COHOMOLOGICAL COMPONENTS OF MODULES

Valdemar V. Tsanov

Ruhr-Universität Bochum

47th Seminar Sophus Lie
Schloss Rauischholzhausen
29 - 31 May 2014

Idea

$\iota : G \hookrightarrow \widetilde{G}$ connected semisimple complex Lie groups.

Branching representations: $\widetilde{V} = \bigoplus V_j$

Embeddings of flag varieties: $G/B \hookrightarrow \widetilde{G}/\widetilde{B}$, if $B = G \cap \widetilde{B}$.

$$\begin{array}{ccc} H(\widetilde{G}/\widetilde{B}, \mathcal{L}) & \xrightarrow{\pi} & H(G/B, \mathcal{L}) \\ \| & & \| \\ \widetilde{V}^* & & V^* \end{array}$$

$$V \xrightarrow{\pi^*} \widetilde{V}$$

Cohomological component

Borel-Weil-Bott Theorem

$G \supset B \supset T$ Borel and Cartan subgroups; W Weyl group.

$\Lambda \supset \Lambda^+$ weight lattice and dominant chamber.

$\Delta = \Delta^+ \sqcup \Delta^-$ root system; $\rho = \frac{1}{2}\langle \Delta^+ \rangle$.

$w \cdot \lambda = w(\lambda + \rho) - \rho$ shifted action of W on Λ.

$\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$ line bundle on G/B.

BWB Theorem:

$$H^q(G/B, \mathcal{L}_\lambda) \cong \begin{cases} V^*_{w\cdot\lambda} & , \textit{ if } w\cdot\lambda \in \Lambda^+ \textit{ and } \ell(w) = q \\ 0 & , \textit{ otherwise.} \end{cases}$$

$$\mu \in \Lambda^+ \quad , \quad V^*_\mu = H^{\ell(w)}(G/B, \mathcal{L}_{w^{-1}\cdot\mu})$$

Setting and tasks

Choose $\widetilde{T} \supset \widetilde{B} \supset \widetilde{G}$ with $B = G \cap \widetilde{B}$ and $T = G \cap \widetilde{T}$.

$$\widetilde{\Lambda} \xrightarrow{\iota^*} \Lambda \quad , \quad \widetilde{\Lambda}^+ \xrightarrow{\iota^*} \Lambda^+$$

Fix $\widetilde{\lambda} \in \widetilde{\Lambda}$ and put $\lambda = \iota^*(\widetilde{\lambda})$, $\quad \widetilde{\mu} = \widetilde{w} \cdot \widetilde{\lambda}$, $\quad \mu = w \cdot \lambda$.

$$\widetilde{V}^*_{\widetilde{\mu}} \cong H(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}}) \xrightarrow{\pi_{\widetilde{\lambda}}} H(G/B, \mathcal{L}_\lambda) = V^*_\mu$$

Setting and tasks

Choose $\widetilde{T} \supset \widetilde{B} \supset \widetilde{G}$ with $B = G \cap \widetilde{B}$ and $T = G \cap \widetilde{T}$.

$$\widetilde{\Lambda} \xrightarrow{\iota^*} \Lambda \quad , \quad \widetilde{\Lambda}^+ \xrightarrow{\iota^*} \Lambda^+$$

Fix $\widetilde{\lambda} \in \widetilde{\Lambda}$ and put $\lambda = \iota^*(\widetilde{\lambda})$, $\quad \widetilde{\mu} = \widetilde{w} \cdot \widetilde{\lambda}$, $\quad \mu = w \cdot \lambda$.

$$\widetilde{V}^*_{\widetilde{\mu}} \cong H(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}}) \xrightarrow{\pi_{\widetilde{\lambda}}} H(G/B, \mathcal{L}_\lambda) = V^*_\mu$$

Questions:

(I) $\pi_{\widetilde{\lambda}} \neq 0 \iff$???

(II) Given $\widetilde{V}_{\widetilde{\mu}} = \oplus V_j$, which components are cohomological?

(III) $\mathcal{C} = \{ \binom{\widetilde{\mu}}{\mu} \in \widetilde{\Lambda}^+ \times \Lambda^+ : V_\mu \subset \widetilde{V}_{\widetilde{\mu}} \ \textit{coho.} \} = ?$

First remarks

Global sections: If $\widetilde{\lambda} \in \widetilde{\Lambda}^+$, then $\lambda = \iota^*(\widetilde{\lambda}) \in \Lambda^+$ and

$$\widetilde{V}^*_{\widetilde{\lambda}} \cong H^0(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}}) \xrightarrow{\pi} H^*(G/B, \mathcal{L}_\lambda) = V^*_\lambda$$

$\pi^* : V_\lambda = \mathfrak{U}(\mathfrak{g}) v^{\widetilde{\lambda}} \subset \widetilde{V}_{\widetilde{\lambda}}$ Cartan component

$\mathcal{C} \supset \mathcal{C}^0 = \{ \binom{\widetilde{\lambda}}{\iota^* \widetilde{\lambda}} : \widetilde{\lambda} \in \widetilde{\Lambda}^+ \}$

First remarks

Global sections: If $\widetilde{\lambda} \in \widetilde{\Lambda}^+$, then $\lambda = \iota^*(\widetilde{\lambda}) \in \Lambda^+$ and

$$\widetilde{V}^*_{\widetilde{\lambda}} \cong H^0(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}}) \xrightarrow{\pi} H^*(G/B, \mathcal{L}_\lambda) = V^*_\lambda$$

$\pi^* : V_\lambda = \mathfrak{U}(\mathfrak{g}) v^{\widetilde{\lambda}} \subset \widetilde{V}_{\widetilde{\lambda}}$ Cartan component

$\mathcal{C} \supset \mathcal{C}^0 = \{ \binom{\widetilde{\lambda}}{\iota^* \widetilde{\lambda}} : \widetilde{\lambda} \in \widetilde{\Lambda}^+ \}$

Higher degree: $SO_5 \hookrightarrow SO_5 \times SO_5$ diagonal.

$$\begin{array}{ccc} H^2(\frac{SO_5}{B} \times \frac{SO_5}{B}, \mathcal{L}_{s_1 \cdot \omega_1} \boxtimes \mathcal{L}_{s_1 \cdot \omega_1}) & \xrightarrow{\pi = 0} & H^2(\frac{SO_5}{B}, \mathcal{L}_{s_1 s_2 \cdot 3\omega_1}) \\ \| & & \| \\ V_{\omega_1} \otimes V_{\omega_1} & & V_{3\omega_1} \end{array}$$

Example: diagonal embeddings and tensor products

$\iota = \delta : G \hookrightarrow G \times G = \widetilde{G}$, then $\widetilde{B} = B \times B$, $\widetilde{W} = W \times W$ ...

Put $X = G/B$, $\lambda = \lambda_1 + \lambda_2$

$$H(X, \mathcal{L}_{\lambda_1}) \otimes H(X, \mathcal{L}_{\lambda_2}) \cong H(X \times X, \mathcal{L}_{\lambda_1} \boxtimes \mathcal{L}_{\lambda_2}) \xrightarrow{\pi} H(X, \mathcal{L}_\lambda)$$

$$V_{w \cdot \lambda} \xrightarrow{\pi^*} V_{w_1 \cdot \lambda_1} \otimes V_{w_2 \cdot \lambda_2}$$

Example: diagonal embeddings and tensor products

$\iota = \delta : G \hookrightarrow G \times G = \widetilde{G}$, then $\widetilde{B} = B \times B$, $\widetilde{W} = W \times W$ ...

Put $X = G/B$, $\lambda = \lambda_1 + \lambda_2$

$$H(X, \mathcal{L}_{\lambda_1}) \otimes H(X, \mathcal{L}_{\lambda_2}) \cong H(X \times X, \mathcal{L}_{\lambda_1} \boxtimes \mathcal{L}_{\lambda_2}) \xrightarrow{\pi} H(X, \mathcal{L}_\lambda)$$

$$V_{w\cdot\lambda} \xrightarrow{\pi^*} V_{w_1\cdot\lambda_1} \otimes V_{w_2\cdot\lambda_2}$$

Theorem [Dimitrov & Roth]

$$\pi \neq 0 \iff \Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2} \quad , \quad \Phi_w = \Delta^+ \cap w^{-1}(\Delta^-)$$

For classical groups (conjecturally always):
Cohomological comp. = PRV comp. of stable multiplicity 1.

Bott's reciprocity and Kostant's harmonics

$$\mathrm{Mult}_G(V_\mu, H^q(G/B, \mathcal{L}_\lambda)) = \mathrm{Mult}_T(\mathbb{C}_\lambda, H^q(\mathfrak{n}, V_\mu))$$

Bott's reciprocity and Kostant's harmonics

$$\mathrm{Mult}_G(V_\mu, H^q(G/B, \mathcal{L}_\lambda)) = \mathrm{Mult}_T(\mathbb{C}_\lambda, H^q(\mathfrak{n}, V_\mu))$$

$$\begin{aligned} H^q(G/B, \mathcal{L}_\lambda) \cong H^{0,q}(G/B, \mathcal{L}_\lambda) \quad & \cong H^q(\mathfrak{n}, \mathbb{C}[G] \otimes \mathbb{C}_{-\lambda})^T \\ & \cong \bigoplus_\mu H^q(\mathfrak{n}, V_\mu^* \otimes V\mu \otimes \mathbb{C}_{-\lambda})^T \\ & \cong \bigoplus_\mu V_\mu^* \otimes [H^q(\mathfrak{n}, V_\mu) \otimes \mathbb{C}_{-\lambda}]^T \end{aligned}$$

Bott's reciprocity and Kostant's harmonics

$$\mathrm{Mult}_G(V_\mu, H^q(G/B, \mathcal{L}_\lambda)) = \mathrm{Mult}_T(\mathbb{C}_\lambda, H^q(\mathfrak{n}, V_\mu))$$

$$\begin{aligned} H^q(G/B, \mathcal{L}_\lambda) \cong H^{0,q}(G/B, \mathcal{L}_\lambda) &\cong H^q(\mathfrak{n}, \mathbb{C}[G] \otimes \mathbb{C}_{-\lambda})^T \\ &\cong \bigoplus_\mu H^q(\mathfrak{n}, V_\mu^* \otimes V\mu \otimes \mathbb{C}_{-\lambda})^T \\ &\cong \bigoplus_\mu V_\mu^* \otimes [H^q(\mathfrak{n}, V_\mu) \otimes \mathbb{C}_{-\lambda}]^T \end{aligned}$$

$$H^q(\mathfrak{n}, V_\mu) = \bigoplus_{\ell(w)=q} H(\mathfrak{n}, V_\mu)^{w^{-1}\cdot\mu} \quad , \quad \mathcal{H}(\mathfrak{n}, V_\mu)^{w^{-1}\cdot\mu} = \mathbb{C}e_w^* \otimes v^{w^{-1}(\mu)}$$

$$e_w^* = \wedge_{\alpha\in\Phi_w} e_\alpha^* \in \Lambda\mathfrak{n}^*$$

Main result

Bott-Kostant: $H(G/B, \mathcal{L}_\lambda) \cong e_w^* \otimes \left(V_{w\cdot\lambda}^* \otimes v^{w^{-1}(w\cdot\lambda)}\right) \otimes z_{-\lambda}$

$$\widetilde{V}_{\widetilde{\mu}}^* \;\cong\; H(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}}) \xrightarrow{\pi} H(G/B, \mathcal{L}_\lambda) \;\cong\; V_\mu^*$$

Main result

Bott-Kostant: $H(G/B, \mathcal{L}_\lambda) \cong e_w^* \otimes (V_{w\cdot\lambda}^* \otimes v^{w^{-1}(w\cdot\lambda)}) \otimes z_{-\lambda}$

$$\widetilde{V}_{\widetilde{\mu}}^* \;\cong\; H(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}}) \;\xrightarrow{\pi}\; H(G/B, \mathcal{L}_\lambda) \;\cong\; V_\mu^*$$

Theorem: $\pi \neq 0 \iff \pi(\mathcal{H}arm) \cap \mathcal{H}arm \neq 0 \iff$

(i) $\widetilde{e}_{\widetilde{w}}^* \xmapsto{\iota^*} a e_w^*$, $a \in \mathbb{C}^\times$

(ii) $\mathrm{Hom}_G(V_\mu, \mathfrak{U}(\mathfrak{g})\widetilde{v}^{\widetilde{w}^{-1}(\widetilde{\mu})}) \neq 0$

Main result

Bott-Kostant: $H(G/B,\mathcal{L}_\lambda) \cong e_w^* \otimes (V^*_{w\cdot\lambda} \otimes v^{w^{-1}(w\cdot\lambda)}) \otimes z_{-\lambda}$

$$\widetilde{V}^*_{\widetilde{\mu}} \cong H(\widetilde{G}/\widetilde{B},\mathcal{L}_{\widetilde{\lambda}}) \xrightarrow{\pi} H(G/B,\mathcal{L}_\lambda) \cong V^*_\mu$$

Theorem: $\pi \neq 0 \iff \pi(\mathcal{H}arm) \cap \mathcal{H}arm \neq 0 \iff$
(i) $\widetilde{e}^*_{\widetilde{w}} \overset{\iota^*}{\longmapsto} ae^*_w$, $a \in \mathbb{C}^\times$
(ii) $\mathrm{Hom}_G(V_\mu, \mathfrak{U}(\mathfrak{g})\widetilde{v}^{\widetilde{w}^{-1}(\widetilde{\mu})}) \neq 0$

Furthermore, (i) $\Longrightarrow$ (ii) for $k\mu$, so $\pi_{w^{-1}\cdot k\mu} \neq 0$ for some $k \in \mathbb{N}$.

$$\mathcal{C} = \bigcup_{\widetilde{w},w:\text{(i)}} \mathcal{C}_{\widetilde{w},w}\ ,\ \mathcal{C}_{\widetilde{w},w} = \{\binom{\widetilde{\mu}}{\mu} \in \widetilde{\Lambda}^+ \times \Lambda^+ : \widetilde{w}^{-1}(\widetilde{\mu}) \overset{\iota^*}{\longmapsto} w^{-1}(\mu)\}$$

finitely generated monoid

Diagonal embeddings revisited

$$G \hookrightarrow G \times G \,, \;\; \mathfrak{n} \hookrightarrow \mathfrak{n} \oplus \mathfrak{n}$$

$$\widetilde{V}^*_{\widetilde{w}\cdot\widetilde{\lambda}} = V^*_{w_1\cdot\lambda_1} \otimes V^*_{w_2\cdot\lambda_2} \stackrel{\pi}{\longrightarrow} V^*_{w\cdot(\lambda_1+\lambda_2)}$$

Diagonal embeddings revisited

$$G \hookrightarrow G \times G \,, \quad \mathfrak{n} \hookrightarrow \mathfrak{n} \oplus \mathfrak{n}$$

$$\widetilde{V}^*_{\widetilde{w}\cdot\widetilde{\lambda}} = V^*_{w_1\cdot\lambda_1} \otimes V^*_{w_2\cdot\lambda_2} \stackrel{\pi}{\longrightarrow} V^*_{w\cdot(\lambda_1+\lambda_2)}$$

$$\widetilde{e}^*_{\widetilde{w}} = e^*_{w_1} \otimes e^*_{w_2} \stackrel{\iota^*}{\longmapsto} e^*_{w_1} \wedge e^*_{w_2} \stackrel{?}{=} e^*_{w}$$

Diagonal embeddings revisited

$$G \hookrightarrow G \times G \,, \quad \mathfrak{n} \hookrightarrow \mathfrak{n} \oplus \mathfrak{n}$$

$$\widetilde{V}^*_{\widetilde{w}\cdot\widetilde{\lambda}} = V^*_{w_1\cdot\lambda_1} \otimes V^*_{w_2\cdot\lambda_2} \stackrel{\pi}{\longrightarrow} V^*_{w\cdot(\lambda_1+\lambda_2)}$$

$$\widetilde{e}^*_{\widetilde{w}} = e^*_{w_1} \otimes e^*_{w_2} \stackrel{\iota^*}{\longmapsto} e^*_{w_1} \wedge e^*_{w_2} \stackrel{?}{=} e^*_{w}$$

$$\text{(i)} \iff \Phi_{w_1} \sqcup \Phi_{w_2} = \Phi_w$$

Diagonal embeddings revisited

$$G \hookrightarrow G \times G \, , \;\; \mathfrak{n} \hookrightarrow \mathfrak{n} \oplus \mathfrak{n}$$

$$\widetilde{V}^*_{\widetilde{w}\cdot\widetilde{\lambda}} = V^*_{w_1\cdot\lambda_1} \otimes V^*_{w_2\cdot\lambda_2} \stackrel{\pi}{\longrightarrow} V^*_{w\cdot(\lambda_1+\lambda_2)}$$

$$\widetilde{e}^*_{\widetilde{w}} = e^*_{w_1} \otimes e^*_{w_2} \stackrel{\iota^*}{\longmapsto} e^*_{w_1} \wedge e^*_{w_2} \;\stackrel{?}{=}\; e^*_w$$

$$\text{(i)} \iff \Phi_{w_1} \sqcup \Phi_{w_2} = \Phi_w$$

[PRV, Kumar, Mathieu]: (i) $\implies$ (ii)

Theorem: $\pi \neq 0 \iff \Phi_{w_1} \sqcup \Phi_{w_2} = \Phi_w$

The Belkale-Kumar product on $H(G/B)$

$$H(G/B) \cong H(\mathfrak{n} \oplus \mathfrak{n}^-)^T \cong [H(\mathfrak{n}) \otimes H(\mathfrak{n}^-)]^T \cong \bigoplus_{w \in W} \mathbb{C} e_w^* \otimes e_w$$

The Belkale-Kumar product on $H(G/B)$

$$H(G/B) \cong H(\mathfrak{n} \oplus \mathfrak{n}^-)^T \cong [H(\mathfrak{n}) \otimes H(\mathfrak{n}^-)]^T \cong \bigoplus_{w \in W} \mathbb{C} e_w^* \otimes e_w$$

$S_{w_0 w} = \overline{w^{-1} B w B} \subset G/B$ shifted Schubert var.; $[S_w] \leftrightarrow [e_w^* \otimes e_w]$

The Belkale-Kumar product on $H(G/B)$

$$H(G/B) \cong H(\mathfrak{n} \oplus \mathfrak{n}^-)^T \cong [H(\mathfrak{n}) \otimes H(\mathfrak{n}^-)]^T \cong \bigoplus_{w \in W} \mathbb{C} e_w^* \otimes e_w$$

$S_{w_0 w} = \overline{w^{-1}BwB} \subset G/B$ shifted Schubert var.; $[S_w] \leftrightarrow [e_w^* \otimes e_w]$

$[S_u] \cdot [S_v] = \sum_w c_{uv}^w [S_w]$ cup product.

The Belkale-Kumar product on $H(G/B)$

$$H(G/B) \cong H(\mathfrak{n} \oplus \mathfrak{n}^-)^T \cong [H(\mathfrak{n}) \otimes H(\mathfrak{n}^-)]^T \cong \bigoplus_{w \in W} \mathbb{C} e_w^* \otimes e_w$$

$S_{w_0 w} = \overline{w^{-1} B w B} \subset G/B$ shifted Schubert var.; $[S_w] \leftrightarrow [e_w^* \otimes e_w]$

$[S_u] \cdot [S_v] = \sum_w c_{uv}^w [S_w]$ cup product.

$[S_u] \odot [S_v] = \sum_w d_{uv}^w [S_w] \leftrightarrow [(e_u^* \wedge e_v^*) \otimes (e_u \wedge e_v)]$ BK product.

The Belkale-Kumar product on $H(G/B)$

$$H(G/B) \cong H(\mathfrak{n} \oplus \mathfrak{n}^-)^T \cong [H(\mathfrak{n}) \otimes H(\mathfrak{n}^-)]^T \cong \bigoplus_{w \in W} \mathbb{C} e_w^* \otimes e_w$$

$S_{w_0 w} = \overline{w^{-1} B w B} \subset G/B$ shifted Schubert var.; $[S_w] \leftrightarrow [e_w^* \otimes e_w]$

$[S_u] \cdot [S_v] = \sum_w c_{uv}^w [S_w]$ cup product.

$[S_u] \odot [S_v] = \sum_w d_{uv}^w [S_w] \leftrightarrow [(e_u^* \wedge e_v^*) \otimes (e_u \wedge e_v)]$ BK product.

Theorem [Belkale & Kumar] $\odot$ is a deformation of $\cdot$ and

$$d_{uv}^w \neq 0 \implies d_{uv}^w = c_{uv}^w$$

The Belkale-Kumar product on $H(G/B)$

$$H(G/B) \cong H(\mathfrak{n} \oplus \mathfrak{n}^-)^T \cong [H(\mathfrak{n}) \otimes H(\mathfrak{n}^-)]^T \cong \bigoplus_{w \in W} \mathbb{C} e_w^* \otimes e_w$$

$S_{w_0 w} = \overline{w^{-1} B w B} \subset G/B$ shifted Schubert var.; $[S_w] \leftrightarrow [e_w^* \otimes e_w]$

$[S_u] \cdot [S_v] = \sum_w c_{uv}^w [S_w]$ cup product.

$[S_u] \odot [S_v] = \sum_w d_{uv}^w [S_w] \leftrightarrow [(e_u^* \wedge e_v^*) \otimes (e_u \wedge e_v)]$ BK product.

Theorem [Belkale & Kumar] $\odot$ is a deformation of $\cdot$ and

$$d_{uv}^w \neq 0 \implies d_{uv}^w = c_{uv}^w$$

Remark: BK-pullback $H(\widetilde{G}/\widetilde{B}) \xrightarrow{\iota^\odot} H(G/B)$ gives

$$\tilde{e}_{\tilde{w}}^* \overset{\iota^*}{\longmapsto} a e_w^* \iff [\widetilde{S}_{\tilde{w}}] \overset{\iota^\odot}{\longmapsto} a[S_w]$$

PRV components

$\binom{\widetilde{\mu}}{\mu} \in \widetilde{\Lambda}^+ \times \Lambda^+.$

$$\widetilde{w}^{-1}(\widetilde{\mu}) \xmapsto{\iota^*} w^{-1}(\mu) \quad \stackrel{??}{\Longrightarrow} \quad V_\mu \subset \widetilde{V}_{\widetilde{\mu}}$$

PRV components

$\binom{\tilde{\mu}}{\mu} \in \tilde{\Lambda}^+ \times \Lambda^+$.

$$\tilde{w}^{-1}(\tilde{\mu}) \stackrel{\iota^*}{\longmapsto} w^{-1}(\mu) \quad \stackrel{??}{\Longrightarrow} \quad V_\mu \subset \tilde{V}_{\tilde{\mu}}$$

Parthasarathy-Ranga Rao-Varadarajan, Kumar, Mathieu:

– Yes, for diagonal embeddings.

PRV components

$\binom{\tilde\mu}{\mu} \in \widetilde{\Lambda}^+ \times \Lambda^+$.

$$\widetilde{w}^{-1}(\tilde\mu) \stackrel{\iota^*}{\longmapsto} w^{-1}(\mu) \quad \stackrel{??}{\Longrightarrow} \quad V_\mu \subset \widetilde{V}_{\tilde\mu}$$

Parthasarathy-Ranga Rao-Varadarajan, Kumar, Mathieu:

– Yes, for diagonal embeddings.

Montagard-Pasquier-Ressayre:

– Yes, for more cases. No, in general. And

$$\widetilde{w}^{-1}(\tilde\mu) \stackrel{\iota^*}{\longmapsto} w^{-1}(\mu) \implies \exists k \in \mathbb{N}: \; V_{k\mu} \subset \mathfrak{U}(\mathfrak{g}) v^{\widetilde{w}^{-1}(k\tilde\mu)} \subset \widetilde{V}_{k\tilde\mu}$$

PRV components

$\binom{\widetilde{\mu}}{\mu} \in \widetilde{\Lambda}^+ \times \Lambda^+$.

$$\widetilde{w}^{-1}(\widetilde{\mu}) \stackrel{\iota^*}{\longmapsto} w^{-1}(\mu) \quad \stackrel{??}{\Longrightarrow} \quad V_\mu \subset \widetilde{V}_{\widetilde{\mu}}$$

Parthasarathy-Ranga Rao-Varadarajan, Kumar, Mathieu:

– Yes, for diagonal embeddings.

Montagard-Pasquier-Ressayre:

– Yes, for more cases. No, in general. And

$$\widetilde{w}^{-1}(\widetilde{\mu}) \stackrel{\iota^*}{\longmapsto} w^{-1}(\mu) \implies \exists k \in \mathbb{N} : \; V_{k\mu} \subset \mathfrak{U}(\mathfrak{g}) v^{\widetilde{w}^{-1}(k\widetilde{\mu})} \subset \widetilde{V}_{k\widetilde{\mu}}$$

Remark: $$\left.\begin{array}{l} \widetilde{e}^*_{\widetilde{w}} \stackrel{\iota^*}{\longmapsto} a e^*_w \\ \widetilde{w}^{-1} \cdot \widetilde{\mu} \stackrel{\iota^*}{\longmapsto} w^{-1} \cdot \mu \end{array}\right\} \Longrightarrow \widetilde{w}^{-1}(\widetilde{\mu}) \stackrel{\iota^*}{\longmapsto} w^{-1}(\mu)$$

Regular (root) embeddings

Regular: $\widetilde{H} \subset N_{\widetilde{G}}(G)$. Then $\Delta \subset \widetilde{\Delta}$ and $\Delta^{\pm} = \Delta \cap \widetilde{\Delta}^{\pm}$.

Regular (root) embeddings

Regular: $\widetilde{H} \subset N_{\widetilde{G}}(G)$. Then $\Delta \subset \widetilde{\Delta}$ and $\Delta^{\pm} = \Delta \cap \widetilde{\Delta}^{\pm}$.

$$\text{(i)} \quad \widetilde{e}^*_{\widetilde{w}} \overset{\iota^*}{\longmapsto} a e^*_w \iff \widetilde{\Phi}_{\widetilde{w}} \subset \Delta$$

Regular (root) embeddings

Regular: $\widetilde{H} \subset N_{\widetilde{G}}(G)$. Then $\Delta \subset \widetilde{\Delta}$ and $\Delta^{\pm} = \Delta \cap \widetilde{\Delta}^{\pm}$.

$$\text{(i)} \quad \widetilde{e}^*_{\widetilde{w}} \stackrel{\iota^*}{\longmapsto} ae^*_w \iff \widetilde{\Phi}_{\widetilde{w}} \subset \Delta$$

$$Also: \quad \text{(i)} \implies \text{(ii)} \quad and \quad \iota^* \widetilde{\mu} = \mu$$

Regular (root) embeddings

Regular: $\widetilde{H} \subset N_{\widetilde{G}}(G)$. Then $\Delta \subset \widetilde{\Delta}$ and $\Delta^{\pm} = \Delta \cap \widetilde{\Delta}^{\pm}$.

$$\text{(i)} \;\; \widetilde{e}^*_{\widetilde{w}} \stackrel{\iota^*}{\longmapsto} a e^*_w \iff \widetilde{\Phi}_{\widetilde{w}} \subset \Delta$$

$$\textit{Also}: \;\; \text{(i)} \implies \text{(ii)} \;\; \textit{and} \;\; \iota^* \widetilde{\mu} = \mu$$

Theorem: $\pi \neq 0 \iff \widetilde{\Phi}_{\widetilde{w}} \subset \Delta$.

Furthermore, $\mathcal{C} = \mathcal{C}^0$.

Thus: Cohomological comp. $=$ Cartan comp.

Principal rational curves

$$SL_2 \;\cong\; G \hookrightarrow \widetilde{G} \;\; \text{simple}, \;\; e_+ = \sum_{\alpha>0} \widetilde{e}_\alpha$$

$$\mathbb{P}^1 \hookrightarrow \widetilde{G}/\widetilde{B}$$

Principal rational curves

$$SL_2 \cong G \hookrightarrow \widetilde{G} \text{ simple}, \quad e_+ = \sum_{\alpha > 0} \widetilde{e}_\alpha$$

$$\mathbb{P}^1 \hookrightarrow \widetilde{G}/\widetilde{B}$$

If $\widetilde{G} \neq SL_3$ and $\ell(\widetilde{\lambda}) > 0$, then $\pi \neq 0 \iff -\widetilde{\lambda} = \alpha$ simple root.

$$\mathbb{C} \cong H^1(\widetilde{G}/\widetilde{B}, \mathcal{L}_{-\alpha}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong \mathbb{C}$$

Principal rational curves

$$SL_2 \cong G \hookrightarrow \widetilde{G} \text{ simple}, \quad e_+ = \sum_{\alpha>0} \widetilde{e}_\alpha$$

$$\mathbb{P}^1 \hookrightarrow \widetilde{G}/\widetilde{B}$$

If $\widetilde{G} \neq SL_3$ and $\ell(\widetilde{\lambda}) > 0$, then $\pi \neq 0 \iff -\widetilde{\lambda} = \alpha$ simple root.

$$\mathbb{C} \cong H^1(\widetilde{G}/\widetilde{B}, \mathcal{L}_{-\alpha}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong \mathbb{C}$$

If $\widetilde{G} = SL_3 = SL(\mathfrak{sl}_2)$ and $\ell(\widetilde{\lambda}) > 0$, then $\pi \neq 0 \iff \widetilde{\lambda} = s_i \cdot 2k\omega_i$.

$$S^{2k}(\mathfrak{sl}_2) \cong H^1(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong \mathbb{C}$$

$$\pi^* : \mathbb{C} \longrightarrow \mathbb{C}\kappa^k \subset S^{2k}(\mathfrak{sl}_2)^*$$

Ad-invariant polynomials

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, ..., p_r] \quad , \quad \deg p_j = d_j$$

Ad-invariant polynomials

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, ..., p_r] \quad , \quad \deg p_j = d_j$$

$$G \xrightarrow{\mathrm{Ad}} SL(\mathfrak{g}) = \widetilde{G}$$

Ad-invariant polynomials

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, ..., p_r] \quad , \quad \deg p_j = d_j$$

$$G \xrightarrow{\mathrm{Ad}} SL(\mathfrak{g}) = \widetilde{G}$$

Theorem: There exist $\widetilde{B} \supset B$ and $\widetilde{\lambda}_1, ..., \widetilde{\lambda}_r$ such that $\iota^* \widetilde{\lambda}_j = -2\rho$ and

$$S^{d_j}(\mathfrak{g}) \cong H^{\ell(w_0)}(\widetilde{G}/\widetilde{B}, \mathcal{L}_{\widetilde{\lambda}_j}) \xrightarrow{\pi \neq 0} H^{\ell(w_0)}(G/B, \mathcal{K}) \cong \mathbb{C}$$

$$\pi^* : \mathbb{C} \longrightarrow \mathbb{C} p_j \subset S^{d_j}(\mathfrak{g})^*$$

THE END

THANK YOU!