Umbilic cylinders in General Relativity or the very weird path of trapped photons

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Introduction: Schwarzschild black hole

Light rays (aka null geodesics) are affected by spacetime curvature. They can get deflected, attracted, or "trapped".

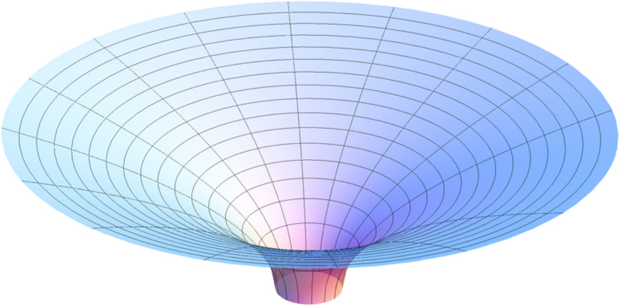
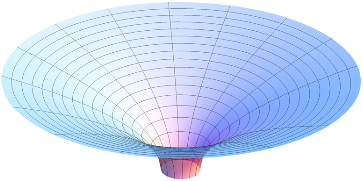


Image credit: Wikipedia.org

Introduction: Schwarzschild black hole ctd.

- deflected light rays: can be detected $\rightarrow$ gravitational lensing
- attracted light rays: cannot be detected
- trapped light rays: cannot be detected



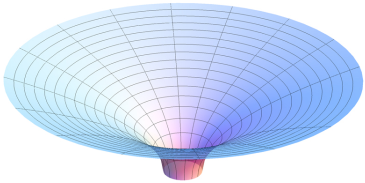
Introduction: Schwarzschild black hole ctd.

The black hole horizon is one of the centered spheres of area radius

$$r_{\mathrm{BH}} = \frac{2GM}{c^2},$$

where $M > 0$ is the mass of the black hole.

$\rightarrow$ This means that the area is $4\pi r_{BH}^2$.



G is the gravitational constant,

c is the speed of light.

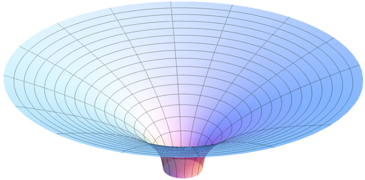
Introduction: Schwarzschild black hole ctd.

The trapped light rays form the

photon sphere.

It is one of the centered spheres of area radius

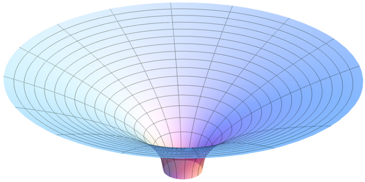
$$r = \frac{3GM}{c^2} = 1.5 r_{\mathrm{BH}}.$$



Introduction: Schwarzschild black hole ctd.

Geometric properties of photon sphere:

- intrinsically round: constant Gauß curvature K
- extrinsically round: trace free second fundamental form h
- constant mean curvature H
- constant static lapse function N
- constant normal derivative $\nu(N)$



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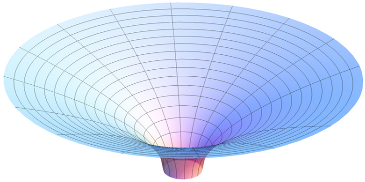

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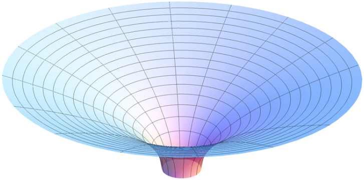

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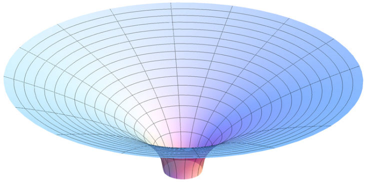

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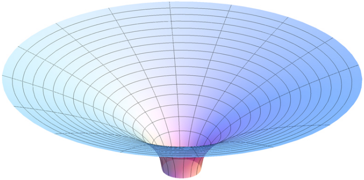

Photon spheres in more generality

- Do photon spheres exist in more general spacetimes?
  $\rightarrow$ Yes! Astrophysical expectation:

  Photon spheres form around any astrophysical object that is suitably compact: $r < \frac{3GM}{c^2}$

- Why are they relevant?
  $\rightarrow$ Central in gravitational lensing.

  Winding number $\cong$ number of gravitational images.

  $\rightarrow$ Crucial for dynamical stability of the Kerr black hole spacetime.

  Long term fate of the universe.

- Do they have hair?
  $\rightarrow$ No ...

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Uniqueness of photon spheres

Theorem (C. 2014)

The Schwarzschild spacetime is the only regularly N-foliated static vacuum asymptotically flat spacetime permitting a photon sphere (up to isometry).

Theorem (C. & Galloway 2015)

There cannot be multiple photon spheres and black holes in a static vacuum asymptotically flat spacetime. The existence of one photon sphere or one black hole already implies that the spacetime is isometric to the Schwarzschild spacetime.

Contents

Setup: mathematical model

Proposition

Generic static spacetimes can be canonically decomposed into

$$\begin{aligned} M^4 &= \mathbb{R} \times M^3 \\ ds^2 &= -N^2 c^2 dt^2 + {}^3g \end{aligned}$$

with induced Riemannian metric ${}^3g$ *and lapse function*

$$N := \frac{1}{c}\sqrt{-ds^2(\partial_t, \partial_t)} > 0.$$

Static geometry: the facts

- All time-slices $\{t = \text{const.}\} = (M^3, {}^3g)$ are isometric.
- They are embedded into $(M^4 = \mathbb{R} \times M^3, ds^2)$ with vanishing second fundamental form.
- The lapse function $N : M^3 \to \mathbb{R}^+$ is independent of "time".

$\Rightarrow$ We think of a static spacetime as a static system: tuple $(M^3, {}^3g, N)$.

Photon spheres

Definition (C. 2014)

Let $(M^3, {}^3g, N)$ be a static system.

A smooth closed surface $\Sigma \subset M^3$ is a photon sphere if $N|_\Sigma \equiv$ const. and if any null geodesic in the corresponding spacetime

$$(\mathbb{R} \times M^3, ds^2 = -c^2N^2 + {}^3g)$$

which is initially tangent to the cylinder $\mathbb{R} \times \Sigma$ remains tangent to it.

More generally:

- Def.: photon surface: umbilic timelike hypersurface $\mathfrak{P}^3 \hookrightarrow \mathbb{R} \times M^3$
- Def.: trapped: $N|_{\mathfrak{P}^3} \equiv$ const. ($\cong$ energy conservation).

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Static vacuum equations

Einstein's equations in vacuum reduce to the

Vacuum static metric equations

$$\begin{aligned} {}^3\triangle N &= 0 \\ N\,{}^3\mathrm{Ric} &= {}^3\nabla^2 N \end{aligned}$$

outside the support of the matter.

Asymptotically flatness

A static system (M^3, g, N) is called asymptotically flat if

- M^3 is diffeomorphic to $\mathbb{R}^3 \setminus K$, K a union of pairwise disjoint balls.
- g, N satisfy fall-off conditions as $r \to \infty$
- Formulated as deviations from the flat (special relativity/ Minkowski) spacetime:

$$g_{ij} = \delta_{ij} + \frac{2m}{r}\delta_{ij} + \mathcal{O}(r^{-1}),$$
$$N = 1 - \frac{m}{r} + \mathcal{O}(r^{-1})$$

- Everything worded in weighted Sobolev spaces.

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- M^3 is diffeomorphic to $\mathbb{R}^3 \setminus K$, K a union of pairwise disjoint balls.
- g, N satisfy fall-off conditions as $r \to \infty$
- Formulated as deviations from the Schwarzschild black hole spacetime ($m := MG/c^2$):

$$\begin{aligned} g_{ij} &= \delta_{ij} + \frac{2m}{r}\delta_{ij} + \mathcal{O}(r^{-1}), \\ N &= 1 - \frac{m}{r} + \mathcal{O}(r^{-1}) \end{aligned}$$

- Everything worded in weighted Sobolev spaces.

Regular N-foliation

Definition

A static asymptotically flat system $(M^3, {}^3g, N)$ is called regularly N-foliated if M^3 is foliated by (spherical) level sets of N.

Remark: This is guaranteed asymptotically by $N = 1 - \frac{m}{r} + \mathcal{O}(\frac{1}{r^2})$ as $r \to \infty$ if $m \neq 0$.

Contents

The theorems once more

Theorem (C. 2014)

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Theorem (C. & Galloway 2015)

There cannot be multiple photon spheres and black holes in a static vacuum asymptotically flat spacetime. The existence of one photon sphere or one black hole already implies that the spacetime is isometric to the Schwarzschild spacetime.

Step 1: Derive (quasi-)local geometric properties of photon spheres in vacuum:

Proposition C. 2014

Let (M^3, g, N) be a vacuum static system with a (connected) embedded photon sphere $\Sigma \hookrightarrow M^3$. Then

- the mean curvature H of $\Sigma \hookrightarrow (M^3, g)$ is constant,
- the embedding $\Sigma \hookrightarrow (M^3, g)$ is totally umbilic: $\mathring{h} = 0$,
- the Gauß curvature K of Σ is constant
- N and its normal derivative $\nu(N)$ are constant on Σ.

Moreover, we find the algebraic identities

$$NH = 2\nu(N), \quad (rH)^2 = \frac{4}{3}, \quad \frac{1}{r^2} = K = \frac{3}{4}H^2,$$

where $r := \sqrt{|\Sigma|/4\pi}$ is the area radius.

Step 2 for first theorem:

Theorem (C. 2014)

The Schwarzschild spacetime is the only regularly N-foliated static vacuum asymptotically flat spacetime permitting a photon sphere (up to isometry).

- Rewrite static vacuum equations as equations on the leaves $\{N \equiv \text{const.}\}$ (via Gauß- and Codazzi-equations).
- Drop terms of the form $|\mathring{h}|^2 \geq 0$ and $|\nabla(\nu(N))|^2 \geq 0$.
- Integrate resulting inequalities over $\{N \equiv \text{const.}\}$ and $N \in [N_\Sigma, 1]$.
- Use Fubini's theorem to obtain algebraic inequalities.
- Use asymptotic decay information about N and g and the quasi-local properties of the photon sphere to conclude rigidity.
- Deduce $\mathring{h} = 0$, $\nu(N) \equiv$ const. on each leaf and thus spherical symmetry of the spacetime. □

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Remark

Similar to Israel's black hole uniqueness proof:

Theorem (Israel 1967)

The Schwarzschild black hole is the only regularly N-foliated static vacuum asymptotically flat spacetime permitting a horizon $(N = 0, H = 0)$ (up to isometry).

Step 2 for second theorem:

Theorem (C. & Galloway 2015)

There cannot be multiple photon spheres and black holes in a static vacuum asymptotically flat spacetime. The existence of one photon sphere or one black hole already implies that the spacetime is isometric to the Schwarzschild spacetime.

- In each photon sphere, glue in a neck piece of a very carefully chosen Schwarzschild (up to the horizon).
- Double the manifold across its horizon boundary.
- Verify the glued manifold is $C^{1,1}$ across the gluing surfaces.
- Conformally modify & one-point-compactify (∞), achieving:
  - non-negative scalar curvature, geodesically complete
  - one AF end with vanishing ADM-mass,
  - enough (Sobolev weak) regularity across gluing surfaces and ∞.
- Apply rigidity case of positive mass theorem (weak version). $\square$

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Remark

Extending Bunting and Masood-ul-Alam's black hole uniqueness proof:

Theorem (Bunting & Masood-ul-Alam 1987)

There cannot be multiple black holes in a static vacuum asymptotically flat spacetime. The existence of one black hole already implies that the spacetime is isometric to the Schwarzschild spacetime.

Consequence for the static n-body problem

Theorem (C. & Galloway 2015)

There are no static equilibrium configurations of n bodies and k black holes with $n + k \geq 1$ in which each body is surrounded by its own photon sphere.