Moduli spaces, integrable systems and applications to surface theory

Lynn Heller
(joint work with S. Heller and N. Schmitt, pictures by N. Schmitt and U. Wagner)

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Surface Theory

$M$ smooth, compact and Riemann surface
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$f : M \rightarrow \mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ conformal immersion
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$f$ is determined by induced metric, mean curvature $H$, and Hopf differential $Q$. 
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Surfaces satisfying variational properties
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  $\leadsto$ constant mean curvature (CMC) surfaces: $H = \text{const}$
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Theorem $f$ has constant mean curvature (CMC) in $\mathbb{R}^3$, $S^3$, $H^3$, $\ldots$ iff its Gauss map is harmonic.
Surface Theory

Surfaces satisfying variational properties

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  $\Rightarrow$ constant mean curvature (CMC) surfaces: $H = \text{const}$
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Theorem
$f$ has constant mean curvature (CMC) in $\mathbb{R}^3, S^3, H^3$ iff its Gauß map is harmonic.
Questions concerning CMC surfaces in $S^3$

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- What are their properties (area, embeddedness,..)?
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- Which compact CMC surfaces exist?
- What are their properties (area, embeddedness,..)?
- Characterize the moduli space of (compact, embedded) CMC surfaces!
State of the art

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► trivial fundamental group (genus 0): only round spheres (Hopf, Almgren)

► abelian fundamental group (genus 1):

► Pinkall & Sterling (89), Hitchin (90), Bobenko (91): all CMC tori via integrable systems

► Brendle, Andrews & Li (2012): embedded CMC tori are rotational symmetric

► non-abelian fundamental group (genus \( \geq 2 \)):

► few examples (Lawson, Karcher-Pinkall-Sterling, Kapouleas)

► no systematic methods for CMC surfaces of higher genus
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The space of embedded CMC tori in $S^3$
Gauge-theoretic description of CMC surfaces

\[ f : M \rightarrow S^3 \text{ via Maurer-Cartan form } \omega = f^{-1}df \in \Omega^1(M, su(2)) \]
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associated family of \( SL(2, \mathbb{C}) \)-connections on \( M \times \mathbb{C}^2 \):

- \( \Phi - \Phi^* = f^{-1}df \) and \( \lambda \in \mathbb{C}_* \)

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\nabla^\lambda = d + \frac{1}{2}(1 + \lambda^{-1})(1 + iH)\Phi - \frac{1}{2}(1 + \lambda)(1 - iH)\Phi^*
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- reconstruction of \( f \) as gauge between \( \nabla^{\lambda_{1/2}}, \lambda_{1/2} \in S^1 \), Sym point conditions
CMC tori (Hitchin)

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- \( \mathbb{C}^2 = L_\lambda \oplus L^*_\lambda \)

- \( \nabla^\lambda = d + \begin{pmatrix} \alpha(\lambda)d\omega - \chi(\lambda)d\bar{\omega} & 0 \\ 0 & -\alpha(\lambda)d\omega + \chi(\lambda)d\bar{\omega} \end{pmatrix} \)

\( L_\lambda, L^*_\lambda \) intersect at finitely many points \( \lambda_i \in \mathbb{C}^* \)

\( \Rightarrow \) Spectral curve \( \Sigma : \xi^2 = \lambda \prod_i (\lambda - \lambda_i) \)

\( \Rightarrow (\chi, \alpha) \) globally well-defined and meromorphic on \( \Sigma \)

\( \Rightarrow \) finite dimensional problem

\( (\Sigma, \chi, \alpha) \) are algebraic and determine \( f \) up to dressing (\( \lambda \)-dependent gauge)
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Non-abelian monodromy

Theorem (S. Heller 09)

For immersed CMC surfaces of higher genus $\nabla^\lambda$ is irreducible for generic $\lambda \in \mathbb{C}_*$. 
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- no straightforward generalization of the spectral curve theory
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For immersed CMC surfaces of higher genus $\nabla^\lambda$ is irreducible for generic $\lambda \in \mathbb{C}_*$. 

- no parallel eigenline bundles
- no straight forward generalization of the spectral curve theory
- gauge equivalence classes $[\nabla^\lambda]$ determine the surface
Reduction to Fuchsian Systems

Observation: $\mathbb{Z}_{g+1}$ symmetric CMC surface (4 fixed points)
$\Rightarrow$ Fuchsian systems $\nabla$ on $M/\mathbb{Z}_{g+1} \cong \mathbb{C}P^1$
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4 singular points $\{0, 1, m, \infty\} \in \mathbb{C}P^1$

$$\nabla = d + A_0 \frac{dz}{z} + A_1 \frac{dz}{z-1}dz + A_m \frac{dz}{z-m}, \quad A_i \in sl(2, \mathbb{C})$$
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Eigenvalue $\pm \rho_i$ of $A_i \rightsquigarrow$ local monodromies conjugated to

$$\begin{pmatrix} \exp(2\pi i \rho_i) & 0 \\ 0 & \exp(-2\pi i \rho_i) \end{pmatrix}$$
Abelianization of Fuchsian Systems

arXiv:1404.7707, joint work with S. Heller

Generic Fuchsian system $\nabla$ with $\rho_i \in ] - \frac{1}{2}, \frac{1}{2}[$ gauge equivalent to

$$d + \begin{pmatrix} -\chi d\bar{\omega} + \alpha d\omega & \beta^- d\omega \\ \beta^+ d\omega & \chi d\bar{\omega} - \alpha d\omega \end{pmatrix}$$

for immersed $Z_{g+1}$ symmetric CMC surfaces $\rho_i = \rho = g_2 g_2 + 2$.
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- for immersed $\mathbb{C}_{g+1}$ symmetric CMC surfaces $\rho_i = \rho = \frac{g}{2g + 2}$
How to obtain CMC surfaces

In order to construct CMC surfaces we need to satisfy:

- Unitarity condition for $\nabla^\lambda$ along $\lambda \in S^1$,
- Asymptotic at $\lambda \to 0$,
- Existence of two connections with trivial monodromy on $S^1$. 

Unitary connections given by Narasimhan-Seshadri section $\alpha_{NS}$:

Jac($T^2$) $\to \mathbb{A}^1$ $\xrightarrow{\alpha}$ is uniquely determined by $\chi$.

Family of Fuchsian systems determined by spectral data $(\Sigma, \chi, \alpha)$ induces CMC surface with boundary $f$: $T^2 \setminus l_1 \cup l_2 \to S^3$. 

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Construction of spectral data

arXiv:1501.01929, joint work with S. Heller, N. Schmitt

Idea: deform spectral data of known surfaces (CMC tori) towards higher genus using $t = 2\rho - \frac{1}{2}$ as parameter!
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Geometric visualisation:

- cut torus along curvature lines
- open the angle $4\pi t$ between curvature lines
Flow of spectral data

t-deformation induces deformation of spectral data:
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- consider $\chi + x : \Sigma^0 \to \text{Jac}(T^2) \cong \mathbb{C}/\Gamma$,
  $x$ in a Banach space of hol. functions on an open RS $\Sigma^0$
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- application of implicit function theorem for Banach spaces
Results


Short time existence of spectral data flow:
- for (stable) homogeneous CMC tori,
Results


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- for 2-lobed Delaunay tori exists two different initial directions.
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*For small rational $t$ there exists (new) families of compact (branched) CMC surfaces.*
Results


*Short time existence of spectral data flow:*
  - for (stable) homogeneous CMC tori,
  - for 2-lobed Delaunay tori exists two different initial directions.


*For small rational t there exists (new) families of compact (branched) CMC surfaces.*

*If the flow exits until $\rho = \frac{g}{2g+2}$, we obtain closed immersed CMC surfaces of genus $g$ with 4 umbilics of order $g - 1$. 
Experimental flows from 2-lobed Delaunay tori

Figure: Deformation of 2-lobed CMC tori in stable direction
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Experimental moduli space of embedded CMC surfaces

arXiv: 1503.07838, joint work with S. Heller, N. Schmitt
Lawson $\xi_{k,l}$ surfaces