-Klein form, if there exists a discrete subgroup $\Gamma \subset G$ acting discontinuosly on G/H and with compact quotient $\Gamma \setminus G/H$.

Benoist theorem

Theorem

If G/H admits compact Clifford-Klein forms, it necessarily admits almost compact ones (but not vice versa).

Purpose of the work

Find a way of checking when certain types of homogenous spaces G/H admit or do not admit almost compact Clifford-Klein forms.

What was known?

A result of Benoist, Ann. Math., 1996

A criterion of existence of non virtually abelian Γ expressed in terms of $\mathfrak{g}$ and $\mathfrak{h}$.

Data required for the Benoist criterion

- reductivity of the pair $(\mathfrak{g},\mathfrak{h})$,
- compatible Cartan decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \;\; \mathfrak{h} = \mathfrak{k}_h \oplus \mathfrak{p}_h$$

- maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}, \mathfrak{a}_h \subset \mathfrak{p}_h$,
- Weyl chamber $\mathfrak{a}^+$ determined by the system of *reduced* roots of $\mathfrak{g}$, its Weyl group and a special convex cone $\mathfrak{b}^+ \subset \mathfrak{a}^+$.

Benoist criterion

Γ exists if and only if

$$\mathfrak{b}^+ \not\subset \cup_{w \in W} w\mathfrak{a}_h.$$

Difficulties in checking it

If one tries to check the condition in terms of $\mathfrak{g}, \mathfrak{h}$, one needs to know the embedding of $\mathfrak{h}$ in $\mathfrak{g}$ expressed in some calculable terms, e.g. in terms of the Satake diagrams. This is not always possible.

What is an added value? B-T, 2014

- Conditions are expressed in terms of Lie algebras $\mathfrak{g}, \mathfrak{h}$,
- They do not depend on the embedding of $\mathfrak{h} \subset \mathfrak{g}$,
- They are expressed directly in terms of an invariant $\tilde{d}(\mathfrak{g})$ ($\tilde{d}(\mathfrak{h})$) called the *a-hyperbolic rank*,
- $\tilde{d}(\mathfrak{g})$ and $\tilde{d}(\mathfrak{h})$ can be read off directly from the Satake diagrams $S_{\mathfrak{g}}$ and $S_{\mathfrak{h}}$.
- New classes of homogeneous spaces appear.

Main result, B-T, 2014

Theorem

Let G be a connected and semisimple Lie group and let H be a reductive subgroup with compact center and finite number of connected components. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the appropriate Lie algebras. Then

1. *If $\tilde{d}(\mathfrak{g}) = \tilde{d}(\mathfrak{h})$ then G/H does not admit almost compact (and, therefore, compact) Clifford-Klein forms.*
2. *If $rank_{\mathbb{R}}(\mathfrak{g}) = rank_{\mathbb{R}}(\mathfrak{h})$, then G/H does not admit almost compact (and, therefore, compact) Clifford-Klein forms.*
3. *If $\tilde{d}(\mathfrak{g}) > rank_{\mathbb{R}}(\mathfrak{h})$ then G/H admits almost compact Clifford-Klein forms.*

Definition of $\mathfrak{b}^+$, preliminaries

Fix a Cartan subalgebra $\mathfrak{j}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$. Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{j}^{\mathbb{C}})$, be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{j}^{\mathbb{C}}$. Consider the subalgebra

$$\mathfrak{j} := \{X \in \mathfrak{j}^{\mathbb{C}} \,|\forall_{\alpha \in \Delta}\ \alpha(X) \in \mathbb{R}\},$$

which is a real form of $\mathfrak{j}^{\mathbb{C}}$. Choose a subsystem Δ^+ of positive roots in Δ. Then

$$\mathfrak{j}^+ := \{X \in \mathfrak{j} \,|\forall_{\alpha \in \Delta^+}\ \alpha(X) \geq 0\}$$

is the closed Weyl chamber for the Weyl group $W_{\mathfrak{g}^{\mathbb{C}}}$ of Δ.

Weighted Dynkin diagrams

Let Π be a simple root system for Δ^+. For every $X \in \mathrm{j}$ we define

$$\Psi_X : \Pi \to \mathbb{R},\ \alpha \to \alpha(X).$$

The above map is called the ***weighted Dynkin diagram*** of $X \in \mathrm{j}$, and the value $\alpha(X)$ is the weight of the node α. Since Π is a base of the dual space j^*, the map

$$\Psi : \mathrm{j} \to \mathit{Map}(\Pi, \mathbb{R}),\ X \to \Psi_X$$

is a linear isomorphism. We see, that

$$\Psi|_{\mathrm{j}^+} : \mathrm{j}^+ \to \mathit{Map}(\Pi, \mathbb{R}_{\geq 0}),\ X \to \Psi_X$$

is bijective.

Definition of $\mathfrak{b}^+$

Let w_0 be the longest element of $W_{\mathfrak{g}^{\mathbb{C}}}$. The action of w_0 sends $\mathfrak{j}^+$ to $-\mathfrak{j}^+$, $X \to -X$. Define

$$-w_0 : \mathfrak{j} \to \mathfrak{j},\ X \to -(wX).$$

This is an involutive automorphism of $\mathfrak{j}$, which preserves $\mathfrak{j}^+$. Then Ψ and $-w$ induce the linear automorphism $\iota = \Psi \circ (-w) \circ \Psi^{-1}$ of $Map(\Pi, \mathbb{R})$.

Fact

$$\iota(\mathfrak{a}^+) \subset \mathfrak{a}^+.$$

Definition

$$\mathfrak{b}^+ = (\mathfrak{a}^+)^\iota$$

this is a convex cone.

The definition of the a-hyperbolic rank

We see that $\iota(\mathfrak{a}^+) = \mathfrak{a}^+$ therefore we can define the convex cone

$$\mathfrak{b}^+ \subset \mathfrak{a}^+$$

as the set of all fixed points of ι in $\mathfrak{a}^+$.

Definition

The dimension of $\mathfrak{a}^+$ is called the real rank ($rank_{\mathbb{R}}(\mathfrak{g})$) of $\mathfrak{g}$. The dimension of $\mathfrak{b}^+$ is called the a-hyperbolic rank of $\mathfrak{g}$ and is denoted by $\tilde{d}(\mathfrak{g})$. Here

$$\dim \mathfrak{b}^+ := \dim \; Span_{\mathbb{R}}(\mathfrak{b}^+).$$

Calculation of $\tilde{d}(\mathfrak{g})$: Satake diagrams

Complex involution $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$. Define the involution σ^* on $(\mathfrak{j}^{\mathbb{C}})^*$ by the formula

$$(\sigma^*\varphi)(X) = \overline{\varphi(\sigma(X))},\ \forall X \in \mathfrak{j}^{\mathbb{C}}.$$

If $\alpha \in \Delta$, then $\sigma^*\alpha \in \Delta$. Let

$$\Delta_0 = \{\alpha \in \Delta \,|\, \alpha|_{\mathfrak{a}} = 0\}.$$

Put $\Delta_1 = \Delta \setminus \Delta_0$. Then $\sigma^*(\Delta_0) \subset \Delta_0$, and $\sigma^*(\Delta_1) = \Delta_1$. Put $\Pi_0 = \Pi \cap \Delta_0$ and $\Pi_1 = \Pi \cap \Delta_1$. Recall that the *Satake diagram* for $\mathfrak{g}$ is defined as follows. One takes the Dynkin diagram for $\mathfrak{g}^{\mathbb{C}}$ and paints vertexes from Π_0 in black and vertexes from Π_1 in white. Next, one shows that σ^* determines an involution $\tilde{\sigma}$ on Π_1 defined by the equation

$$\sigma^*\alpha - \beta = \sum_{\gamma \in \Pi_0} k_\gamma \gamma,\ k_\gamma \geq 0.$$

By definition, if the above equality holds for α and β, then $\tilde{\sigma}\alpha = \beta$. Now the construction of the Satake diagram is completed by joining by arrows the white vertexes transformed into each other by $\tilde{\sigma}$.

Calculation of $\tilde{d}(\mathfrak{g})$

Definition

$\Psi_X \in \mathit{Map}(\Pi, \mathbb{R})$ -weighted Dynkin, $S_\mathfrak{g}$-Satake. Ψ_X *matches* $S_\mathfrak{g}$ if all black nodes in $S_\mathfrak{g}$ have weights equal 0 in Ψ_X, and every two nodes joined by an arrow have the same weights.

Theorem A

The map $\Psi : \mathfrak{j} \to \mathit{Map}(\Pi, \mathbb{R})$ *yields*

$$\Psi|_\mathfrak{a} : \mathfrak{a} \to \{\Psi_X \text{ mathes } S_\mathfrak{g}\}(\cong)$$

Theorem B

$$\psi|_{\mathfrak{b}^+} : \mathfrak{b}^+ \to \{\Psi_X \text{ matches } S_\mathfrak{g}^\iota\}(\cong)$$

Calculation algorithm

Step 1. We calculate the a-hyperbolic rank separately for every simple part of $\mathfrak{g}$ and add results.
Step 2. We calculate the a-hyperbolic rank for simple $\mathfrak{g}$ ($\dim(\mathfrak{g}) = n$) by taking the weighted Dynkin diagrams matching $S_{\mathfrak{g}}$ and preserved by ι. We interpret weights of a given weighted Dynkin diagram as coordinates of a vector in $\mathbb{R}^n$. All vectors constructed this way give us the convex cone which has dimension equal to $\tilde{d}(\mathfrak{g})$.

Part 2: proof of the main theorem

We postpone the explanations of Theorem A and B and explain the main theorem first.

Ingredient 1: Antipodal hyperbolic orbits

Hyperbolic elements

$X \in \mathfrak{g}$ is hyperbolic, if X is semisimple (that is, ad_X is diagonalizable) and all eigenvalues of ad_X are real.

Definition of antypodal hyperbolic orbits

An adjoint orbit $O_X := Ad(G)X$ is said to be hyperbolic if X (and therefore every element of O_X) is hyperbolic. An orbit O_Y is antipodal if $-Y \in O_Y$ (and therefore for every $Z \in O_Y$, $-Z \in O_Y$).

Ingredient 2: 4 facts

Theorem 1

There is a bijective correspondence between vectors X in $\mathfrak{b}^+$ and hyperbolic antipodal orbits O_X

Theorem 2

Any antipodal O_X intersects $\mathfrak{a}$ as a single W-orbit.

Benoist criterion

Γ *exists if and only if* $\mathfrak{b}^+ \not\subset \cup_{w \in W} w\mathfrak{a}_h$.

Theorem 3

If $X \in \mathfrak{b}_h^+$, *then* $Ad(G)(X)$ *is still antipodal and hyperbolic.*

Proof of the main theorem

Choose $X \in \mathfrak{b}_h^+ \implies Ad(G)(X)$ is antipodal and hyperbolic $\implies$ there exists $Y \in \mathfrak{b}^+$ such that $Ad(G)(X) = Ad(G)(Y) \implies$ (by Theorem 2)

$$X = wY,\ w \in W.$$

Hence

$$\mathfrak{b}_h^+ \subset W\, Span(\mathfrak{b}^+)$$

these are convex cones:

$$\mathfrak{b}_h^+ \subset w \cdot Span(\mathfrak{b}^+).$$

By assumption, $\dim\ Span(\mathfrak{b}_h^+) = \dim\ Span\,(\mathfrak{b}^+)$, hence

$$Span(\mathfrak{b}_h^+) = w \cdot Span(\mathfrak{b}^+)$$

thus

$$w \cdot Span(\mathfrak{b}^+) = Span(\mathfrak{b}_h^+) \subset \mathfrak{a}_h$$

and Γ does not exist (the first case of the Theorem).

New examples

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ a closed subgroup.
The following examples are obtained by calculating the a-hyperbolic ranks of the corresponding G and H (according to Table 1).

Examples of non-existence

The following homogeneous spaces do not admit compact Clifford-Klein forms:

$$SL(4k+2l,\mathbb{R})/SO(2k,2k) \times Sp(l,\mathbb{R});$$

$$SL(2k+2l,\mathbb{R})/Sp(k,\mathbb{R}) \times Sp(l,\mathbb{R});$$

$$SL(4k+4l,\mathbb{R})/SO(2k,2k) \times SO(2l,2l);$$

$$SL(4k+2l+1,\mathbb{R})/SO(2k,2k) \times SO(l,l+1);$$

$$SU^*(4k+2)/U(s,r-s)\times Sp(t,2k+1-r-t),\ \text{for } s+t=k+1,\ 1 \leq r \leq 2k+1$$

$$SU^*(4k)/U(s,r-s) \times Sp(t,2k+1-r-t),\ \text{for } s+t=k,\ 1 \leq r \leq 2k.$$

Examples of existence

The following homogeneous spaces admit almost compact Clifford-Klein forms:

New examples

$$SL(2k+2l+2,\mathbb{R})/SO(k,k+1)\times SO(l,l+1);$$

$$SL(2k+2l+2,\mathbb{R})/SO(k,k)\times SO(l,l);$$

$$E_6^I/\{SL(3,\mathbb{C})\times SU(2,1)\}/\mathbb{Z}_3$$

New non-existence for compact Clifford-Klein

Theorem

Assume that $G = E_6^{IV}, SO^*(6), SL(3,\mathbb{R})$ and H is a non-compact subgroup of reductive type. Then G/H does not admit compact Clifford-Klein forms.

Example: Okuda's results on symmetric spaces, J. Different. Geom., 2013

Symmetric spaces

$(G, H), G_0^\sigma \subset H \subset G^\sigma, \sigma \in Aut(G), \sigma^2 = id.$

Okuda's Theorem

There is a complete classification of all pairs (G, H) which admit almost compact Clifford-Klein forms.

Example: 3-symmetric spaces, B-T., 2014

3-symmetric spaces

$(G, H), G_0^\sigma \subset H \subset G^\sigma, \sigma \in Aut(G), \sigma^3 = id.$

Classification theorem, B-T

There is a classification of 3-symmetric (G, H) with simple G admitting almost compact Clifford-Klein forms.

Come back to Theorems 1, 2, 3

1. the correspondence between $\mathfrak{a}^+$ and the set of hyperbolic orbits is "more or less" clear from the definition of $\mathfrak{a}$,
2. It is sufficient to prove that $A \in OX$ if and only if $(-w_0)X = X$.

To prove (2) observe: if $A \in O_X$ (hyperbolic and antipodal) $\implies$ $-X \in -\mathfrak{a}^+$, but both $\mathfrak{a}^+$ and $-\mathfrak{a}^+$ are the Weyl chambers. The Weyl group acts simply transitively on Weyl chambers $\implies$

$$-X = wX \implies w = w_0$$

Hence

$$w_0\mathfrak{a}^+ = -\mathfrak{a}^+ \implies -w_0X = X \implies X \in \mathfrak{b}^+.$$