An epiperimetric inequality approach to the regularity of the free boundary in the thin obstacle problem

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Classical obstacle problem

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Suppose we want to wrap a meatloaf in a plastic wrap.
Here the meatloaf is the obstacle, and the configuration of the plastic wrap, after it adjusts to the geometry of the meatloaf, represents the solution to the obstacle problem.

Formulation of the classical obstacle problem

We are given:

- $\phi \in C^2(D)$, the *obstacle*;
- $\psi \in W^{1,2}(D)$ with $\phi \leq \psi$ on ∂D, the *boundary values*;
- $f \in L^\infty(D)$, the *source term*.

We want to minimize

$$\int_D (|\nabla u|^2 + 2fu)dx$$

over $\mathcal{K} = \{u \in W^{1,2}(D) \ : u = \psi \text{ on } \partial D, u \geq \phi \text{ a.e. in } D\}$.

There exists a unique minimizer u which satisfies:

$$\Delta u = f \text{ in } \{u > \phi\}$$

$$\Delta u = \Delta\phi \ \text{ a.e. on } \{u = \phi\}.$$

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- Coincidence set: $\Lambda_\phi(u) = \{x \in D \mid u(x) = \phi(x)\}$.
- Free boundary: $\Gamma_\phi(u) = \partial\{x \in D \mid u(x) = \phi(x)\}$.

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First fundamental question: How smooth is the solution? The optimal regularity of the solution is $u \in C^{1,1}_{\text{loc}}(D) \cong W^{2,\infty}_{\text{loc}}(D)$.

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Second fundamental question: How smooth is the free boundary? In 1977 Kinderlherer and Nirenberg proved that, if the free boundary is a C^1 hypersurface, then it is C^ω (real analytic). In the same year Caffarelli developed his theory of the regularity of the free boundary and proved Lipschitz regularity, and then proved how to go from Lipschitz to $C^{1,\alpha}$.

The thin obstacle problem

We are given:

- $D \subset \mathbb{R}^n$: bounded domain;
- $\mathcal{M}$: smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$, that divides D into two parts, D_+ and D_-;
- $\phi : \mathcal{M} \to \mathbb{R}$, the *obstacle*;
- $\psi : \partial D \to \mathbb{R}$ with $\psi > \phi$ on $\mathcal{M} \cap \partial D$;
- $A(x) = [a_{ij}(x)]$ in D.

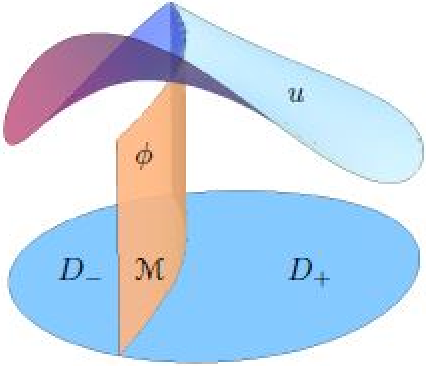
We want to minimize

$$\int_D \langle A(x)\nabla u, \nabla u\rangle dx, \tag{0.1}$$

over the convex set

$$\mathcal{K} = \{u \in W^{1,2}(D) \mid u = \psi \text{ on } \partial D, u \geq \phi \text{ on } \mathcal{M} \cap D\}. \tag{0.2}$$

The thin obstacle problem



Where does the thin obstacle problem appear?

- In elasticity (Signorini), when an elastic body is at rest, partially laying on a surface $\mathcal{M}$.
- It models the flow of a saline concentration through a semipermeable membrane.
- In mathematical finance, when the random variation of an underlying asset changes discontinuously.

Notations, definitions

- $B_1 \subset \mathbb{R}^n$: unit ball; $\mathcal{M} = B_1' = B_1 \cap \{x_n = 0\}$;
- $S_1 \subset \mathbb{R}^n$: unit sphere;
- the obstacle $\varphi \in C^{1,1}(B_1')$;
- $\psi : S_1 \to \mathbb{R}$ such that $\psi > \varphi$ on $S_1 \cap B_1'$
- $A(x) = [a_{ij}(x)]$ defined in B_1.

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Definitions

$$\Lambda_\phi(u) = \{x \in \mathcal{M} \cap D \mid u(x) = \phi(x)\} \quad \textit{coincidence set}$$

$$\Gamma_\phi(u) = \partial_{\mathcal{M}} \Lambda_\phi(u) \quad \textit{free boundary}$$

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- (ellipticity) $\exists \lambda > 0$ such that

$$\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda^{-1} |\xi|^2, \forall x \in B_1, \forall \xi \in \mathbb{R}^n;$$

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$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda^{-1}|\xi|^2, \forall x \in B_1, \forall \xi \in \mathbb{R}^n;$$

- Lipschitz continuity of the a_{ij}, $|a_{ij}(x) - a_{ij}(y)| \leq M|x - y|, \forall i, j.$

Properties of u

Signorini conditions:

- $Lu := \mathrm{div}(A\nabla u) = 0$ in $B_1^+ \cup B_1^-$,
- $u \geq \varphi$ in B_1',
- $\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle \geq 0$ in B_1',
- $(u - \varphi)(\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle) = 0$ in B_1'.
  ambiguous boundary conditions

Regularity of the solution

Caffarelli (1979):

When $\mathcal{M}$ is a hyperplane, ϕ is $C^{2,\alpha}$ for some $0 < \alpha < \frac{1}{2}$ and $a_{ij} \in C^{1,1}_{loc}$: solution is $C^{1,\alpha}_{\text{loc}}(D_\pm \cup \mathcal{M})$.

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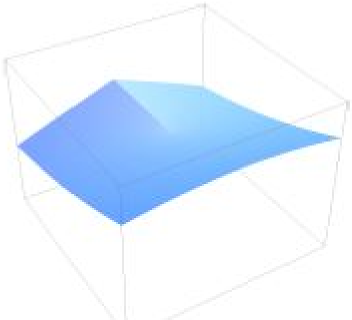
Arkhipova and Uraltseva (1985-87):

Same conclusion when $a_{ij} \in W^{1,p}_{\text{loc}}$ and $\phi \in W^{2,p}_{\text{loc}}$, for some $p > n$. This includes, in particular, $a_{ij} \in W^{1,\infty}_{\text{loc}} = C^{0,1}_{loc}$.

What about optimal regularity?

Even when $A(x) \equiv I$, $\mathcal{M}$ is flat and $\phi = 0$ the best one can hope for is $C^{1,\frac{1}{2}}_{loc}(D_\pm \cup \mathcal{M})$. One has in fact the following global solution to the Signorini problem with $\mathcal{M} = \{x_n = 0\}$, and $\phi \equiv 0$

$$u(x) = \Re(x_1 + i|x_n|)^{3/2} \in C^{1,\frac{1}{2}}_{loc}(D_\pm \cup \mathcal{M}).$$



Optimal Regularity

Athanasopoulos & Caffarelli (2004): case $A(x) \equiv I$

Major breakthrough: they proved that when $\mathcal{M}$ is flat and $\phi = 0$, then the solution u to the Signorini problem is $C^{1,\frac{1}{2}}_{loc}(D_\pm \cup \mathcal{M})$ for any $n \geq 2$.

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Had already proved this result when $n = 2$. His method is based on complex analysis and does not extend to $n \geq 3$.

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Guillen (2009): $A(x) \in C^{1,\gamma}_{loc}$, for some $\gamma > 0$

Proved $C^{1,\frac{1}{2}}_{loc}(D_\pm \cup \mathcal{M})$ regularity when $\mathcal{M}$ is flat and ϕ in $C^{1,\beta}$ for some $\beta > \frac{1}{2}$.

Garofalo & Smit Vega Garcia (2013), $A(x) \in C^{0,1}_{loc}$

By means of some new monotonicity formulas, we have established the $C^{1,\frac{1}{2}}_{\text{loc}}(D_\pm \cup \mathcal{M})$ regularity when $\phi \in C^{1,1}$ and the manifold $\mathcal{M}$ is flat.

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Theorem (Garofalo & Smit Vega Garcia:)

Let u be the solution of the Signorini problem (0.1), (0.2) in B_1, with $A(x) \in C^{0,1}(B_1)$, and $\phi \in C^{1,1}(B_1')$. If $0 \in \Gamma_\phi(u)$, then $u \in C^{1,\frac{1}{2}}_{\mathrm{loc}}(B_1^\pm \cup B_1')$ and

$$||u||_{C^{1,\frac{1}{2}}(B^\pm_{\frac{1}{2}} \cup B'_{\frac{1}{2}})} \leq C(n, \lambda, M, ||u||_{W^{1,2}(B_1)}).$$

Historical background: Almgren's monotonicity formula

The crucial tool introduced to study the regularity of the minimizer is a fundamental monotonicity formula proved in 1979 by F. Almgren, who proved that if $\Delta u = 0$ in B_1, then the frequency of u, given by

$$r \to N(u, r) = \frac{rD(r)}{H(r)} = \frac{r \int_{B_r} |\nabla u|^2}{\int_{S_r} u^2},$$

is increasing in $(0, 1)$. Furthermore, $N(r) \equiv \kappa \iff u$ is homogeneous of degree κ, i.e., $u(rx) = r^\kappa u(x)$.

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Theorem (Athanasopoulos, Caffarelli and Salsa (2007))

Let u be the solution of the Signorini problem in B_1 when $A(x) \equiv I$, with zero obstacle and flat thin manifold $\mathcal{M}$. Then, the frequency $r \to N(u,r)$ is increasing in $(0,1)$. Moreover, $N(u,r) \equiv \kappa$ for $0 < r < 1 \iff u$ is homogeneous of degree κ in B_1.

Structure of the proof in the zero obstacle case

Let us consider the general Signorini problem (0.1), (0.2), with $A(x) \in C^{0,1}(B_1)$. When the obstacle is zero, we introduce the following

$$\text{Frequency function:} \quad N(r) = \frac{rI(r)}{H(r)}, \; 0 < r < 1,$$

where

$$I(r) = \int_{S_r} u\langle A(x)\nabla u, \nu\rangle = \int_{B_r} \langle A(x)\nabla u, \nabla u\rangle,$$

$$H(r) = \int_{S_r} u^2 \mu d\sigma, \quad \mu(x) = \frac{\langle A(x)x, x\rangle}{|x|^2}.$$

Recall the original frequency function:

$$N(r) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{S_r} u^2}.$$

Crucial result:

Theorem (Monotonicity formula for zero obstacle)

There exists a universal $C > 0$ such that $r \to \tilde{N}(r) = e^{Cr} N(r)$ is monotone nondecreasing for $0 < r < 1$.

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Important observation: the above theorem is not true for a general Lipschitz matrix $A(x)$. We prove it under the assumption that

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However, we show that there exists a local $C^{1,1}$ diffeomorphism which reduces the general Signorini problem (0.1), (0.2) to one for which (0.3) holds. So (0.3) is not restrictive!

Blow-ups

Assume $0 \in \Gamma(u)$.
Non-homogeneous rescalings of u:

$$u_r(x) = \frac{u(rx)}{d_r}, \; x \in B_{\frac{1}{r}}, \text{ where } d_r = \left(\frac{1}{r^{n-1}} \int_{S_r} u^2 \mu \right)^{1/2}$$

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Using the monotonicity of $e^{Cr} N(r)$: $\exists u_0 \in W^{1,2}(B_1)$ such that (up to a subsequence),

- $u_{r_j} \to u_0$ weakly in $W^{1,2}(B_1)$,
- $u_{r_j} \to u_0$ in $L^2(S_1, d\sigma)$ and
- $u_{r_j} \to u_0$ in $C^1_{\text{loc}}(B_1^{\pm} \cup B_1')$

Properties of the limit

- $\Delta u_0 = 0$ in $B_1^+ \cup B_1^-$;
- $u_0 \geq 0$ on B_1',
- $\langle \nabla u_0, \nu_+ \rangle + \langle \nabla u_0, \nu_- \rangle \geq 0$ on B_1',
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- u_0 is homogeneous of degree $N(0+)$.
- $N(0+) = \frac{3}{2}$ or $N(0+) \geq 2$ (Petrosyan, Shahgholian & Uraltseva).

Blow up and optimal regularity

Lemma

Let u solve the thin obstacle problem in B_1 with $\varphi = 0$, $0 \in \Gamma(u)$ and $N(0+) \geq \kappa$. Then $\exists C_0 = C_0(n, \lambda, M, ||u||_{W^{1,2}(B_1)}) > 0$ such that

$$\sup_{B_r} |u| \leq C_0 r^{\kappa}, 0 < r < r_0.$$

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$$\sup_{B_r} |u| \leq C_0 r^{\kappa}, 0 < r < r_0.$$

Now recall that we have $\kappa \geq \frac{3}{2}$!

Challenges of the non-zero obstacle case

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Now $Lv = f$ outside B_1', with $f \in L^\infty$ and life is not so easy...
When $L = \Delta$, then

$$\frac{r}{2} \frac{d}{dr} \log H(r) = N(r) + \frac{n-1}{2}. \tag{0.4}$$

Challenges of the non-zero obstacle case

For a general L one has

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The term

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is the one to be tamed. Its potential bad oscillations could destroy the monotonicity of the adjusted frequency.

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New idea: work with a truncated version of a suitable normalization of $H(r)$.

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We introduce functions:

$$\psi(r), \quad \sigma(r) : (0,1] \to (0,\infty),$$

where the role of $\psi(r)$ is to kill the term

$$\frac{r}{2}\frac{\int_{S_r} v^2 L|x|}{H(r)},$$

whereas the role of the function $\sigma(r)$ is to clean up the blood introduced in the process.

A new frequency function

We define the renormalized height and energy

$$M(r) = \frac{1}{\psi(r)} H(r), \qquad J(r) = \frac{1}{\psi(r)} I(r)$$

and introduce the generalized frequency of v

$$\Phi(r) = \frac{\sigma(r) J(r)}{M(r)} = \frac{\sigma(r) I(r)}{H(r)},$$

Partial Monotonicity

Recall that

$$\Phi(r) = \frac{\sigma(r)J(r)}{M(r)} = \frac{\sigma(r)I(r)}{H(r)}.$$

Theorem (Partial monotonicity of the generalized frequency)

Given $\delta \in (0,1)$, there exist $r_0, K > 0$ such that $r \to e^{Kr^{\frac{1-\delta}{2}}}\Phi(r)$ is non-decreasing in the open set

$$\left\{ r \in (0, r_0) \mid M(r) > r^{3+\delta} \right\}.$$

Full Monotonicity

Theorem (Monotonicity of the truncated frequency)

Given $\delta \in (0,1)$, there exist $r_0, K > 0$ (depending on δ and $||f||_{L^\infty}$), such that

$$N(r) = \frac{\sigma(r)}{2} e^{Kr^{\frac{1-\delta}{2}}} \frac{d}{dr} \log \max \left\{ M(r), r^{3+\delta} \right\}.$$

is monotone non-decreasing on $(0, r_0)$.

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Define

$$\tilde{N}(r) = \frac{r}{\sigma(r)} N(r).$$

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Define

$$\tilde{N}(r) = \frac{r}{\sigma(r)} N(r).$$

Since $\lim_{r \to 0+} \frac{r}{\sigma(r)} = \alpha > 0$, $\tilde{N}(0+)$ exists.

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Then, since $\tilde{N}(0+) \geq \frac{3}{2}$, we obtain that

$$\sup_{B_r} |v| \leq Cr^{\frac{3}{2}},$$

which gives optimal regularity.

Regularity of the regular part of the free boundary

With v as before and $x_0 \in \Gamma(v)$, denote

$$v_{x_0}(x) = v(x_0 + A^{1/2}(x_0)x) - b_{x_0} x_n,$$
$$b_{x_0} = \langle A^{1/2}(x_0) \nabla v(x_0), e_n \rangle.$$

Definition

$$\Gamma_{3/2}(v) = \left\{ x_0 \in \Gamma(v) \ : \ \tilde{N}(0+, v_{x_0}) = \frac{3}{2} \right\}$$ is the regular set.

The regular set

In the Laplacian case, the regular set is $C^{1,\alpha}$ regular:

Proposition (Caffarelli, Salsa, Silvestre, 2008:)

Let u solve the thin obstable problem in B_1 with $A \equiv I$ and $\varphi \in C^{2,1}$. Then $\Gamma_{3/2}(u)$ is locally a $C^{1,\alpha}$-regular $(n-2)$-dimensional surface.

Weiss monotonicity formula

We introduce a functional suited for the study of the blowups at regular free boundary points by analysing the homogeneous rescallings of v:
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Weiss type functional: $W_L(v,r) = \frac{\sigma(r)}{r^3}\left\{J_L(v,r) - \frac{3}{2r}M_L(v,r)\right\}$.

Theorem (Garofalo, Petrosyan & Smit Vega Garcia)

Assume $0 \in \Gamma_{3/2}(v)$. There exists a universal constant $C > 0$ such that

$$\frac{d}{dr}\left(W_L(v,r) + Cr^{1/2}\right) \geq \frac{2}{r^{n+1}}\int_{S_r}\left(\frac{\langle A\nabla v, \nu\rangle}{\sqrt{\mu}} - \frac{3\sqrt{\mu}}{2r}v\right)^2. \qquad (0.5)$$

Hence, $r \mapsto W_L(v,r) + Cr^{1/2}$ is monotone nondecreasing, therefore the limit $W_L(0+,v) \overset{def}{=} \lim_{r\to 0} W_L(v,r)$ exists.

We recall the definition of the homogeneous rescallings of v:

$$C_r(x) = \frac{v(rx)}{r^{3/2}}.$$

Lemma

Let $0 \in \Gamma_{3/2}(v)$. Given $r_j \to 0, \exists C_0 \in C^{1,\alpha}_{\text{loc}}((\mathbb{R}^n)^\pm \cup \{0\}), \forall \alpha \in (0, 1/2)$, such that

$$C_{r_j} \to C_0 \text{ in } C^{1,\alpha}_{\text{loc}}((\mathbb{R}^n)^\pm \cup \{0\}).$$

C_0 is a global solution in $\mathbb{R}^n$ of the Signorini problem with zero thin obstacle, homogeneous of degree $3/2$.

Epiperimetric inequality

Let

$$h(x) = \Re(x_1 + i|x_n|)^{\frac{3}{2}}.$$

In the Laplacian case, our Weiss functional takes an easier form:

$$W_\Delta(v) := W_\Delta(v, 1) = \int_{B_1} |\nabla v|^2 - \frac{3}{2} \int_{S_1} v^2 d\mathcal{H}^{n-1}.$$

Theorem (Garofalo, Petrosyan, Smit Vega Garcia)

There exists $\kappa \in (0,1)$ and $\theta \in (0,1)$ such that if $w \in W^{1,2}(B_1)$ is homogeneous of degree $\frac{3}{2}$, $w \geq 0$ on B_1' and $\|w - h\|_{W^{1,2}(B_1)} \leq \theta$, then there exists $\zeta \in W^{1,2}(B_1)$ such that $\zeta = w$ on S_1, $\zeta \geq 0$ on B_1' and

$$W_\Delta(\zeta) \leq (1 - \kappa) W_\Delta(w).$$

Uniquess of blowups

For $r > 0$ and $x_0 \in \Gamma_{3/2}(v)$, we define the homogeneous rescalings $C_{x_0,r}(x) = \frac{v(x_0+A^{1/2}(x_0)x)}{r^{3/2}}$.

Proposition

Let v be as before with $x_0 \in \Gamma_{3/2}(v)$. Then, $\exists\ r_0 = r_0(x_0), \eta_0 = \eta_0(x_0) > 0$ such that

$$\Gamma(v) \cap B'_{\eta_0}(x_0) \subset \Gamma_{3/2}(v).$$

Moreover, if $C_{\bar{x},0}$ is any blow up of v at $\bar{x} \in \Gamma(u) \cap B'_{\eta_0}(x_0)$, then

$$\int_{S_1} |C_{\bar{x},r} - C_{\bar{x},0}| \leq Cr^{\gamma}, \quad \text{for all } r \in (0, r_0),$$

where C and $\gamma > 0$ are universal constants.
In particular, the blow-up limit $C_{\bar{x},0}$ is unique.

Regularity of the regular set

Theorem (Garofalo, Petrosyan, Smit Vega Garcia, 2015)

Let v be a solution of the thin obstacle problem with $x_0 \in \Gamma_{3/2}(v)$. Then there exists $\eta_0 = \eta_0(x_0) > 0$ such that

$$B'_{\eta_0}(x_0) \cap \Gamma(v) \subset \Gamma_{3/2}(v)$$

and

$$B'_{\eta_0}(x_0) \cap \Lambda(v) = B'_{\eta_0}(x_0) \cap \{x_{n-1} \leq g(x'')\}$$

for $g \in C^{1,\beta}(\mathbb{R}^{n-2})$ with a universal exponent $\beta \in (0,1)$, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$.

Thank you!