Parabolic equations and the bounded slope condition

Verena Bögelein

Fachbereich Mathematik
Paris-Lodron Universität Salzburg

Minimizer of the area functional

$\Omega \subset \mathbb{R}^n$ bounded domain, $\varphi \in C^0(\partial\Omega)$.

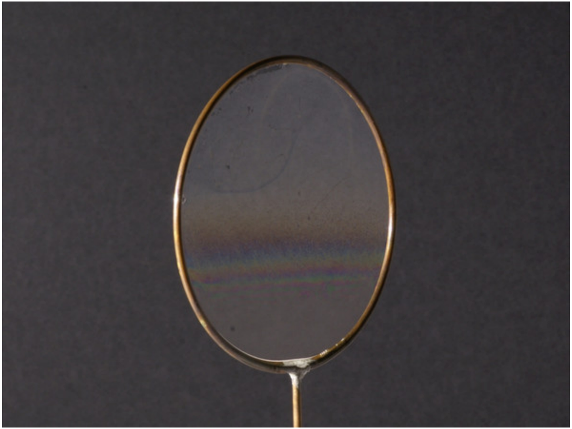
Classical problem: Find a function $u\colon \Omega \to \mathbb{R}$ whose graph

$$G_u := \left\{(x, u(x)) : x \in \Omega\right\}$$

has minimal area

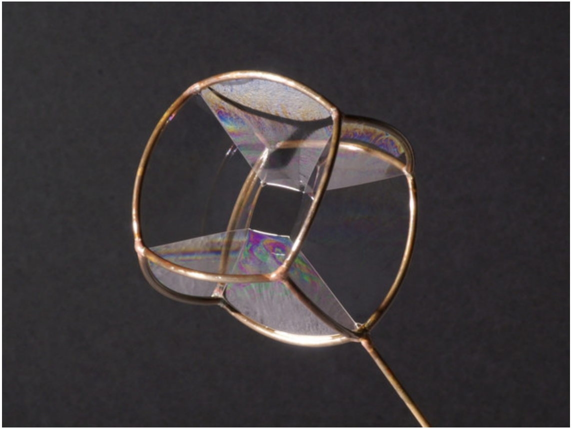
$$\text{Area}(G_u) = A_\Omega(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

Experiments with soap films



picture from Homepage of F. Duzaar (Erlangen)

Experiments with soap films



picture from Homepage of F. Duzaar (Erlangen)

Minimizer of the area functional

$\Omega \subset \mathbb{R}^n$ bounded domain, $\varphi \in C^0(\partial\Omega)$.

Classical problem: Find a function $u\colon \Omega \to \mathbb{R}$ whose graph

$$G_u := \left\{ (x, u(x)) : x \in \Omega \right\}$$

has minimal area

$$\mathrm{Area}(G_u) = A_\Omega(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

$\Rightarrow$ Find a minimizer u of the area functional A_Ω, i.e.

$$A_\Omega(u) \le A_\Omega(u + \eta) \qquad \forall \eta \in C^1(\Omega), \mathrm{spt}\, \eta \subset\subset \Omega$$

Minimizer of the area functional

$\Omega \subset \mathbb{R}^n$ bounded domain, $\varphi \in C^0(\partial\Omega)$.

Classical problem: Find a function $u\colon \Omega \to \mathbb{R}$ whose graph

$$G_u := \{(x, u(x)) : x \in \Omega\}$$

has minimal area

$$\text{Area}(G_u) = A_\Omega(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

$\Rightarrow$ Find a minimizer u of the area functional A_Ω, i.e.

$$A_\Omega(u) \le A_\Omega(u + \eta) \qquad \forall \eta \in C^1(\Omega), \text{spt}\, \eta \subset\subset \Omega$$

Minimal surface equation

In the minimality condition replace η by $\varepsilon\eta$:

$$A_\Omega(u) \le A_\Omega(u + \varepsilon\eta) \qquad \forall \varepsilon \in \mathbb{R}.$$

$\Rightarrow$ First variation of A_Ω:

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A_\Omega(u+\varepsilon\eta) = \int_\Omega \frac{\nabla u \cdot \nabla \eta}{\sqrt{1+|\nabla u|^2}}\,dx = -\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\eta\,dx$$

Non parametric minimal surface equation: For given $\varphi \in C^0(\partial\Omega)$ find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with

$$\begin{cases} \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Minimal surface equation

In the minimality condition replace η by $\varepsilon\eta$:

$$A_\Omega(u) \le A_\Omega(u + \varepsilon\eta) \qquad \forall \varepsilon \in \mathbb{R}.$$

$\Rightarrow$ First variation of A_Ω:

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A_\Omega(u+\varepsilon\eta) = \int_\Omega \frac{\nabla u \cdot \nabla \eta}{\sqrt{1+|\nabla u|^2}}\,dx = -\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\eta\,dx$$

Non parametric minimal surface equation: For given $\varphi \in C^0(\partial\Omega)$ find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with

$$\begin{cases} \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Minimal surface equation

In the minimality condition replace η by $\varepsilon\eta$:

$$A_\Omega(u) \le A_\Omega(u + \varepsilon\eta) \qquad \forall \varepsilon \in \mathbb{R}.$$

$\Rightarrow$ First variation of A_Ω:

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A_\Omega(u+\varepsilon\eta) = \int_\Omega \frac{\nabla u \cdot \nabla \eta}{\sqrt{1 + |\nabla u|^2}}\, dx = -\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)\eta\, dx$$

Non parametric minimal surface equation: For given $\varphi \in C^0(\partial\Omega)$ find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with

$$\begin{cases} \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Minimization of the area functional

- Existence of solutions in general not so easy
- Approach by Haar (Math. Ann., 1927): existence of Lipschitz minima if $\partial\Omega$ and φ satisfy the bounded slope condition

Minimization of the area functional

- Existence of solutions in general not so easy
- Approach by Haar (Math. Ann., 1927): existence of Lipschitz minima if $\partial\Omega$ and φ satisfy the bounded slope condition

Solution of the gradient constrained problem

$u: \Omega \to \mathbb{R}$ is Lipschitz continuous in Ω: There exists $L > 0$ s.t.

$$|u(x) - u(y)| \leq L|x - y| \qquad \forall x, y \in \Omega;$$

$\mathrm{Lip}(u)$: smallest possible constant L.

Define

$$\mathrm{Lip}(\Omega; \varphi) := \left\{ u \in C^0(\overline{\Omega}) : u \text{ bounded, Lipschitz cont. in } \Omega,\ u|_{\partial\Omega} = \varphi \right\},$$

$$\mathrm{Lip}_R(\Omega; \varphi) := \left\{ u \in \mathrm{Lip}(\Omega; \varphi) : \mathrm{Lip}(u) \leq R \right\}$$

Theorem. There exists a unique minimum u_R of A_Ω in the class $\mathrm{Lip}_R(\Omega; \varphi)$.

Solution of the gradient constrained problem

$u: \Omega \to \mathbb{R}$ is Lipschitz continuous in Ω: There exists $L > 0$ s.t.

$$|u(x) - u(y)| \leq L|x - y| \qquad \forall x, y \in \Omega;$$

$\mathrm{Lip}(u)$: smallest possible constant L.

Define

$$\mathrm{Lip}(\Omega;\varphi) := \left\{u \in C^0(\overline{\Omega}) : u \text{ bounded, Lipschitz cont. in } \Omega,\ u|_{\partial\Omega} = \varphi\right\},$$

$$\mathrm{Lip}_R(\Omega;\varphi) := \left\{u \in \mathrm{Lip}(\Omega;\varphi) : \mathrm{Lip}(u) \leq R\right\}$$

Theorem. There exists a unique minimum u_R of A_Ω in the class $\mathrm{Lip}_R(\Omega;\varphi)$.

The bounded slope condition

Problem: remove the constraint $\mathrm{Lip}(u_R) \le R$

Idea of Haar: find a geometric constraint on Ω, φ such that $\mathrm{Lip}(u_R) \le Q$ for any $R > 0$, with some constant $0 < Q < \infty$.

Definition. Ω, φ satisfy the bounded slope condition (bsc) with constant $0 < Q < \infty$ if for any $x_o \in \partial\Omega$ there exist two affine functions $w^{\pm}_{x_o}$ such that

i) $w^{\pm}_{x_o}(x_o) = \varphi(x_o)$

ii) $w^{-}_{x_o}(x) \le \varphi(x) \le w^{+}_{x_o}(x) \quad \forall x \in \partial\Omega,$

iii) $|\nabla w^{\pm}_{x_o}| \le Q$

The bounded slope condition

Problem: remove the constraint $\mathrm{Lip}(u_R) \le R$

Idea of Haar: find a geometric constraint on Ω, φ such that $\mathrm{Lip}(u_R) \le Q$ for any $R > 0$, with some constant $0 < Q < \infty$.

Definition. Ω, φ satisfy the bounded slope condition (bsc) with constant $0 < Q < \infty$ if for any $x_o \in \partial\Omega$ there exist two affine functions $w^{\pm}_{x_o}$ such that

i) $w^{\pm}_{x_o}(x_o) = \varphi(x_o)$

ii) $w^{-}_{x_o}(x) \le \varphi(x) \le w^{+}_{x_o}(x) \quad \forall x \in \partial\Omega,$

iii) $|\nabla w^{\pm}_{x_o}| \le Q$

The bounded slope condition

Problem: remove the constraint $\mathrm{Lip}(u_R) \le R$

Idea of Haar: find a geometric constraint on Ω, φ such that $\mathrm{Lip}(u_R) \le Q$ for any $R > 0$, with some constant $0 < Q < \infty$.

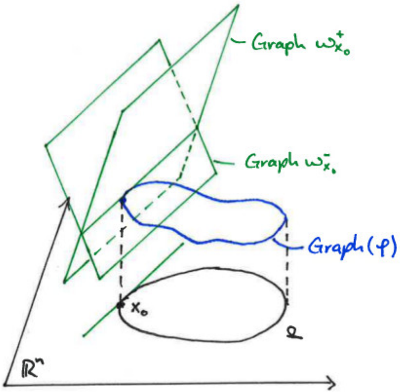
Definition. Ω, φ satisfy the bounded slope condition (bsc) with constant $0 < Q < \infty$ if for any $x_o \in \partial\Omega$ there exist two affine functions $w^{\pm}_{x_o}$ such that

i) $w^{\pm}_{x_o}(x_o) = \varphi(x_o)$

ii) $w^{-}_{x_o}(x) \le \varphi(x) \le w^{+}_{x_o}(x) \quad \forall x \in \partial\Omega,$

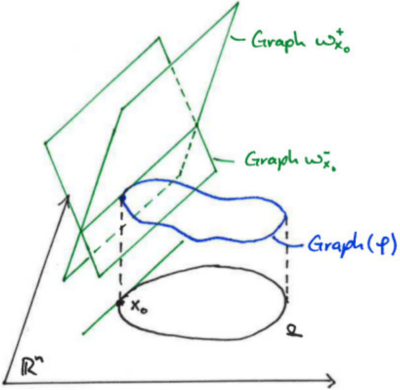
iii) $|\nabla w^{\pm}_{x_o}| \le Q$

The bounded slope condition



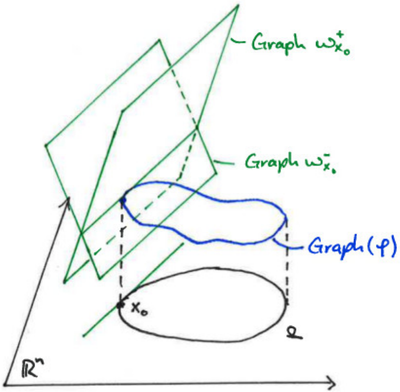
- bsc $\Rightarrow$ Ω is convex
- Ω uniformly convex $C^{1,1}$-domain:
  bsc $\Leftrightarrow$ φ is $C^{1,1}$ on $\partial\Omega$

The bounded slope condition



- bsc $\Rightarrow$ Ω is convex
- Ω uniformly convex $C^{1,1}$-domain:
  bsc $\Leftrightarrow$ φ is $C^{1,1}$ on $\partial\Omega$

The bounded slope condition



- bsc $\Rightarrow$ Ω is convex
- Ω uniformly convex $C^{1,1}$-domain:
  bsc $\Leftrightarrow$ φ is $C^{1,1}$ on $\partial\Omega$

The existence result of Haar

Theorem. Let $\Omega \subset \mathbb{R}^n$ convex and bounded, $\varphi: \partial\Omega \to \mathbb{R}$ Lipschitz continuous and (Ω, φ) satisfy the bsc with constant $Q < \infty$. Then, for any $R > Q$ the function u_R satisfies $\mathrm{Lip}(u_R) \leq Q$. Therefore, it is the unique minimum of A_Ω in $\mathrm{Lip}(\Omega; \varphi)$.

Remark. The same result holds true for variational functionals

$$F(u) := \int_\Omega f(Du)\, dx,$$

with $f: \mathbb{R}^n \to \mathbb{R}$ strictly convex.

The existence result of Haar

Theorem. Let $\Omega \subset \mathbb{R}^n$ convex and bounded, $\varphi\colon \partial\Omega \to \mathbb{R}$ Lipschitz continuous and (Ω, φ) satisfy the bsc with constant $Q < \infty$. Then, for any $R > Q$ the function u_R satisfies $\mathrm{Lip}(u_R) \leq Q$. Therefore, it is the unique minimum of A_Ω in $\mathrm{Lip}(\Omega; \varphi)$.

Remark. The same result holds true for variational functionals

$$F(u) := \int_\Omega f(Du)\, dx,$$

with $f\colon \mathbb{R}^n \to \mathbb{R}$ strictly convex.

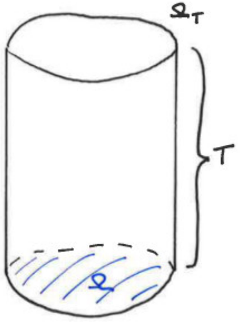
Parabolic problem

Is there a similar existence result for parabolic equations?

$$\begin{cases} \partial_t u - \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 & \text{in } \Omega_T := \Omega \times (0,T), \\ u = u_o & \text{on } \partial_P \Omega_T := (\overline{\Omega} \times \{0\}) \cap (\partial\Omega \times (0,T)), \end{cases}$$

with $u_o \in \mathrm{Lip}(\Omega;\varphi)$, or more generally

$$\begin{cases} \partial_t u - \operatorname{div} Df(Du) = 0 & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_P \Omega_T. \end{cases}$$



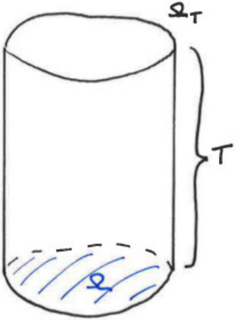
Parabolic problem

Is there a similar existence result for parabolic equations?

$$\begin{cases} \partial_t u - \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 & \text{in } \Omega_T := \Omega \times (0,T), \\ u = u_o & \text{on } \partial_P \Omega_T := (\overline{\Omega} \times \{0\}) \cap (\partial\Omega \times (0,T)), \end{cases}$$

with $u_o \in \mathrm{Lip}(\Omega; \varphi)$, or more generally

$$\begin{cases} \partial_t u - \operatorname{div} Df(Du) = 0 & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_P \Omega_T. \end{cases}$$



Variational inequality

Multiply both sides of the diff. eq. by $v - u$, integrate over Ω_T:

$$\underbrace{\iint_{\Omega_T} \partial_t u(v-u)\,dxdt}_{=:I} - \underbrace{\iint_{\Omega_T} \operatorname{div} Df(Du)(v-u)\,dxdt}_{=:II} = 0.$$

For the time term compute

$$\begin{aligned} I &= \iint_{\Omega_T} \partial_t v(v-u)\,dxdt - \tfrac{1}{2}\int_0^T \int_\Omega \partial_t |v-u|^2\,dxdt \\ &= \iint_{\Omega_T} \partial_t v(v-u)\,dxdt - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)}. \end{aligned}$$

Integration by parts and convexity of f:

$$-II = \iint_{\Omega_T} Df(Du)\cdot D(v-u)\,dxdt \le \iint_{\Omega_T} \big[f(Dv) - f(Du)\big]\,dxdt.$$

Variational inequality

Multiply both sides of the diff. eq. by $v - u$, integrate over Ω_T:

$$\underbrace{\iint_{\Omega_T} \partial_t u(v-u)\,dxdt}_{=:I} - \underbrace{\iint_{\Omega_T} \operatorname{div} Df(Du)(v-u)\,dxdt}_{=:II} = 0.$$

For the time term compute

$$\begin{aligned} I &= \iint_{\Omega_T} \partial_t v(v-u)\,dxdt - \tfrac{1}{2}\int_0^T \int_\Omega \partial_t |v-u|^2\,dxdt \\ &= \iint_{\Omega_T} \partial_t v(v-u)\,dxdt - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)}. \end{aligned}$$

Integration by parts and convexity of f:

$$-II = \iint_{\Omega_T} Df(Du)\cdot D(v-u)\,dxdt \le \iint_{\Omega_T} \big[f(Dv) - f(Du)\big]\,dxdt.$$

Variational inequality

Multiply both sides of the diff. eq. by $v - u$, integrate over Ω_T:

$$\underbrace{\iint_{\Omega_T} \partial_t u (v-u)\, dxdt}_{=:I} - \underbrace{\iint_{\Omega_T} \operatorname{div} Df(Du)(v-u)\, dxdt}_{=:II} = 0.$$

For the time term compute

$$\begin{aligned} I &= \iint_{\Omega_T} \partial_t v (v-u)\, dxdt - \tfrac{1}{2} \int_0^T \int_\Omega \partial_t |v-u|^2\, dxdt \\ &= \iint_{\Omega_T} \partial_t v (v-u)\, dxdt - \tfrac{1}{2} \|v(T) - u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2} \|v(0) - u_o\|^2_{L^2(\Omega)}. \end{aligned}$$

Integration by parts and convexity of f:

$$-II = \iint_{\Omega_T} Df(Du) \cdot D(v-u)\, dxdt \le \iint_{\Omega_T} \big[f(Dv) - f(Du) \big]\, dxdt.$$

Variational inequality

$$\iint_{\Omega_T} f(Du)\,dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + f(Dv)\right] dxdt$$
$$+ \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)}$$

for any $v\colon \Omega_T \to \mathbb{R}$ with $v = u_o$ on $\partial\Omega \times (0,T)$.

Function spaces: Since $\mathrm{Lip}(\Omega) \cong W^{1,\infty}(\Omega)$, we consider

$$K(\Omega_T) := \left\{ v \in L^\infty(\Omega_T) \cap C^0([0,T];L^2(\Omega)) : Dv \in L^\infty(\Omega_T,\mathbb{R}^n) \right\}$$

and

$$K_{u_o}(\Omega_T) := \left\{ v \in K(\Omega_T) : v(t) = u_o \text{ on } \partial\Omega \text{ for a.e. } t \in (0,T) \right\}$$

and

$$K^{(R)}_{u_o}(\Omega_T) := \left\{ v \in K_{u_o}(\Omega_T) : \|Dv\|_{L^\infty} \le R \right\}$$

Variational inequality

$$\iint_{\Omega_T} f(Du)\,dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + f(Dv)\right] dxdt + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)}$$

for any $v\colon \Omega_T \to \mathbb{R}$ with $v = u_o$ on $\partial\Omega \times (0,T)$.

Function spaces: Since $\mathrm{Lip}(\Omega) \cong W^{1,\infty}(\Omega)$, we consider

$$K(\Omega_T) := \left\{ v \in L^\infty(\Omega_T) \cap C^0([0,T]; L^2(\Omega)) : Dv \in L^\infty(\Omega_T, \mathbb{R}^n) \right\}$$

and

$$K_{u_o}(\Omega_T) := \left\{ v \in K(\Omega_T) : v(t) = u_o \text{ on } \partial\Omega \text{ for a.e. } t \in (0,T) \right\}$$

and

$$K^{(R)}_{u_o}(\Omega_T) := \left\{ v \in K_{u_o}(\Omega_T) : \|Dv\|_{L^\infty} \le R \right\}$$

Existence of variational solutions

Definition. Let $u_o \in W^{1,\infty}(\Omega)$. A function $u \in K_{u_o}(\Omega_T)$ is a variational solution if

$$\iint_{\Omega_T} f(Du)\,dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + f(Du)\right] dxdt + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)}$$

holds true for any $v \in K_{u_o}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$.

Theorem (B., Duzaar, Marcellini, Signoriello). Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex, $u_o \in W^{1,\infty}(\Omega)$, $(u_o|_{\partial\Omega}, \Omega)$ satisfy the bsc with some constant $Q > 0$. Then, there exists a unique variational solution u with

$$\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le \max\left\{Q, \|Du_o\|_{L^\infty(\Omega_T,\mathbb{R}^n)}\right\}.$$

Existence of variational solutions

Definition. Let $u_o \in W^{1,\infty}(\Omega)$. A function $u \in K_{u_o}(\Omega_T)$ is a variational solution if

$$\iint_{\Omega_T} f(Du)\,dxdt \le \iint_{\Omega_T} \big[\partial_t v(v-u) + f(Du)\big]\,dxdt + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)}$$

holds true for any $v \in K_{u_o}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$.

Theorem (B., Duzaar, Marcellini, Signoriello). Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex, $u_o \in W^{1,\infty}(\Omega)$, $(u_o|_{\partial\Omega}, \Omega)$ satisfy the bsc with some constant $Q > 0$. Then, there exists a unique variational solution u with

$$\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le \max\big\{Q, \|Du_o\|_{L^\infty(\Omega_T,\mathbb{R}^n)}\big\}.$$

Existence of variational solutions

- Contrary to the elliptic case, uniqueness is guarantied even if f is (not uniformly) convex.
- Possible integrands:
  - Area integrand: $f(\xi) = \sqrt{1 + |\xi|^2}$;
  - Integrands with exponential growth: $f(\xi) = \exp(|\xi|^2)$;
  - Orlicz type functionals: $f(\xi) = |\xi| \log(1 + |\xi|)$.

Existence of variational solutions

- Contrary to the elliptic case, uniqueness is guarantied even if f is (not uniformly) convex.
- Possible integrands:
  - Area integrand: $f(\xi) = \sqrt{1 + |\xi|^2}$;
  - Integrands with exponential growth: $f(\xi) = \exp(|\xi|^2)$;
  - Orlicz type functionals: $f(\xi) = |\xi| \log(1 + |\xi|)$.

Idea of the proof

- Solve the constrained problem: For $R > 0$ there exists a function $u_R \in K_{u_o}^{(R)}(\Omega_T)$ satisfying the varaitional inequality

$$\iint_{\Omega_T} f(Du)\,dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + f(Du)\right] dxdt \\ - \tfrac{1}{2}\|v(T) - u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0) - u_o\|^2_{L^2(\Omega)}$$

  for any $v \in K_{u_o}^{(R)}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$.

- Remove the constraint: For $R > Q$ prove that $\|Du_R\|_{L^\infty} < Q$.

Idea of the proof

- Solve the constrained problem: For $R > 0$ there exists a function $u_R \in K_{u_o}^{(R)}(\Omega_T)$ satisfying the varaitional inequality

$$\iint_{\Omega_T} f(Du)\,dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + f(Du)\right] dxdt - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)}$$

  for any $v \in K_{u_o}^{(R)}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$.

- Remove the constraint: For $R > Q$ prove that $\|Du_R\|_{L^\infty} < Q$.

Solve the constrained problem

- Consider on $\Omega_T \subset \mathbb{R}^{n+1}$ the convex variational integral

$$F_\varepsilon(v) := \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\tfrac{1}{2} |\partial_t v|^2 + \tfrac{1}{\varepsilon} f(Dv) \right] dxdt.$$

- Existence of a minimizer u^ε in the class $K_{u_o}^{(R)}(\Omega_T)$ with $\partial_t u^\varepsilon \in L^2(\Omega_T)$, $u^\varepsilon(0) = u_o$
- Formally compute first variation

$$0 = \frac{d}{ds}\bigg|_{s=0} F_\varepsilon(u^\varepsilon + s\varphi) = \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\partial_{tt} u^\varepsilon - \tfrac{1}{\varepsilon} \partial_t u^\varepsilon + \tfrac{1}{\varepsilon} \operatorname{div} Df(Du^\varepsilon) \right] \varphi \, dxdt$$

- u^ε formally solves the differential equation

$$\varepsilon \partial_{tt} u^\varepsilon - \partial_t u^\varepsilon + \operatorname{div} Df(Du^\varepsilon) = 0$$

- Heuristically, u^ε should converge in the limit $\varepsilon \downarrow 0$ to a solution u_R of the parabolic pde.
- Rigorous proof on the level of variational solutions

Solve the constrained problem

- Consider on $\Omega_T \subset \mathbb{R}^{n+1}$ the convex variational integral

$$F_\varepsilon(v) := \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\tfrac{1}{2} |\partial_t v|^2 + \tfrac{1}{\varepsilon} f(Dv) \right] dxdt.$$

- Existence of a minimizer u^ε in the class $K^{(R)}_{u_o}(\Omega_T)$ with $\partial_t u^\varepsilon \in L^2(\Omega_T)$, $u^\varepsilon(0) = u_o$
- Formally compute first variation

$$0 = \frac{d}{ds}\bigg|_{s=0} F_\varepsilon(u^\varepsilon + s\varphi) = \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\partial_{tt} u^\varepsilon - \tfrac{1}{\varepsilon} \partial_t u^\varepsilon + \tfrac{1}{\varepsilon} \operatorname{div} Df(Du^\varepsilon) \right] \varphi \, dxdt$$

- u^ε formally solves the differential equation

$$\varepsilon \partial_{tt} u^\varepsilon - \partial_t u^\varepsilon + \operatorname{div} Df(Du^\varepsilon) = 0$$

- Heuristically, u^ε should converge in the limit $\varepsilon \downarrow 0$ to a solution u_R of the parabolic pde.
- Rigorous proof on the level of variational solutions

Solve the constrained problem

- Consider on $\Omega_T \subset \mathbb{R}^{n+1}$ the convex variational integral

$$F_\varepsilon(v) := \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\tfrac{1}{2} |\partial_t v|^2 + \tfrac{1}{\varepsilon} f(Dv) \right] dxdt.$$

- Existence of a minimizer u^ε in the class $K_{u_o}^{(R)}(\Omega_T)$ with $\partial_t u^\varepsilon \in L^2(\Omega_T)$, $u^\varepsilon(0) = u_o$
- Formally compute first variation

$$0 = \frac{d}{ds}\bigg|_{s=0} F_\varepsilon(u^\varepsilon + s\varphi) = \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\partial_{tt} u^\varepsilon - \tfrac{1}{\varepsilon} \partial_t u^\varepsilon + \tfrac{1}{\varepsilon} \operatorname{div} Df(Du^\varepsilon) \right] \varphi \, dxdt$$

- u^ε formally solves the differential equation

$$\varepsilon \partial_{tt} u^\varepsilon - \partial_t u^\varepsilon + \operatorname{div} Df(Du^\varepsilon) = 0$$

- Heuristically, u^ε should converge in the limit $\varepsilon \downarrow 0$ to a solution u_R of the parabolic pde.
- Rigorous proof on the level of variational solutions

Solve the constrained problem

- Consider on $\Omega_T \subset \mathbb{R}^{n+1}$ the convex variational integral

$$F_\varepsilon(v) := \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\tfrac{1}{2} |\partial_t v|^2 + \tfrac{1}{\varepsilon} f(Dv) \right] dxdt.$$

- Existence of a minimizer u^ε in the class $K_{u_o}^{(R)}(\Omega_T)$ with $\partial_t u^\varepsilon \in L^2(\Omega_T)$, $u^\varepsilon(0) = u_o$
- Formally compute first variation

$$0 = \frac{d}{ds}\bigg|_{s=0} F_\varepsilon(u^\varepsilon + s\varphi) = \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\partial_{tt} u^\varepsilon - \tfrac{1}{\varepsilon} \partial_t u^\varepsilon + \tfrac{1}{\varepsilon} \operatorname{div} Df(Du^\varepsilon) \right] \varphi \, dxdt$$

- u^ε formally solves the differential equation

$$\varepsilon \partial_{tt} u^\varepsilon - \partial_t u^\varepsilon + \operatorname{div} Df(Du^\varepsilon) = 0$$

- Heuristically, u^ε should converge in the limit $\varepsilon \downarrow 0$ to a solution u_R of the parabolic pde.
- Rigorous proof on the level of variational solutions

Solve the constrained problem

- Consider on $\Omega_T \subset \mathbb{R}^{n+1}$ the convex variational integral

$$F_\varepsilon(v) := \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\tfrac{1}{2} |\partial_t v|^2 + \tfrac{1}{\varepsilon} f(Dv) \right] dxdt.$$

- Existence of a minimizer u^ε in the class $K_{u_o}^{(R)}(\Omega_T)$ with $\partial_t u^\varepsilon \in L^2(\Omega_T)$, $u^\varepsilon(0) = u_o$
- Formally compute first variation

$$0 = \frac{d}{ds}\bigg|_{s=0} F_\varepsilon(u^\varepsilon + s\varphi) = \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \left[\partial_{tt} u^\varepsilon - \tfrac{1}{\varepsilon} \partial_t u^\varepsilon + \tfrac{1}{\varepsilon} \operatorname{div} Df(Du^\varepsilon) \right] \varphi \, dxdt$$

- u^ε formally solves the differential equation

$$\varepsilon \partial_{tt} u^\varepsilon - \partial_t u^\varepsilon + \operatorname{div} Df(Du^\varepsilon) = 0$$

- Heuristically, u^ε should converge in the limit $\varepsilon \downarrow 0$ to a solution u_R of the parabolic pde.
- Rigorous proof on the level of variational solutions

Solve the constrained problem

- Consider on $\Omega_T \subset \mathbb{R}^{n+1}$ the convex variational integral

$$F_\varepsilon(v) := \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \Big[\tfrac{1}{2} |\partial_t v|^2 + \tfrac{1}{\varepsilon} f(Dv) \Big] \, dxdt.$$

- Existence of a minimizer u^ε in the class $K^{(R)}_{u_o}(\Omega_T)$ with $\partial_t u^\varepsilon \in L^2(\Omega_T)$, $u^\varepsilon(0) = u_o$
- Formally compute first variation

$$0 = \frac{d}{ds}\Big|_{s=0} F_\varepsilon(u^\varepsilon + s\varphi) = \iint_{\Omega_T} e^{-\frac{t}{\varepsilon}} \Big[\partial_{tt} u^\varepsilon - \tfrac{1}{\varepsilon} \partial_t u^\varepsilon + \tfrac{1}{\varepsilon} \operatorname{div} Df(Du^\varepsilon) \Big] \varphi \, dxdt$$

- u^ε formally solves the differential equation

$$\varepsilon \partial_{tt} u^\varepsilon - \partial_t u^\varepsilon + \operatorname{div} Df(Du^\varepsilon) = 0$$

- Heuristically, u^ε should converge in the limit $\varepsilon \downarrow 0$ to a solution u_R of the parabolic pde.
- Rigorous proof on the level of variational solutions

Remove the constraint $|Du_R| \le R$

- Affine functions w are variational solutions with initial-boundary data $u_o = w$.
- For $x_o \in \partial\Omega$ take $w^{\pm}_{x_o}$ from the bsc, so that $w^{-}_{x_o} \le u_o \le w^{+}_{x_o}$ in Ω
- Maximum principle:

$$w^{-}_{x_o}(x) \le u_R(x) \le w^{+}_{x_o}(x) \qquad \forall\, x \in \Omega$$

$$\Rightarrow |u_R(x) - u_o(x_o)| \le Q|x - x_o| \qquad \forall\, x \in \Omega, x_o \in \partial\Omega$$

Remove the constraint $|Du_R| \le R$

- Affine functions w are variational solutions with initial-boundary data $u_o = w$.
- For $x_o \in \partial\Omega$ take $w^{\pm}_{x_o}$ from the bsc, so that $w^-_{x_o} \le u_o \le w^+_{x_o}$ in Ω
- Maximum principle:

$$w^-_{x_o}(x) \le u_R(x) \le w^+_{x_o}(x) \qquad \forall\, x \in \Omega$$

$$\Rightarrow |u_R(x) - u_o(x_o)| \le Q|x - x_o| \qquad \forall\, x \in \Omega, x_o \in \partial\Omega$$

Remove the constraint $|Du_R| \le R$

- Affine functions w are variational solutions with initial-boundary data $u_o = w$.
- For $x_o \in \partial\Omega$ take $w^{\pm}_{x_o}$ from the bsc, so that $w^{-}_{x_o} \le u_o \le w^{+}_{x_o}$ in Ω
- Maximum principle:

$$w^{-}_{x_o}(x) \le u_R(x) \le w^{+}_{x_o}(x) \qquad \forall\, x \in \Omega$$

$$\Rightarrow |u_R(x) - u_o(x_o)| \le Q|x - x_o| \qquad \forall\, x \in \Omega, x_o \in \partial\Omega$$

Remove the constraint $|Du_R| \le R$

- So far, we know

$$|u_R(x) - u_o(x_o)| \le Q|x - x_o| \qquad \forall\, x \in \Omega, x_o \in \partial\Omega$$

- Consider $x, \tilde{x} \in \Omega$, let $y := \tilde{x} - x$ and define

$$v_R(x) := u_R(x - y) \qquad \text{for } x \in \tilde{\Omega} := y + \Omega$$

- Maximum principle: there exists $x_o \in \partial(\Omega \cap \tilde{\Omega})$ such that

$$|u_R(\tilde{x}) - v_R(\tilde{x})| \le |u_R(x_o) - v_R(x_o)| = |u_R(x_o) - u_R(x_o - y)|$$

- Either $x_o \in \partial\Omega$ or $x_o - y \in \partial\Omega$; moreover $v_R(\tilde{x}) = u_R(x)$

$$\Rightarrow |u_R(\tilde{x}) - u_R(x)| \le Q|\tilde{x} - x| \qquad \Rightarrow \mathrm{Lip}(u_R) \le Q.$$

Remove the constraint $|Du_R| \leq R$

- So far, we know

$$|u_R(x) - u_o(x_o)| \leq Q|x - x_o| \qquad \forall\, x \in \Omega, x_o \in \partial\Omega$$

- Consider $x, \tilde{x} \in \Omega$, let $y := \tilde{x} - x$ and define

$$v_R(x) := u_R(x - y) \qquad \text{for } x \in \tilde{\Omega} := y + \Omega$$

- Maximum principle: there exists $x_o \in \partial(\Omega \cap \tilde{\Omega})$ such that

$$|u_R(\tilde{x}) - v_R(\tilde{x})| \leq |u_R(x_o) - v_R(x_o)| = |u_R(x_o) - u_R(x_o - y)|$$

- Either $x_o \in \partial\Omega$ or $x_o - y \in \partial\Omega$; moreover $v_R(\tilde{x}) = u_R(x)$

$$\Rightarrow |u_R(\tilde{x}) - u_R(x)| \leq Q|\tilde{x} - x| \qquad \Rightarrow \mathrm{Lip}(u_R) \leq Q.$$

Remove the constraint $|Du_R| \le R$

- So far, we know

$$|u_R(x) - u_o(x_o)| \le Q|x - x_o| \qquad \forall\, x \in \Omega, x_o \in \partial\Omega$$

- Consider $x, \tilde{x} \in \Omega$, let $y := \tilde{x} - x$ and define

$$v_R(x) := u_R(x - y) \qquad \text{for } x \in \tilde{\Omega} := y + \Omega$$

- Maximum principle: there exists $x_o \in \partial(\Omega \cap \tilde{\Omega})$ such that

$$|u_R(\tilde{x}) - v_R(\tilde{x})| \le |u_R(x_o) - v_R(x_o)| = |u_R(x_o) - u_R(x_o - y)|$$

- Either $x_o \in \partial\Omega$ or $x_o - y \in \partial\Omega$; moreover $v_R(\tilde{x}) = u_R(x)$

$$\Rightarrow |u_R(\tilde{x}) - u_R(x)| \le Q|\tilde{x} - x| \qquad \Rightarrow \mathrm{Lip}(u_R) \le Q.$$

Remove the constraint $|Du_R| \le R$

- So far, we know

$$|u_R(x) - u_o(x_o)| \le Q|x - x_o| \qquad \forall\, x \in \Omega, x_o \in \partial\Omega$$

- Consider $x, \tilde{x} \in \Omega$, let $y := \tilde{x} - x$ and define

$$v_R(x) := u_R(x - y) \qquad \text{for } x \in \tilde{\Omega} := y + \Omega$$

- Maximum principle: there exists $x_o \in \partial(\Omega \cap \tilde{\Omega})$ such that

$$|u_R(\tilde{x}) - v_R(\tilde{x})| \le |u_R(x_o) - v_R(x_o)| = |u_R(x_o) - u_R(x_o - y)|$$

- Either $x_o \in \partial\Omega$ or $x_o - y \in \partial\Omega$; moreover $v_R(\tilde{x}) = u_R(x)$

$$\Rightarrow |u_R(\tilde{x}) - u_R(x)| \le Q|\tilde{x} - x| \qquad \Rightarrow \mathrm{Lip}(u_R) \le Q.$$

The total variation flow

Initial-boundary value preoblem for the total variation flow

$$\begin{cases} \partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_P \Omega_T. \end{cases}$$

Existence results by Andreu, Ballester, Caselles, Mazón.

Variational inequality:

$$\iint_{\Omega_T} |Du|\, dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + |Dv|\right] dxdt - \tfrac{1}{2}\|v(T)-u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)}$$

for any (sufficiently regular) comparison function $v\colon \Omega_T \to \mathbb{R}$ with $v = u_o$ on the lateral boundary $\partial\Omega \times (0,T)$.

Rigorous formulation: repalce $|Du|$ by total variation $\|Du(t)\|(\Omega)$.

The total variation flow

Initial-boundary value preoblem for the total variation flow

$$\begin{cases} \partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_P \Omega_T. \end{cases}$$

Existence results by Andreu, Ballester, Caselles, Mazón.

Variational inequality:

$$\iint_{\Omega_T} |Du|\,dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + |Dv|\right] dxdt$$
$$- \tfrac{1}{2}\|v(T) - u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0) - u_o\|^2_{L^2(\Omega)}$$

for any (sufficiently regular) comparison function $v \colon \Omega_T \to \mathbb{R}$ with $v = u_o$ on the lateral boundary $\partial\Omega \times (0,T)$.

Rigorous formulation: repalce $|Du|$ by total variation $\|Du(t)\|(\Omega)$.

The total variation flow

Initial-boundary value preoblem for the total variation flow

$$\begin{cases} \partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_P \Omega_T. \end{cases}$$

Existence results by Andreu, Ballester, Caselles, Mazón.

Variational inequality:

$$\begin{aligned} \iint_{\Omega_T} |Du| \, dxdt \le & \iint_{\Omega_T} \left[\partial_t v (v - u) + |Dv| \right] dxdt \\ & - \tfrac{1}{2} \| v(T) - u(T) \|^2_{L^2(\Omega)} + \tfrac{1}{2} \| v(0) - u_o \|^2_{L^2(\Omega)} \end{aligned}$$

for any (sufficiently regular) comparison function $v \colon \Omega_T \to \mathbb{R}$ with $v = u_o$ on the lateral boundary $\partial\Omega \times (0, T)$.

Rigorous formulation: repalce $|Du|$ by total variation $\|Du(t)\|(\Omega)$.

The total variation flow

Initial-boundary value preoblem for the total variation flow

$$\begin{cases} \partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_P \Omega_T. \end{cases}$$

Existence results by Andreu, Ballester, Caselles, Mazón.

Variational inequality:

$$\iint_{\Omega_T} |Du|\, dxdt \le \iint_{\Omega_T} \left[\partial_t v(v-u) + |Dv|\right] dxdt \\ - \tfrac{1}{2}\|v(T) - u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|v(0) - u_o\|^2_{L^2(\Omega)}$$

for any (sufficiently regular) comparison function $v\colon \Omega_T \to \mathbb{R}$ with $v = u_o$ on the lateral boundary $\partial\Omega \times (0, T)$.

Rigorous formulation: repalce $|Du|$ by total variation $\|Du(t)\|(\Omega)$.

Existence of variational solutions to the obstacle problem for the total variation flow

Theorem (B., Duzaar, Scheven).

- Assume that

$$\psi \in L^2(\Omega_T) \cap L^1_{w^*}(0,T;\mathrm{BV}_{u_o}(\Omega)).$$

  Then, there exists a variational solution u of the obstacle problem to the total variation flow with $u \geq \psi$ a.e. in Ω_T.

- Assume that ψ is upper semicontinuous. Then, there exists a (generalized) variational solution u of the obstacle problem to the total variation flow with $u(t) \geq \psi(t)$ at least outside a set of Hausdorff-dimension $\leq n-1$ for a.e. $t \in (0,T)$.

Existence of variational solutions to the obstacle problem for the total variation flow

Theorem (B., Duzaar, Scheven).

- Assume that

$$\psi \in L^2(\Omega_T) \cap L^1_{w^*}(0,T;\mathrm{BV}_{u_o}(\Omega)).$$

  Then, there exists a variational solution u of the obstacle problem to the total variation flow with $u \geq \psi$ a.e. in Ω_T.
- Assume that ψ is upper semicontinuous. Then, there exists a (generalized) variational solution u of the obstacle problem to the total variation flow with $u(t) \geq \psi(t)$ at least outside a set of Hausdorff-dimension $\leq n-1$ for a.e. $t \in (0,T)$.