Moduli spaces, integrable systems and applications to surface theory

Lynn Heller

(joint work with S. Heller and N. Schmitt, pictures by N. Schmitt and U. Wagner)

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EBERHARD KARLS
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Surface Theory

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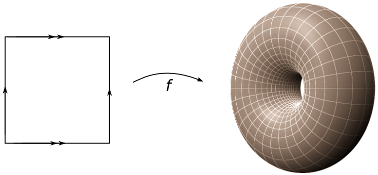
$f : M \to \mathbb{R}^3,\ \mathbb{S}^3,\ \mathbb{H}^3$ conformal immersion



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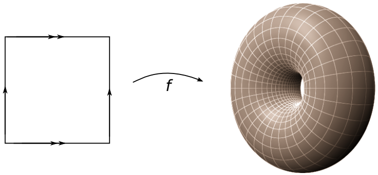

f is determined by induced metric, mean curvature H, and Hopf differential Q.

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Theorem

f has constant mean curvature (CMC) in $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ iff its Gauß map is harmonic.

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- Which compact CMC surfaces exist?
- What are their properties (area, embeddedness,..)?
- Characterize the moduli space of (compact, embedded) CMC surfaces!

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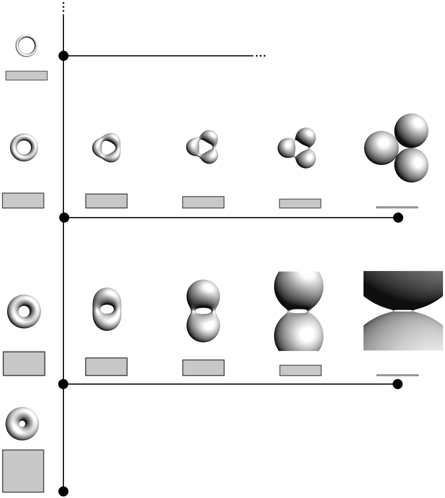
  - no systematic methods for CMC surfaces of higher genus

The space of embedded CMC tori in $\mathbb{S}^3$



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- $\Phi - \Phi^* = f^{-1}df$ and $\lambda \in \mathbb{C}_*$

$$\nabla^\lambda = d + \tfrac{1}{2}(1+\lambda^{-1})(1+iH)\Phi - \tfrac{1}{2}(1+\lambda)(1-iH)\Phi^*$$

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- reconstruction of f as gauge between $\nabla^{\lambda_{1,2}}$, $\lambda_{1/2} \in S^1$, Sym point conditions

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- (Σ, χ, α) are algebraic and determine f up to dressing
  (λ-dependent gauge)

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Theorem (S. Heller 09)

For immersed CMC surfaces of higher genus ∇^λ *is irreducible for generic* $\lambda \in \mathbb{C}_*$.

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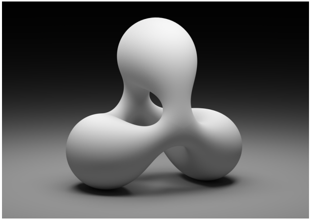
- gauge equivalence classes $[\nabla^\lambda]$ determine the surface

Reduction to Fuchsian Systems

Observation: $\mathbb{Z}_{g+1}$ symmetric CMC surface (4 fixed points)
$\Rightarrow$ Fuchsian systems ∇ on $M/\mathbb{Z}_{g+1} \cong \mathbb{C}P^1$



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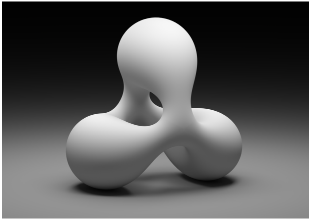

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$$\nabla = d + A_0 \frac{dz}{z} + A_1 \frac{dz}{z-1} dz + A_m \frac{dz}{z-m}, \quad A_i \in sl(2, \mathbb{C})$$

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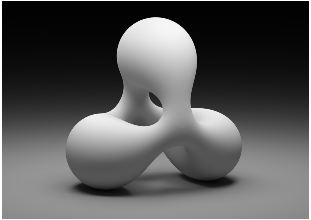

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Eigenvalue $\pm\rho_i$ of $A_i \rightsquigarrow$ local monodromies conjugated to

$$\begin{pmatrix} \exp(2\pi i \rho_i) & 0 \\ 0 & \exp(-2\pi i \rho_i) \end{pmatrix}$$

Abelianization of Fuchsian Systems

arXiv:1404.7707, joint work with S. Heller

Generic Fuchsian system ∇ with $\rho_i \in]-\frac{1}{2}, \frac{1}{2}[$ gauge equivalent to

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- for immersed $\mathbb{Z}_{g+1}$ symmetric CMC surfaces $\rho_i = \rho = \frac{g}{2g+2}$

How to obtain CMC surfaces

In order to construct CMC surfaces we need to satisfy:

- Unitarity condition for ∇^λ along $\lambda \in S^1$,
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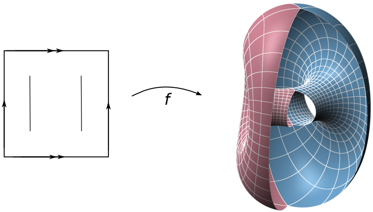
Family of Fuchsian systems determined by spectral data (Σ, χ, α) induces CMC surface with boundary $f : T^2 \setminus l_1 \cup l_2 \to S^3$.



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arXiv:1501.01929, joint work with S. Heller, N. Schmitt

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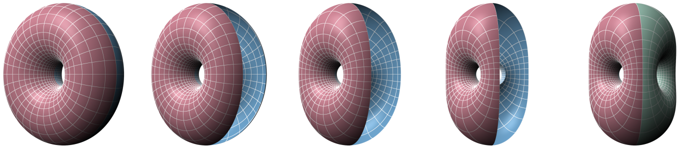
Geometric visualisation:

- cut torus along curvature lines
- open the angle $4\pi t$ between curvature lines



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- application of implicit function theorem for Banach spaces

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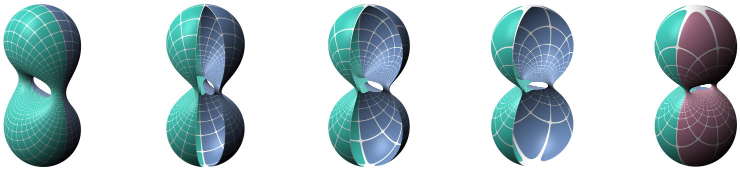
Experimental flows from 2-lobed Delaunay tori



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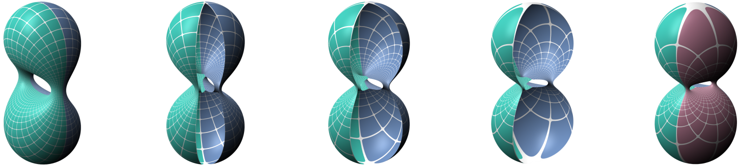

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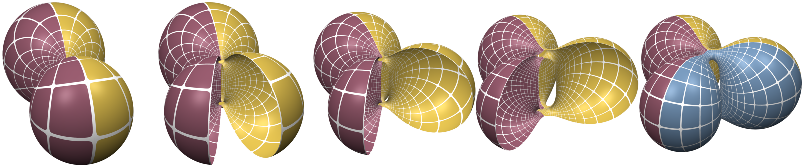

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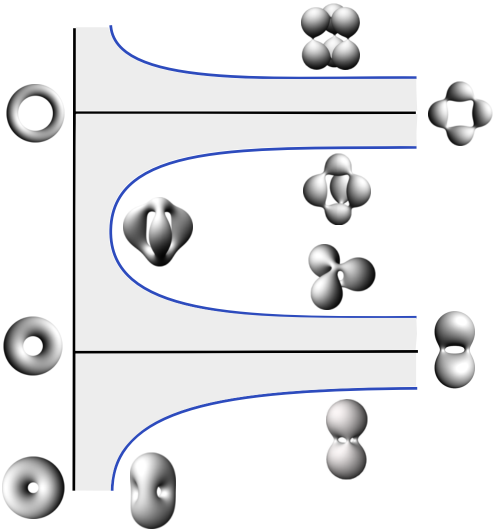
Experimental moduli space of embedded CMC surfaces

arXiv: 1503.07838, joint work with S. Heller, N. Schmitt



Lawson $\xi_{k,l}$ surfaces

arXiv: 1503.00969, joint work with S. Heller, N. Schmitt