Riemannian Geometry with Skew Torsion

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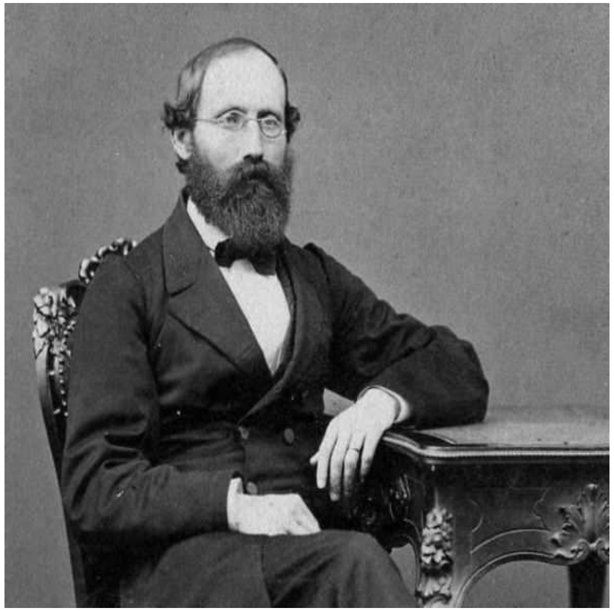
Three major players...



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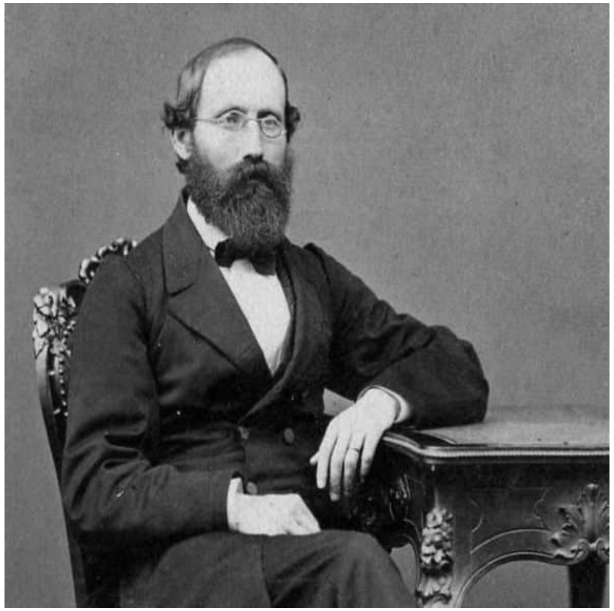

Bernhard Riemann
(1826-1866)

Albert Einstein
(1879-1955)

Élie Cartan
(1869-1951)

Small fish in a big shark tank...

Riemannian geometry with skew torsion

Ana Cristina Castro Ferreira
St Cross College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Hilary Term 2010

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The center of the world...

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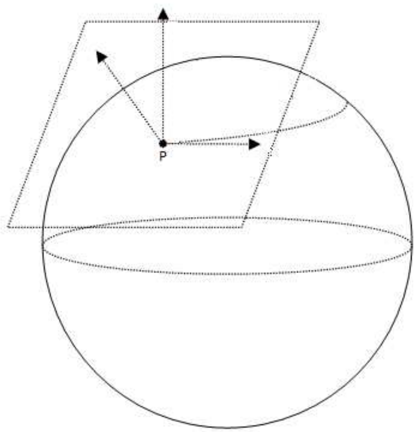
Philipps-Universität Marburg

Basic ingredients

Riemannian manifolds (M^n, g)

B. Riemann's Habilitationsvortrag (Göttingen, 1854) " Über die Hypothesen, welche der Geometrie zugrunde liegen"



M^n – a manifold of dimension n
g – a metric, i.e. a scalar product on each tangent space

Connections

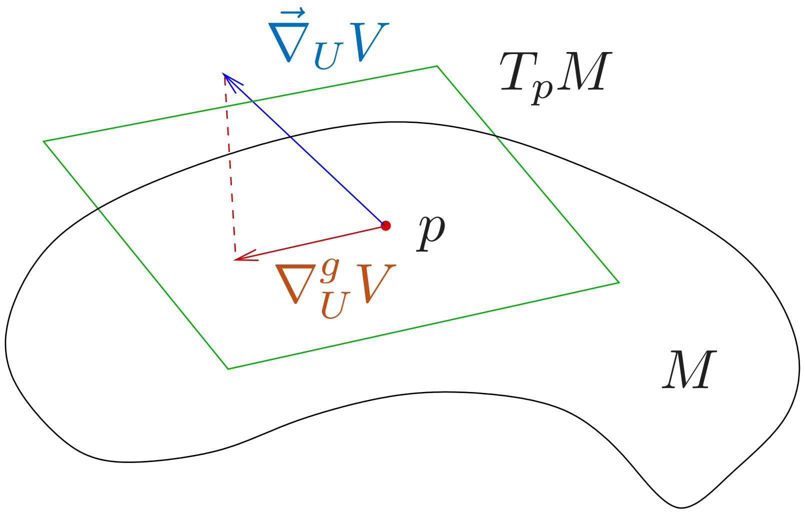
Throwback to calculus: Direccional derivative of vector-valued smooth functions $f : \mathbb{R}^p \longrightarrow \mathbb{R}^q \longleftrightarrow \vec{\nabla}$

Connection ∇: abstract derivation rule satisfying all formal properties of dir. derivative

different name: 'covariant derivative'

Ex. Projection $\nabla^g_U V$ of dir. derivative $\vec{\nabla}_U V$ to tangent plane

= 'Levi-Civita connection' ∇^g



[∇^g: completely determined by g]

Connections

However... not only possibility
→ connection with torsion [Dfn: Cartan, 1925]

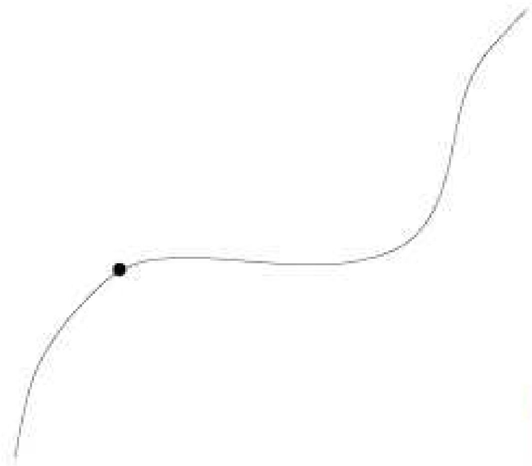
Ex. Electrodynamics: $\nabla_U V := \vec{\nabla}_U V + \frac{ie}{\hbar} A(U) V$ ($\Leftrightarrow$ $\nabla_\mu = \partial_\mu + \frac{ie}{\hbar} A_\mu$)
A: gauge potential = electromagnetic potential

Ex. If $n = 3$: $\nabla_U V := \vec{\nabla}_U V + U \times V$
additional term gives space an 'internal angular momentum', a *torsion*

Fact: 3 types of torsion: vectorial, skew symmetric, and [something else].

Classical general relativity and electromagnetism

point particle



moves along
a curve γ

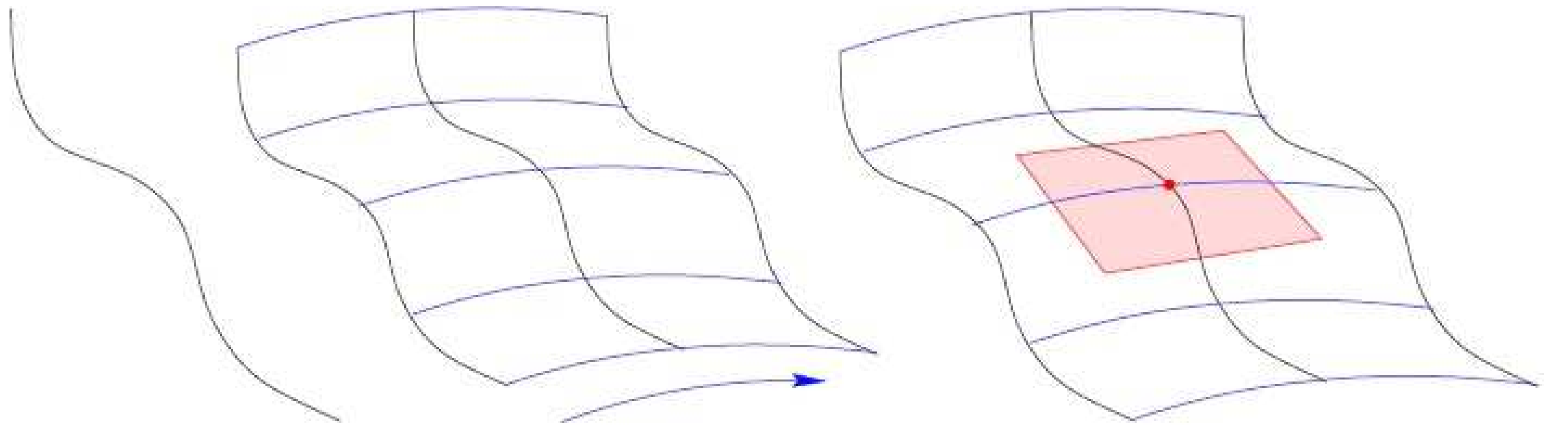
physical action: $\int_\gamma A$

for a potential A : 1-form

field strength
$F = dA$: 2-form $\Leftrightarrow$ geometric concept
of curvature

curvature measures deviation from vacuum !

Modern unified models



string particle

moves along
a surface S

physical action: $\int_S \tilde{A}$ for
a higher order potential $\tilde{A}$: 2-Form

higher order field strength
$F = d\tilde{A}$: 3-form

$\Leftrightarrow$

geometric concept
of torsion

torsion measures deviation from vacuum ("integrable case") !

Torsion and curvature

$x_1, \cdots, x_n$ coordinates on M^n; vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$

$$T\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$$

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \qquad \left[\frac{\partial^2}{\partial x_i \partial x_j} \stackrel{?}{=} \frac{\partial^2}{\partial x_j \partial x_i}\right]$$

Also: Ricci curvature, scalar curvature...

New data: (M^n, g, T)

More adapted to certain geometries:

- Lie groups, KT manifolds, Generalized geometry, Homogeneous spaces
- Almost Hermitian manifolds – almost Kähler, nearly Kähler – almost contact metric manifolds – quasi-Sasaki – cocalibrated G2

Einstein metrics

Field Equations: M^4 spacetime, g signature (3,1)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Mathematical Einstein equations:

(M^n, g) Riemannian manifold

$$R_{\mu\nu} = \frac{R}{n} g_{\mu\nu}$$

Topological obstruction – known only in dimension 4

Hitchin-Thorpe Inequality: $2\chi \geq 3|\tau|$ [1969/1974]

Einstein metrics with skew torsion

Dfn: In dimension 4, based on the phenomenon of self-duality [F. '11]

- Hitchin-Thorpe Inequality holds $\surd$
- $*T$ is a Killing field

Question: How to generalize to higher dimensions?

Einstein metrics with skew torsion

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Question: How to generalize to higher dimensions?

$$¿\ \boxed{R_{\mu\nu} = \frac{R}{n} g_{\mu\nu}}\ ?$$

$\rightarrow$some 'issues' with this definition...

Einstein metrics with parallel skew torsion

Assume $\nabla T = 0$. Then [Agricola-F. '13]

$\rightarrow$ $R_{\mu\nu}$ is symmetric

$\rightarrow$ scalar curvature is a constant

$\rightarrow$ $R > 0 \Rightarrow$ M compact and π_1 finite

$\longrightarrow$ Best possible analogy with the Riemannian case $\longleftarrow$

- Hordes of examples of manifolds with parallel torsion:
  Sasaki, nearly Kähler, nearly parallel G_2, naturally reductive spaces...

- Systematic investigation of such examples $\surd$

Homogeneous spaces

→Links algebraic theory of Lie groups and geometric notions such as isometry and curvature

→good examples in Riemannian geometry

→classification of (Riemannian) symmetric spaces [Cartan 1926]

However classification without further assumptions is impossible

- very small dimensions
- positive curvature
- isotropy irreducible (examples of Einstein manifolds)

Our class:

naturally reductive homogeneous spaces

Homogeneous spaces

- (M, g) plus $G \subseteq \text{Iso}(M)$ s.t. G acts on M transitively.
  K stabilizer of a point $p \in M \longrightarrow M = G/K$.

Thm. [Ambrose-Singer 1958]

(M, g) is homogeneous iff exists torsion T s.t. $\nabla T = \nabla \mathcal{R} = 0$

T skew symmetric $\longleftrightarrow$ (M, g) naturally reductive

Previous classifications of nat. red. spaces

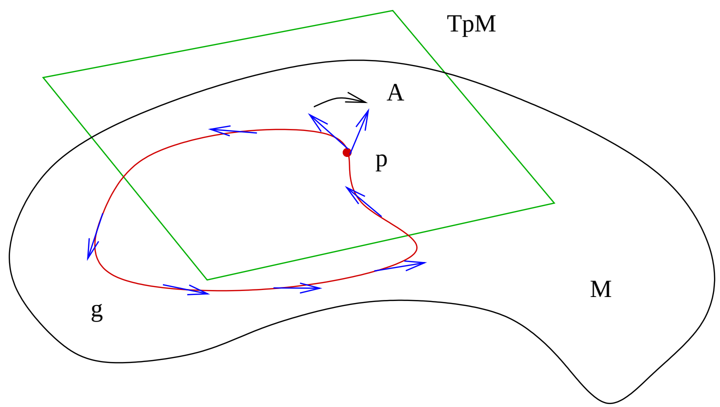
$\longrightarrow$Dim 3 (Tricerri-Vanhecke, 1983)

$\longrightarrow$Dim 4 and 5 (Kowalski-Vanhecke, 1983, 1985)

Our methods

[Agricola-F.-Friedrich '2015]

- Look at the parallel torsion as the fundamental object.
- Use recent development in the holonomy theory of connections with parallel skew torsion.
- Determine the "geometric nature" of M



An important tool

- $\sigma_T := \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T) \;=\; \overset{X,Y,Z}{\mathfrak{S}}\; g(T(X,Y), T(Z,V))$ $\quad$ $(= 0 \text{ if } n \leq 4)$

$\rightarrow$For "non-degenerate" torsion, the connection in (AS) is the characteristic connection for some known geometry (almost contact, almost Hermitian...).

Classifications

$\sigma_T = 0$**:** ($n \geq 5$) M is a compact simple Lie group or its dual noncompact symmetric space

$\sigma_T \neq 0$**:**

dim.	space	remarks
		$T \sim \mathrm{dvol}$
	$\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$	spaces forms
	$\mathrm{SU}(2), \mathrm{SL}(2,\mathbb{R}), H^3$	left. inv. metric
4		$*T$ parallel field
	$N^3 \times \mathbb{R}$	N^3 nat. red.
5		$*\sigma_T$ Reeb field
	H^5	quasi-Sasaki
	$(G_1 \times G_2)/\mathrm{SO}(2)$	
	$SU(3)/SU(2), SU(2,1)/SU(2)$	α-Sasaki

Classifications

dim.	space	remarks
6		$*\sigma_T$ skew-sym. endomorph.
	$G_1 \times G_2$	$\mathrm{rk}(*\sigma_T) = 2$
	cannot occur	$\mathrm{rk}(*\sigma_T) = 4$
		$\mathrm{rk}(*\sigma_T) = 6$
	S^6, $S^3 \times S^3$, $\mathbb{C}P^3$, $\mathrm{U}(3)/U(1)^3$	type $\mathcal{W}_1$
	$\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{S}^3 \times \mathbb{R}^3, \mathbb{S}^3 \ltimes \mathbb{R}^3$ $\mathbb{S}^3 \times \mathbb{S}^3, \mathrm{SL}(2, \mathbb{C})$	type $\mathcal{W}_1 \oplus \mathcal{W}_3$

¡ Vielen Dank für Ihre Aufmerksamkeit !